Computational Celtic Canvas for Zoomorphs and Knotworks

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ABSTRACT

This paper presents *Celtic Canvas* -- a framework towards computationally generating patterns similar to simple Celtic artwork. As Celtic patterns are highly structured, a robust model of curvature dynamics is introduced in the paper to allow the evolution of space curves as a foundation for line drawing and structural anatomy. Cellular automata integrates several features including -- lines and forms, allocation of space, a solution of crossing states, the occlusion processing, depth cues, and line weighing. Lattice gas automata assist in generating varying width patterns. Original digitally available artwork provides shapes and forms as line-art. Our novel contribution is that we are successful in generating examples of knotwork and zoomorphs (animal designs shapes) that mimic the characteristics of the Celtic art forms. Future research areas are also identified.

1. INTRODUCTION

Reconstruction of Celtic design methods could perhaps be traced to the work of J. Romilly Allen's [All93] 1903's survey which was followed by the studies of George Bain [Bai51] in 1951. As part of her survey of Celtic interlace in Northumbria (Northern England, 500-1100), Gwenda Adcock [Adc74] developed the system now used for archeological description and analysis. Iain Bain in 1986 [Bai86] transformed many of his father's construction techniques into concise procedures or algorithms. Peter Cromwell [Cro93] followed with his examination of frieze patterns, and is credited with considering interlace as being traced by a ray reflected by bounding structure [Fun07]. Paul Gerdes described this generator in detail, first for Tchokwe and Tamil pictograms [Ger90] and later for Celtic interlace [Ger99]. A theory of mirror curves emerged in [Jab95]. Adcock's research [Adc74] discussed evidence of templates. While the approach is workable in typography, it also helped illustrate another of Adcock's observations, that a skilled artist is needed to prevent monotonous patterns. Christians Mercat [Mer97] observed that crossings in Celtic interlace comprise an encoding (tetravalent, of four converging cord segments) enabling the specification of arbitrary interlace in terms of planar graphs. Frank Drewes [Dre89] demonstrated graph grammars where terminal symbols are associated with tiles. This resembles the approach taken in Lindenmayer-Systems [Pru90] as well. One of the characteristics of the illustrations is to appear hand drawn. So the following guidance was developed: (a) reduce artifacts that may be perceived as the result of a rendering process, (b) portray materials accurately according to context; (c) understand what to emphasize; (d) avoid patterns and regularities; (e) follow established rules of traditional rendering techniques – how the lines are placed and use visual cues such as line weight and depth variation.

Development in the theory of alternating knots [Bro05, Chu05] and tangles now extends to classification and enumeration [Bae07]. Adcock [Adc02] completed a study of interlaced animal designs (zoomorphs) in Bernician sculpture and their relationship to work produced by the monastery at Lindisfarne. One significant difference between zoomorphs and interlace is that interlace involves strictly closed curves and animal designs more often involve open curves. The Isenberg [Ise06] study recommends experimentation with approaches to line weight and cuing and line shape as that can convey emotion [Fre03]. Knotwork, such as work by David Llewellyn-Jones, posted on the web, does not have a zoomorph implementation. NPR research today includes modeling for pencil [Mer08] and ink on paper [Chu05], brush and paint on canvas [Bax04], medieval manuscripts [Bro04] and collections at Trinity College Dublin [Qui07]. Parametric curves, such as Bezier, Hermite, or B-Splines [Far97] tend to be less expressive than those drawn by hand, yet managing continuity is always an important piece which may affect the artwork. Work by Kurt Fleischer [Fle97] on self-organizing geometry is relevant for generating adaptive and curved lattice gas models [Har71], and can be useful due to the evolution of a trefoil shaped filament [Mal96] shapes and shape grammars. In this paper, we focus on providing research on how to computationally generate knotwork [LAr07] and zoomorphs (animal design forms). Our research and results are novel and promising in the sense that we have been successful in capturing some of the innate aesthetics and simplicity observed in knotwork and zoomorphs of Celtic Art form. These are shown as crossing (Figures 2, 4, 12, and 13), design elements (Figure 6-8), and zoomorphs (Figures 9 and 14). These are basic elements found in Celtic Art, and according to best of our knowledge, not captured by any other existing *computational* method.

2. BACKGROUND

To conform to the style of insular design from the 9th century, proportions for interlace are found in Adcock [Adc74; 02]. Alternatively these can also be measured from lithographs, photography from the manuscript departments of universities, or measured from digital photography available on the web. Similar measurements may be developed for zoomorphs, plant designs, humans, key patterns, spirals, and so forth.

The key idea in complex systems theory is that local interactions give rise to organized global behavior. Brush-strokes by an artist are examples of local control creating global emergence, which is the overall painting. Modeling of these systems involves simulation over assemblies of discrete elements such as brush-strokes. Broad surveys have been done [Fle95]. Organizations include lattice gas automata (LGA) [Har76]. Other related work are in [Sim94] but for very different applications.

Differential Geometry

The nib of a pen traces a line on a writing surface. The point of contact may be thought of as a particle moving along a space curve in the plane of the writing surface. From the fundamental theorem of space curves, if curvature and torsion are differentiable over the arc length and curvature is everywhere positive, there exists a unit speed space curve with those properties. The initial unit tangent and unit normal vectors may be assigned as long as their dot product is everywhere zero. For a planar space curve, torsion is everywhere zero. In this special case, the curvature function may be signed, allowing turns left and right. The formulation is interesting to us as varying shape curve can be generated given an initial position X, unit tangent (T), unit normal (N), and B, and assigned curvature K(s) and torsion (TO(s)) values which provide the rate of change of present position of the curve using curve parameter (s) based on differential equation formulation described in [Gra98].

Mirror Curve Generator

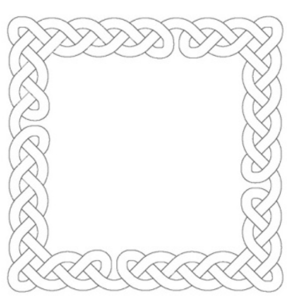

Figure 1: Single knot three cord border.

Part of the fascination of mirror curves in design is that an outcome may not be immediately obvious. For example, while placing breaks in a variation on a grid studied by Aidan Meehan [Mee03], it was not apparent that the preceding knot-work in Figure 1 uses a *single* cord. Designs in this study, following the example of George Bain, are first developed using medial lines with a density of a one cord per square. As a cord passes through a square, its departure is designated by a chain code. Chain code directions on the Moore neighborhood are shown in Figure 2.

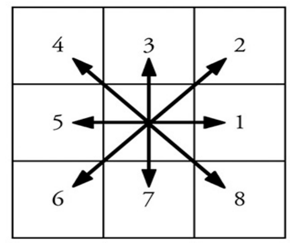

Figure 2: chain code directions.

Using a varying grid, an automaton, called a lattice gas automaton, for left reflection is derived by examining part of the following knot in the following work (Figure 3) using chain codes:

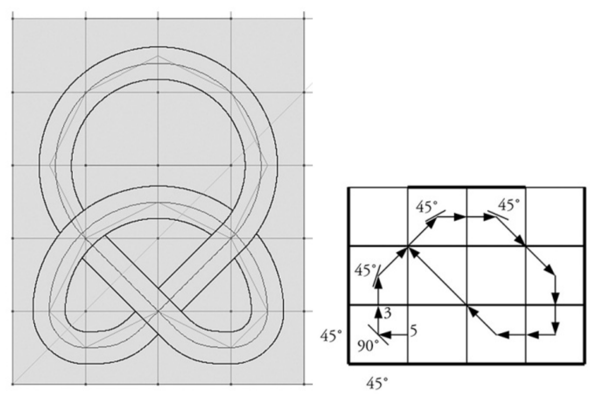

Figure 3: (left) Derivation of left mirror (bottom 3 by 4) moves as (Right) chain codes (3 by 4).

For left reflection assume that mirrors are placed to the left of the particle path in the lattice gas particle automaton. Begin with the particle in the lower left corner heading in chain code direction five (Figure 3). The model describes a particle that interprets a break as a 45-degree direction change. Angles are additive. Addition of random moves can provide more variety as the particle moves, offering variation of patterns.

The motion of the particle simulates the generation of a mirror curve as chain codes Each mirror curve defines the intersections of a guide curve with the grid. If the path traced by the particle is closed in the desire curve, Thurston's theorem says that an alternating knot covering that path exists. A GCA particle tracing the edges of the major grid, interpreting breaks as mirrors, can use Mercat's algorithm to find a coding for the crossings (Figure 4). Alternatively, the encoding can be determined using a mobile cellular automaton that traces an image of the solution for each guide curve across the drawing surface to interpret a composite image of all the guide curves in the composition.

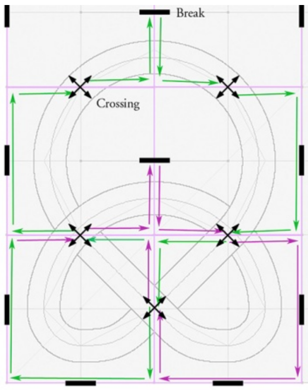

Figure 4: A path traced by Mercat's coding algorithm.

From this all of the intersections are identified. Each intersection, as with Mercat's algorithm, is used to hold the identification of guide curves that pass through it and the reservation of the above or below state for each intersection of the knot. If, while traversing the image of a guide curve, the mobile automaton encounters an intersection where the needed reservation is already taken, the solution fails. In the case of designs involving only knots, resolution of these failures may be resolved through a divide and conquer strategy. However, if the design contains tangles, a solution may not exist. Where a crossing solution exists, it is encoded as a function of arc length on each guide curve in the composition. The coding is adjusted so that state transitions occur half way between intersections.

The lattice gas particle path is drawn as a polygon joining the centers of cell entry and exit points. This automatically generates results in Glassner's visualization [Gla99]. Since the grid is square, the generating sequence for a space curve covering the path can be assembled by collecting sequences from a small set of standard curve segments, indexed by which turn is taken by a particle on exiting a cell relative to the direction on entry. When the grid is irregular, a suitable curve for each segment must be adaptively produced. For example, models for the curve segment are selected and then, in the worst case, fitted by evolutionary search on the ranges of their parameters and the one with the best-fit wins. Our observation is that the behaviour of the particle with such set interfaces resembles what an artist may be doing when judging how to draw a line. In studying the lines of Celtic interlace some of these shapes are familiar. Their incorporation may also be important as suggested by a study on the effect of line shapes on the perception of emotion or style [Fre03].

3. INFLATING THE CURVES

Once the lines representing the Celtic-canvas are satisfactory, closed or open curves represented by cords or the lines can be inflated. For interlace, a cord must be fitted to the guide curve. This process is often described as inflating the cord. If the left and right edges of the cord are assumed to be lines parallel to the guide curve, the parallel line theorem from differential geometry may be used [Gra98]:

$$c_{c\parallel\alpha}(\alpha,s,t) = \frac{sJ\alpha'(t)}{\|\alpha'(t)\|}$$

Tangent velocity of the curve is normalized at t, and scaled by distance s; J is the two-dimensional Jacobian [Gra98; Doy08]. The cord is inflated in the direction of the curve's normal and then repeated in the opposite direction to create cords from lines in a cylindrical fashion. The radius of the cord is used as the scale factor and can vary based on parameter t as shown above. Parameter t could also be used for mapping textures in future.

Design Elements as Forms and Sleeves

Design elements are described as forms and sleeves, as detailed in [Doy08]. Each is represented by line art for the shape of a given element around symmetries to be provided by guide curves which are generated from the given celtic art piece as explained below. Forms are used primarily for zoomorphs and plants in the Celtic Art. Sleeves are specialized forms that provide the shape for cords in the knotwork. The reason for using line art is to allow a designer the freedom to draw these shapes as needed without relying on any particular technology. These line art can then be inflated to create knotwork supplementing the computer generated forms. Design elements provide an efficient mean to provide templates, and can be used as tiles to create continuous curves if properly used.

A form (input shape) is specified in terms of four files: anatomy, symmetry, segmentation, and pivots. Figure 5 shows such an arrangement for our implementation. Anatomy defines what a form looks like (leftmost Figure 5). Symmetry defines a skeleton (left-middle Figure 5). Segmentation defines where a form can be split and stretched along symmetry (right-middle Figure 5). Pivots define where parts of the anatomy may be independently turned and are left for future study (rightmost Figure 5).

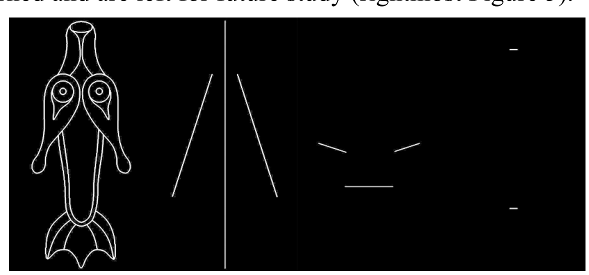

Figure 5: Celtic Snake anatomy, symmetries, segmentation and pivots (left to right).

Extraction of lines from the anatomy drawing is supported by connected region enumeration and a simple set algebra, so desired lines can be obtained (Figure 6). Extracted lines are sampled as distance functions along the arc length of the corresponding symmetry and then coded as part of the genotype for the shape. To resolve occlusion during rendering, lines that form the outermost outline of the form

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are marked as exterior. Sleeves are handled similarly and only have one symmetry. Sleeves are fitted to cover a portion of the arc length of a guide curve. Examples of cord sleeves with their specifications are as follows (Figure 7).

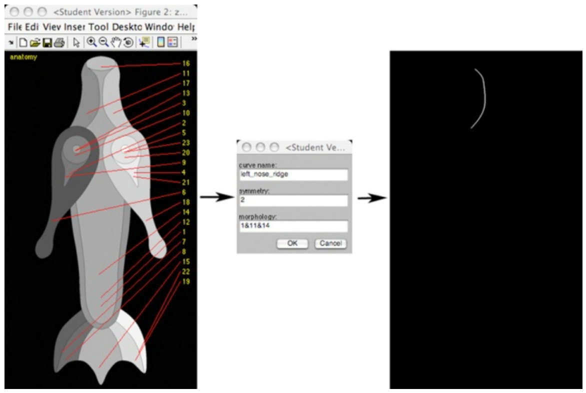

Figure 6: Left Nose ridges extracted from anatomy.

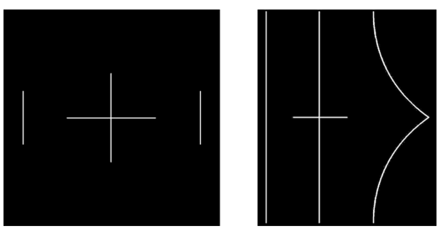

Figure 7: Chord sleeve, left, chevron sleeve right turn, right.

The well structured basic shapes representing genotypes (code for the line-art) for a composite sleeve are assembled on request when binding to the guide curve for a knot. Generative system implementations where genotype or binary bit pattern representing the eventual shape can be expanded to present line art (phenotypes) has been also used in our implementation. An appropriate sleeve (cord, left or right chevron) is selected for each segment of arc length of the guide curve, where a segment is the portion of the guide curve that passes through each cell in the grid.

Artifacts of hand drawn line and rendering

Noise or small random variations in basic line-art code shape or genotype, are used in our implementation to create variety of shapes or phenotypes which then can be represented in line form and inflated as explained earlier. Some of the issues identified by Isenberg [Ise06] have been included in this study. Vector noise accounts for variations in the movement of a designer's hand when drawing. We assume that variation in the motion of the hand, perpendicular to the direction in which a line is being drawn, varies as an arrival problem. In other words random variation occurs at some rate, and can be modeled as an exponential distribution. The scaling term from the Brownian Bridge is borrowed to assure that the ends of walks meet as follows:

$$W_s = \sqrt{\frac{s(s_1 - s)}{s_1}} \cdot f(N_{0.1}), \quad 0 < s < s_1$$

where s_1 is the length of the respective walk and s is the parameter within the walk. This is implemented by genetically generating piecewise continuous curves whose positions left and right of zero are randomly varied by the normal distribution at the rate of the exponential distribution. The generator joining two points is zero seeking. These functions are expensive (time consuming) to generate; so, they are kept short and looped as needed. This noise process is added to the widths of lines centered on guide curves:

$$w_{noisy}(s) = w(s) + W_s$$

The lengths of these noise functions, the amount of their variation, and the rate at which variations occur are set by the designer. The above value of w provides variation of cord width simulating human hand drawing. The depth cue parameter is implemented as a multiplier, d(s), on the width function. Again the curvature dynamics system is used to develop easements in and out of these cues. The combined effect of depth cue and hand motion variation becomes:

$$w_{noisy\,\&\,depth}(s) = d(s) \cdot w(s) + W_s$$

The designer can set an arc length distance from the guide curve being crossed to start a depth cue, as well, as separate parameters for the width of above and below states. The two pass rendering algorithm is implemented. The first pass draws all lines having the *above* state. Lines with the *below* state are drawn on the second pass. When the image of an exterior line having the above state is encountered on the second pass, the state of the lower line is changed to "occluded." This state continues until the next exterior line of the form being passed under is reached. The Sousa pencil model [Sou03] simulates the abrasion and deposition of a graphite source when rubbed against paper. We also implemented an alternate approach for deriving guide curves as patterns generated from a genetic interpretation of a Perlin [42] noise process (Figure 8).

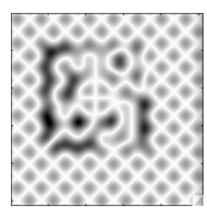

Figure 8: Cross in circle pattern using Perlin Noise.

A number of exotic problems arise first in attempting to find these patterns and then extract them for use in assembling interlace. Although promising, more is needed to computationally extract zoomorphs and knot-work from Perlin noise, especially representing the aesthetics and complexity of Celtic art. More details of this work are in [Doy08].

4. IMPLEMENTATION

Based on methods from the previous Section, our algorithm is summarized as follows:

1: Define structures (grids) with breaks on a canvas

2: Define design elements for the grids

3: Arrange and pose elements within the structure

4: Solve alternating crossings for knots and tangles, ensuring continuity as the curve crosses grid elements

5: Render a result

Multiple grids provide structures for local interactions that are consistent with neighbors joining two points across grid elements, thus providing consistency across grid boundaries. Quadrilateral meshes for structure grids were based on the templates of Matthias Muller-Hannemann [Mul97]. Design elements provide the context and include chain-code shapes (genotypes) or line-art from Celtic art pieces as guidance curves. Complete Celtic-Canvas system diagram is shown in Figure 9.

The system is organized around a set of user interface tools. Inputs to the system include mattes of the drawing surfaces(s), under drawings (optional), line drawings of forms (animals, plants, etc.), and of sleeves. Morphological analysis is used to provide an initial interpretation of drawing surfaces. Tools are provided for defining and editing grids and placing breaks, assisting with the design of guide curves (lattice gases and freehand drawing) and applying them to posing forms.

Celtic Canvas provides an opportunity to view such an art form as a complex system using the work of Mitchell [Mit09]. Complex forms are simulated using mutation and variation of genotypes, and rendering them as phenotypes. Specifically, The forms and guide curves are easily identified as individuals (genotypes). Their phenotypes consist of both structural and visual components including variation of depth and width. Design elements are first formed by a skeleton of guide curves which are derived from their symmetries. The chromosome of a composite sleeve might contain hundreds of sequences, including the description of each thing to be drawn and it's binding to a guide curve. The genotype of a form provides everything needed to solve and pose its anatomy on the supporting guide curve(s).

A Celtic-composition in this system presented contains grids and design elements. It contains a constructor (ribosome) that builds phenotypes of its elements under the guidance of the designer. Its phenotype is the rendering of its design-elements which are results of our implementations as presented in Figures 1, 5, 8, and 10-14. The reason that the shapes generated by our implementation resemble Celtic-art forms is because design elements used in our system are either guided by Celtic-Art form or are designed that way.

Although our focus has been to computationally generate Celtic knotwork and zoomorphs, the formulation presented in Section 2 is also capable of producing three-dimensional shapes and forms if we use 3D design elements and suitable phenotypes. In addition, the formulation presented in this paper can be extended to *non-existent* new interpretations of Celtic-Art forms based on 3D Space filling arbitrary shapes using either solid or hollow three-dimensional cords and zoomorph as design elements.

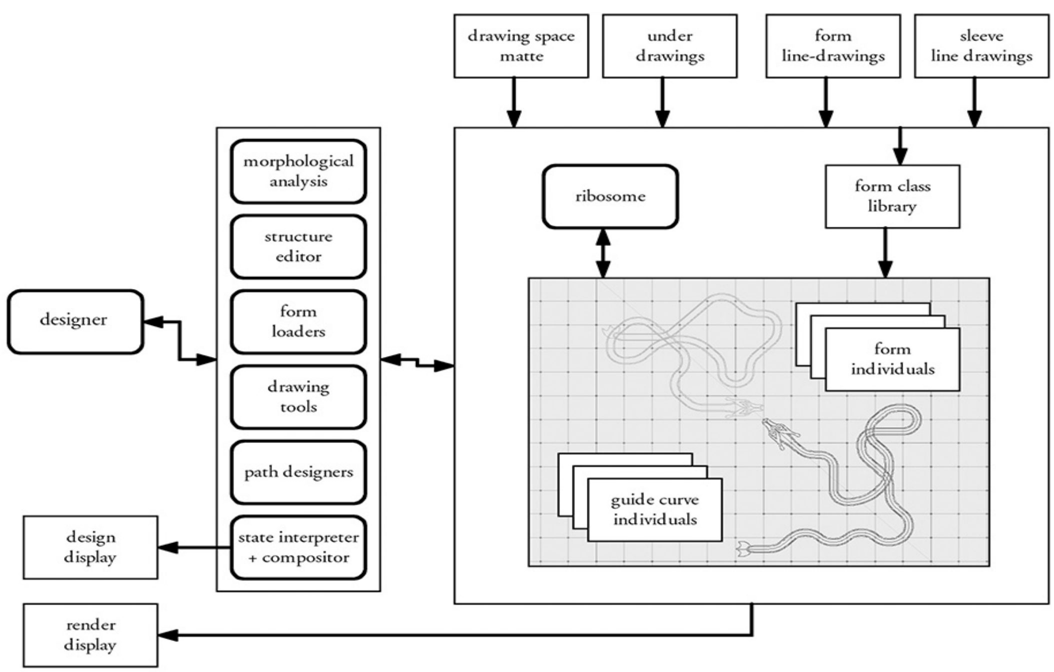

Figure 9: Celtic-Canvas: system diagram.

5. RESULTS

This section describes results of our implementation using Celtic-Canvas (Figure 10 [Doy08]). They are presented with examples from the Book of Kells and the Lindisfarne Gospels [Hen74, Mee94]. The primary references for comparison with hand-drawn examples are those developed by Adcock and the illustrations of Iain Bain. Our review of the literature found that automatic means for evaluating the quality of a visual design remains an open question [Ise06].

Our results in Figures 10-14 show major improvement over existing computational methods presented in [Gla99, Kap03], and take few minutes as symmetry, shape, skeletion and pivot points are defined, as discussed in [Doy08]. Since curvature is such a natural attribute of space curves, one of our first attempt was a Celtic spiral. Our reconstruction (left) uses a progression based on the golden ratio indicated (right) in Figure 11. The design was inspired by a triple spiral on folio 34r of the Book of Kells. It is presented here (left) without artifacts and in the original (Figure 10).

Figure 10: Triple spiral from MS 58 f34r, Board of Trinity College Dublin (right) with permission.

Lindisframe Knotwork (Bain): Although knotwork is not part of the lozenge in folio 34r of the Book of Kells, it is fundamental to most Celtic design. Using Celtic Canvas, a reconstruction using Celtic Canvas of a design from the Lindisfarne Gospels (f. 11) [7] is shown in Figure 11 using with (right) and without (left) slight artifact. Both are produced with our implementation.

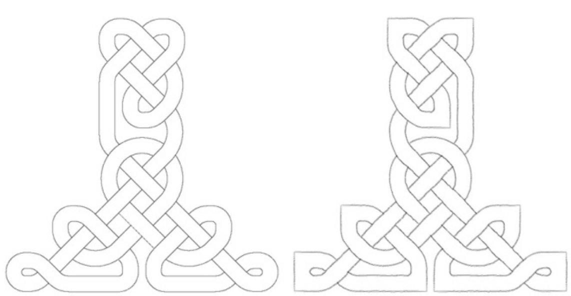

Figure 11: Lindisfarne (f. 11) reconstruction, left, and with added artifacts, right.

Lindisframe Knotwork (Adcock)

A more complex example of knotwork is a fragment from the Lindisfarne Gospels (f. 26) [Adc73, Bai51]. The design was not drawn from a single cord, but rather three. An omitted break revealed a design using only two cords. A single cord would be Celtic symbolism for eternity. A reconstruction with chevrons and artifacts is in Figure 12. Patterns for curves with larger radii can be recognized when curve segment sequences are symbolically coded. This figure includes curve sweetening by substitution of larger radii curves. These beautiful computationally generating curves using Celtic-Canvas can be further improved by evolving the parameters that affect their proportional fit in the grid.

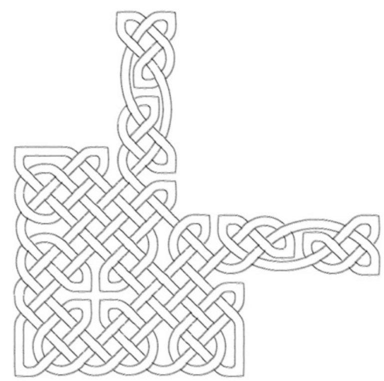

Figure 12: Lindisfarne Gospels (f. 26) corner with artifacts

Reptile (snake) based from Lozenge of MS 58 f 34r: A motivator for our research was the lozenge in cross of the Chi in folio 34r of the Book of Kells, first studied by George Bain in 1951[Bai51]. An example of a snake posed on a guide curve designed by a mirror curve using Celtic Canvas is in Figure 13.

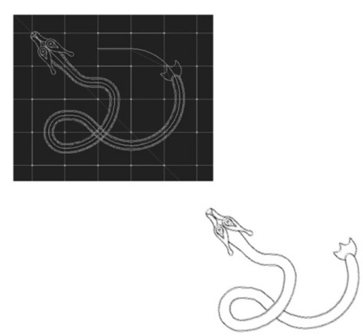

Figure 13: Posed reptile with artifacts

A significant portion of the design in the lozenge in folio 34r is devoted to a tangle of creatures. A series of reptiles were assembled as a test of the solver for crossings in alternating knots and tangles. An LGA (Lattice Gas Automata) with a higher degree of freedom was used to design guide curves to pose each of the snakes. The result of our implementation without artifacts is in Figure 14 and shows complex tangled designs Celtic Canvas can generate, specially

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the visual variation in each resulting snake is interesting.

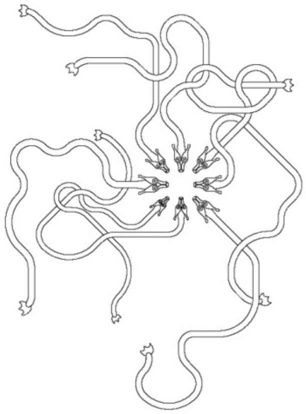

Figure 14: Tangle of reptiles.

6. CONCLUSIONS

In this paper, we have presented a novel implementation to computationally generate interlacing knotwork and zoomorphs seen in historical Celtic Art. The complex system based implementation integrates some of the aesthetics and characteristics of the original art form as its design elements. Significant contribution of our work is that shapes and forms, mimicking the original art, emerge as shown by several examples of knotwork (Figures 11, 12) and zoomorphs (Figure 14). In addition, we also developed a solution for alternating crossings and rendering. An implementation of additive vector noise based on our research provides hand-drawn quality to our work. We were not able to computationally generate complexities in Figure 15. George Bain's reconstruction of the lozenge at the crossing of strokes in the Chi of folio 34r in the Book of Kells, the XPI monogram page (Figure 15), is indeed remarkable and worth pursuing further. Appearing as a mass of tangles, it contains little of the regular structure found in Celtic interlace and would require further study.

Figure 15: Lozenge of MS 58 f34r, Board of Trinity College Dublin with permission.

Computationally generating and measuring the aesthetic quality of generated art form [Yev02] and identifying it to be similar to Celtic-art, and occlusion shading is another research challenge for the future. A line of inquiry, which is of interest to us, is the work by Itti, Dhavale and Pighin [Itt04] and Santella [San05]. These will be further investigated and are still open areas of research at this time.

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