The Elucidation of Planar Aesthetic Curves

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ABSTRACT
A compact formula for Logarithmic Curvature Histogram (LCH) and its gradient for planar curves have been proposed. Using these entities and the analysis of Generalized Cornu Spiral (GCS), the mathematical definition for a curve to be aesthetic has been introduced to overcome the ambiguity that occurs in measuring the beauty of a curve. In the last section, detailed examples are shown on how LCH and its gradient represented as a straight line equation can be used to measure the aesthetic value of planar curves.

Keywords
Shape Interrogation technique; Logarithmic Curvature Histogram; Aesthetic Curves; Spiral; Fairing; Curvature Profile.

1. INTRODUCTION
A potential customer judges the aesthetic appeal of a product before its physical performance [Pug91]. This clearly indicates the importance of aesthetic shapes for the success of an industrial product.

Geometric modeling is the study of free-form curve and surface design. It is one of the basic foundations in the product design environment. Shape interrogation technique is the process of information extraction from geometrical model [Pat98]. The inspection of a curvature profile is an example of shape interrogation technique that is vital for product manufacturing in order to verify a product's functionalities and aesthetic shapes are met.

There are many studies indicating the importance of the curvature profile to characterize planar curves (see [Nut88] & [Sap94] and references therein). A curvature profile is a graph plotted with the values of parameter t representing the x-axis against its corresponding signed curvature values representing the y-axis [Far96]:

$$\kappa(t)=\frac{x'(t)y''(t)-x''(t)y'(t)}{(x'(t)^2+y'(t)^2)^{3/2}} \quad (1)$$

Hence, the curvature profile has been highlighted as a shape interrogation tool to fair B-spline curves and surfaces [Sap90]. The designer arrives to the desired curve by interactively or automatically tweaking the control points and concurrently inspecting the curvature profile. However, the curvature profile alone is insufficient to identify the aesthetic value of planar curves.

Planar curves which are visually pleasing has been denoted with many terms, e.g., fair curves, beautiful curves, aesthetic curves, spirals, monotonic curvature curves and etc. In this paper, the term aesthetic curve has been used to denote a visually pleasing curve.

Recently, a constructive mathematical formula denoted Logarithmic Curvature Histogram (LCH) and its gradient have been used to define a visually pleasing. In this case, an aesthetic curve refers to a curve with a constant gradient of LCH [Har99, Kan06, Yosh02, Yos06, Yos07, Yos08].

This paper promotes simplification of the formula for the LCH and its gradient. The second part of the work focuses on the mathematically elucidating the definition of the aesthetic curve. This is contrary to recent researches on LCH and constant gradient [Har99, Kan06, Yosh02, Yos06, Yos07, Yos08].

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LDDC leading to LCH

Harada et.al have proposed the use of Logarithmic Distribution Diagram of Curvature (LDDC) to analyze the characteristics of planar curves with monotonic curvature [Har99]. LDDC is the relationship between the length frequencies of a segmented curve with regards to its radius of curvature is plotted in log-log coordinate system. These graphs can be used to identify the aesthetic value of a curve [Kan03, Har99] for automobile design.

LDDC is generated to mathematically obtain the locus of the interval of radius of curvature and its corresponding length frequency. Thus, two curves with different length would generate distinct LDDC regardless of the similarities of the shape of curvature profile.

For example, two circular arcs with the same radius but different length would generate similar curvature profile; but in the case of LDDC, different shapes would be generated [Har99].

The steps involved in generating LDDC are as follows [Yosh02]:

1. Compute the value of radius of curvature for the curve,
2. Segregate the radius of curvature based on the formulated classes and count its frequency,
3. Plot the frequency against the radius of curvature divided by the arc length on a log-log coordinate system,
4. Compute the outline of this histogram and estimate its gradient.

Some of the drawbacks of the LDDC method are; it is computationally expensive, tedious and cannot be used to investigate arbitrary monotone curves (see [Har99 & Yosh02] for details) as well as high numerical errors may occur due to approximation process.

In 2003, Kanaya et.al proposed the generation of Logarithmic Curvature Histogram (LCH) to substitute LDDC [Kan03]. LCH is an analytical way of obtaining the relationship between the interval of curvature radius and its corresponding length frequency.

Generalized Cornu Spiral

Nutbourne et.al developed a technique for constructing planar curves by integrating their curvature profile functions [Nut72]; this technique is well known as curve synthesis. There have been vast interests in carrying out curve synthesis for linear curvature functions whereby segments of Cornu spiral are generated. Pal & Nutbourne [Pal77], Schechter [Sch78] and Meek & Walton [Mee92] have carried out extensive investigation on curvature function consisting of piecewise linear curvature functions.

Let a curve defined in the interval of $0 \leq s \leq S$, and its curvature function is represented by a bilinear curvature element function or denoted as BLINCE as:

$$\kappa(s) = \frac{ps+q}{rs+S} \quad (2)$$

where p, q, (r > -1) and (S >0) are the free parameters of the curve segment. The resultant curve upon curve synthesis is a family of Generalized Cornu Spiral (GCS) [Jam99]. It is noted that GCS contains straight lines (p=q=0), circular arcs (q=r=0), Logarithmic spiral (q=0, r≠0) and clothoid (q≠0, r=0). The values of r is restricted to (r > -1) in order to ensure $\kappa(s)$ is well behaved and continuous over the stated interval.

Let the arc length of the GCS curve segment be S and the end curvatures are κ_0 and κ_1, then at s=0 and s=S, we obtain $\kappa(0)=\kappa_0$ and $\kappa(S)=\kappa_1$. Thus, equation (2) can further be reduced to the following set of equations [Jam99]:

$$q = S\kappa_0 \quad (3)$$

$$pS + q = S(1+r)\kappa_1 \quad (4)$$

Solving for p in terms of r, we get $p = (1+r)\kappa_1 - \kappa_0$. Finally, by substituting p and q in equation (2), the curvature function becomes:

$$\kappa(s) = \frac{(\kappa_1 - \kappa_0 + r\kappa_1)s + \kappa_0 S}{rs+S}, \; 0 \leq s \leq S \text{ and } r > -1 \quad (5)$$

A GCS segment can be obtained by substituting the derived curvature equation from (5) into equation (6):

$$GCS(s) = \left\{x(0) + \int_0^s \cos\left[\theta(0) + \int_0^t \kappa(u)du\right]dt,\; y(0) + \int_0^s \sin\left[\theta(0) + \int_0^t \kappa(u)du\right]dt\right\} \quad (6)$$

Depending on the selection for end curvatures, a GCS segment may only have one inflection point and the monotonicity of the curvature function is always preserved. Furthermore, it is claimed as a high quality curve and has been used for a number of applications [Cri03]. Thus, the measurement of aesthetic value of GCS may further offer insights on defining aesthetic curves.

2. LOGARITHMIC CURVATURE HISTOGRAM

Theorem 1: Let a planar curve be defined as $C(t) = \{x(t), y(t)\}$ and its radius of curvature and arc length function is defined as $\rho(t)$ and $s(t)$ respectively. The LCH for $C(t)$ can be obtained using:

$$LCH(t) = \left\{Log[\rho(t)], Log\left[\frac{\rho(t)s'(t)}{\rho'(t)}\right]\right\} \quad (7)$$

Proof. An analytical model of LDDC can be derived when the number of segments→ ∞ and the number of radius of curvature classes→ ∞ (as proposed in [Kan03]):

$$LCH(t) = \{Log[\rho(t)], Log\left[\frac{\Delta s(t)}{\Delta\, Log[\rho(t)]}\right]\}$$

The vertical value of equation can further be simplified as [Kan03]:

$$\frac{\Delta s(t)}{\Delta\, Log[\rho(t)]} = \frac{ds(t)/dt}{d(\,Log[\rho(t)])/dt} = \frac{\rho(t)s'(t)}{\rho'(t)} \quad \square$$

3. THE GRADIENT OF LCH

Theorem 2: Consider a planar curve given as $C(t)$ and the first derivative of LCH for $C(t)$ exists. Let $\rho(t)$ and $s(t)$ be its radius of curvature and arc length function respectively, then the gradient of LCH denoted as $\lambda(t)$ can be defined as:

$$\lambda(t) = 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho(t)'s''(t)}{s'(t)} - \rho''(t)\right) \tag{8}$$

Proof. The first derivative of *LCH(t)* is:

$$\frac{dLCH(t)}{dt} = \left\{\frac{dLog[\rho(t)]}{dt}, \frac{dLog\left[\frac{\rho(t)s'(t)}{\rho'(t)}\right]}{dt}\right\}$$

Hence, the gradient of LCH in Leibniz notation:

$$\lambda(t) = \frac{\frac{dLog\left[\frac{\rho(t)s'(t)}{\rho'(t)}\right]}{dt}}{\frac{dLog[\rho(t)]}{dt}} = \frac{dLog\left[\frac{ds(t)}{d(Log[\rho(t)])}\right]}{dLog[\rho(t)]}$$

$$\lambda(t) = 1 + \frac{\rho(t)}{\rho'(t)^2}\left(\frac{\rho(t)'s''(t)}{s'(t)} - \rho''(t)\right) \quad \square$$

Based on the gradient of LCH, the following definitions are constructed.

4. ELUCIDATING AESTHETIC CURVE

Hypothesis 1: A curve is said to be an aesthetic curve if the gradient of LCH of the curve is constant. The aesthetic value of a curve increases when the gradient of LCH approximates to a constant value.

Hypothesis 1 has been proposed and well accepted by researchers involved in the development of LDDC and LCH [Har99, Kan06, Yosh02, Yos06, Yos07, Yos08]. It is used as a standard definition to coin aesthetic curves. However, the elucidation of what makes a curve aesthetic is revealed upon the investigation of GCS as found in the following section.

The LCH of GCS

The specification of radius of curvature and arc length for GCS in order to obtain the LCH function and its gradient are as follows:

$$\rho_{GCS}(t) = \frac{rs+S}{(\kappa_1-\kappa_0+r\kappa_1)s+\kappa_0 S},\ s_{GCS}(t) = t \tag{9}$$

Hence, by substituting equation (9) and its' derivatives into equation (7) and (8), the LCH function and its gradient are obtained respectively:

$$LCH_{GCS}(t) = \begin{array}{l}\{Log\left[\left\|\frac{rt+S}{(\kappa_1-\kappa_0+r\kappa_1)t+\kappa_0 S}\right\|\right],\\ Log\left[\left\|\frac{(S+r\,t)\,(S\,\kappa_0+t\,(-\kappa_0+\kappa_1+r\kappa_1))}{(1+r)\,S\,(\kappa_0-\kappa_1)}\right\|\right]\}\end{array} \tag{10}$$

$$\lambda_{GCS}(t) = \frac{((-1+r)\,S-2\,r\,t)\,\kappa_0+(1+r)\,(S+2\,r\,t)\,\kappa_1}{(1+r)\,S\,(\kappa_0-\kappa_1)} \tag{11}$$

Equation (11) can further be simplified in a general straight line equation as $y = mx + c$:

$$\lambda_{GCS} = \left(\frac{2\,r\,(-\kappa_0+\kappa_1+r\,\kappa_1)}{(1+r)\,S\,(\kappa_0-\kappa_1)}\right)t + \left(\frac{2\,r\,\kappa_0}{(1+r)\,(\kappa_0-\kappa_1)} - 1\right) \tag{12}$$

whereby λ_{GCS} represents y, x represents t, m represents the slope and c represents the y intercept:

$$m = \left(\frac{2\,r\,(-\kappa_0+\kappa_1+r\,\kappa_1)}{(1+r)S\,(\kappa_0-\kappa_1)}\right),\ c = \left(\frac{2\,r\,\kappa_0}{(1+r)\,(\kappa_0-\kappa_1)} - 1\right) \tag{13}$$

Equation (12) indicates that the gradient of LCH can be represented as a straight line. This element prevails as a vital criterion for identifying aesthetic curves.

Definition 1: A curve is said to be an aesthetic curve if the gradient of LCH of the curve is either constant (which denotes a horizontal line) or the gradient is represented as a straight line with a certain degree of slope. The aesthetic value of a curve increases when the gradient of LCH approximates to a straight line.

Definition 2: The classification of the three patterns of aesthetic curves is made based on the gradient of LCH [Kan08]:

1. Convergent: the gradient of LCH is positive,
2. Divergent: the gradient of LCH is negative,
3. Neutral: the path of LCH is flat whereby the gradient is zero.

5. NUMERICAL EXAMPLES

The examples are categorized based on three types of gradients namely planar curves with constant gradient, straight line function and inconsistent gradient (the representation of gradients other than constant and straight line function).

Planar Curves with Constant Gradient

Three types of spiral are analyzed in this section; Cornu spiral ($CS(t)$), circle involute ($CI(t)$) and Logarithmic spiral ($LS(t)$). The respective formulas in parametric form are:

$$CS(t) = \pi B \begin{pmatrix}\int_0^t \cos\left[\frac{\pi u^2}{2}\right]du\\ \int_0^t \sin\left[\frac{\pi u^2}{2}\right]du\end{pmatrix} \tag{14}$$

where B is positive, parameter t is non-negative and the integrals are Fresnel integrals.

$$CI(t) = \{\cos[t] + t\sin[t], \sin[t] - t\cos[t]\} \tag{15}$$

where parameter t represents the winding angle of a circle.

$$LS(t) = \{ae^{bt}\cos[t], ae^{bt}\sin[t]\} \quad (16)$$

where t is the angle from the x-axis, a and b are arbitrary constants.

Table 1 shows the analysis of three types of natural spiral which is considered aesthetic and these curves have constant gradient of LCH.

Curves	$LCH(t)$	$\lambda(t)$
Cornu Spiral	$\{Log\left[\frac{B}{t}\right], Log[B\pi t]$	-1
Circle Involute	$\{Log[t], 2Log[t]\}$	2
Logarithmic Spiral	$\{Log[a\sqrt{1+b^2e^{bt}}, Log[\frac{a\sqrt{1+b^2e^{bt}}}{b}\}$	1

Table 1. Three types of spirals and its $\lambda(t)$

Planar Curves with Gradient as a Straight Line Function

In this section, GCS curve segment with various configuration are examined. Figure 1 shows the curvature function of GCS with the values of r ranging from {100, 5, 2, 1, 0, -0.5,-0.9,-0.99} with $\kappa_0 = 0$, $\kappa_1 = 2$ and $S = \pi$. Figure 2 shows corresponding GCS segments for the BLINCE. Figure 3 shows the corresponding LCH for GCS segments and Figure 4 shows the corresponding gradients of LCH for GCS segments shown in Figure 3.

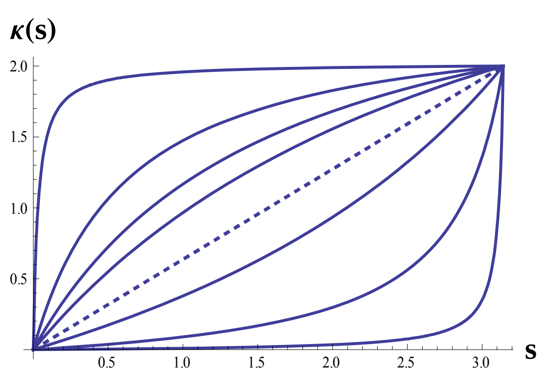

Figure 1. The dashed line is obtained when r=0 and the curvature function goes higher as r increases (r=1, 2, 5,100) and goes lower as r decreases (r=-0.5, -0.9, -0.99).

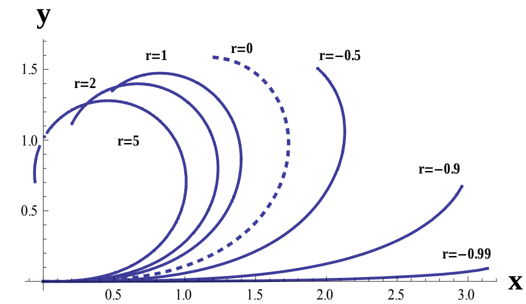

Figure 2. Cornu spiral is obtained when r=0 and the GCS segment curls to the left as r increases (r=1, 2, 5) and flattens as r decreases to (r=-0.5, -0.9, -0.99).

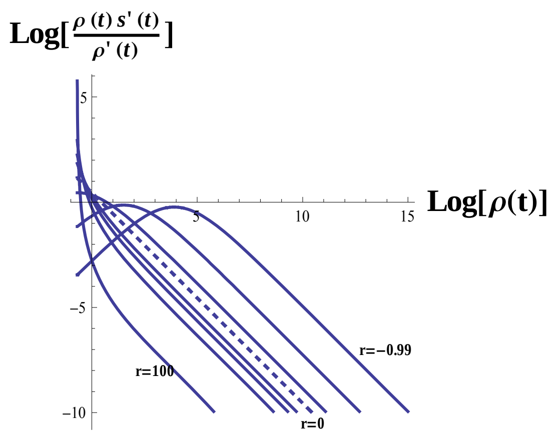

Figure 3. The LCH straightens as r decreases from r=100 to r=0 and bends as r decreases further to {-0.5, -0.9, -0.99}.

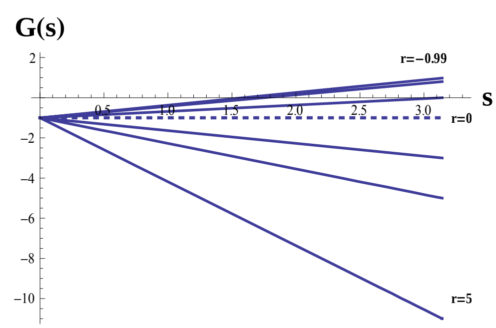

Figure 4. The slope of LCH's gradient becomes negative as r increases (r=1, 2, 5) and positive as r decreases to (r=-0.5, -0.9,-0.99).

From Figure 4, it is clear that the gradient of LCH for GCS is always a straight line regardless of the shape of the LCH (Figure 3) and this example confirms the validity of Definition 1. Hence, an aesthetic curve does not necessarily has a constant value of $\boldsymbol{\lambda(t)}$, but may comprise of $\boldsymbol{\lambda(t)}$ represented as a straight line.

In general, GCS segments comprised of two types of aesthetic curve which can be determined based on the

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selection of its shape factor r, κ_0, κ_1 and S, whereby the point of sign change for $\lambda(t)$ occurs when $t = \frac{1}{2}S\left(\frac{k_0}{k_0 - k_1(1+r)} - \frac{1}{r}\right)$ with $r > -1$ and $0 \le t \le S$. From the configuration of $\{\kappa_0 = 0, \kappa_1 = 2, S = \pi\}$, $\lambda(t)$ changes sign at $t = -\frac{\pi}{2r}$. GCS segments with the defined variables consist of the following types of aesthetic curve:

i. Divergent : $r \ge 0.5$,
ii. Divergent-Convergent : $-1 < r < -0.5$

Planar Curves with Inconsistent Gradient

In this section, two types of planar curves are described namely parabola and logarithmic curve. The general equation of LCH and its gradient are derived followed by a numerical example for each curve.

Parabola

A Parabola is defined in parametric form as:

$$Pb(t) = \{t, at^2\} \qquad (17)$$

where a is positive constant and parameter t is non-negative. Upon algebraic simplification, the LCH for parabola can be written as:

$$LCH_{Pb}(t) = \{Log\left[\frac{(1+4a^2t^2)^{\frac{3}{2}}}{2a}\right], Log\left[\frac{(1+4a^2t^2)^{\frac{3}{2}}}{12a^2t}\right]\} \qquad (18)$$

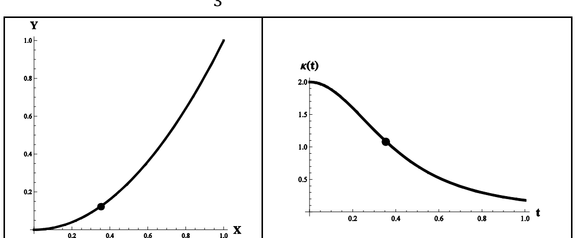
and the gradient for parabola is:

$$\lambda_{Pb}(t) = \frac{2}{3} - \frac{1}{12a^2t^2} \qquad (19)$$

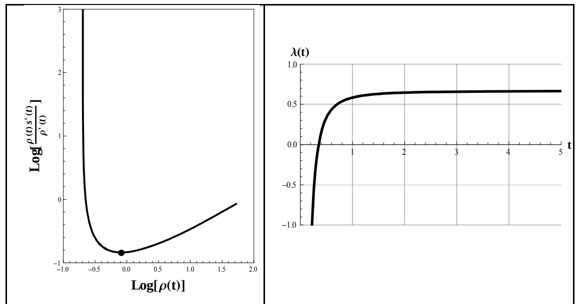
Since $\lambda_{Pb}(t)$ is in a quadratic form, it has a critical point at $t_c = \frac{1}{2\sqrt{2a}}$ which can be obtained by solving $\lambda_{Pb}(t) = 0$. Hence, $\lambda_{Pb}(t)$ changes sign as follows:

i. $\lambda_{Pb}(t) > 0$ when $t_c > \frac{1}{2\sqrt{2a}}$
ii. $\lambda_{Pb}(t) < 0$ when $t_c < \frac{1}{2\sqrt{2a}}$

Figure 5 is an example of parabola with a=1 and for simplification purpose, only the first quadrant is shown here since it is symmetrical. Figure 5(d) shows a graph plotted using equation (19) whereby the gradient changes sign at $t_c = \frac{1}{2\sqrt{2}}$. Parabola has high esthetic value in the range of $\frac{1}{2\sqrt{2}} < t < \infty$ whereby $\lambda_{Pb}(t) \to \frac{2}{3}$.



5(a) Parabola | 5(b) Curvature Profile



5(c) The LCH of parabola | 5(d) The gradient of parabola becomes constant when t→ ∞

Figure 5. Parabola defined in $0 \le t \le 1$ with a black dot indicates t_c.

Logarithmic Curve

A Logarithmic curve is defined in parametric form as follows:

$$Lc(t) = \{t, a\,Log[t]\} \qquad (20)$$

where $a, t > 0$. The LCH and gradient of LCH for Logarithmic curve is stated in equation (21) and (22) respectively:

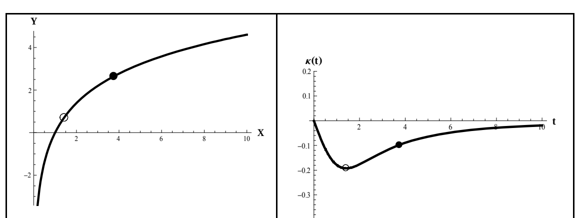
$$LCH_{Lc}(t) = \left\{Log\left[t^2\left(1 + \frac{1}{t^2}\right)^{\frac{3}{2}}\right], Log\left[\frac{(1+t^2)^{\frac{3}{2}}}{2t^2-1}\right]\right\} \qquad (21)$$

$$\lambda_{Lc}(t) = -\frac{7a^2t^2 + 2t^4}{(a^2 - 2t^2)^2} \qquad (22)$$

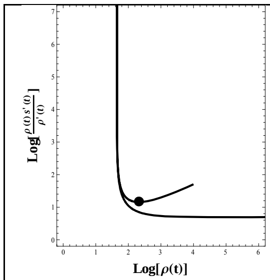
Similar to parabola, the $\lambda_{Lc}(t)$ for Logarithmic curve can be classified as:

i. $\lambda_{Lc}(t) > 0$ when $t_c > a\sqrt{\frac{7}{2}}$
ii. $\lambda_{Lc}(t) < 0$ when $0 < t_c < \frac{a}{\sqrt{2}}$ and $\frac{a}{\sqrt{2}} < t_c < a\sqrt{\frac{7}{2}}$

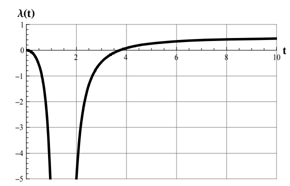
Figure (6) illustrates an example of Logarithmic curve where $a = 2$, its curvature profile, $LCH_{Lc}(t)$ and $\lambda_{Lc}(t)$. There are two critical points occurring at $t_c = \sqrt{2}$ and $t_c = \sqrt{14}$. Figure 6(d) indicates that this curve is aesthetically high in the range of $\sqrt{14} < t < \infty$, in which $\lambda_{Lc}(t) \to \frac{1}{2}$.



6(a) Logarithmic curve | 6(b) Curvature Profile



6(c) The LCH of Logarithmic curve



6(d) The gradient of Logarithmic curve becomes constant when $t \to \infty$

Figure 6. Parabola defined in $0 \le t \le 1$ with a black dot and a white dot indicates t_c.

6. CONCLUSION

In this paper, we propose a simple formula to obtain LCH and its gradient in order to identify the aesthetic value of planar curves. Based on the gradient of LCH, we elucidate the identification of aesthetic curves mathematically and classify these curves into three groups. For numerical understanding, three types of LCH gradient has been carried out to illustrate the idea of measuring the aesthetic value of planar curves. In the first case the natural spirals, namely Cornu spiral, circle involute and logarithmic spiral have constant gradient. The second case indicates that the gradient of LCH for GCS can always be represented as a straight line. Since the latter is a general case as compared to the former, this research concludes that the straight line representation of LCH gradient is is the key identification of aesthetic curves. The last case is illustrated for the purpose of analyzing any arbitrary planar curves in terms of LCH and its gradient.

7. ACKNOWLEDGMENTS

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8. REFERENCES

[Cri03] R. J. Cripps. Algorithms to support point-based cadcam. International Journal of Machine Tools and Manufacture, Volume 43, Issue 4, 425-432, 2003.

[Far96] Farin, G. Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide. Second Edition, Academic Press, 1996.

[Har99] Harada,T., Yoshimoto F., & Moriyama M. An aesthetic curve in the field of industrial design. IN Visual Language99, 38-47, 1999.

[Jam99] Jamaludin, M. A., Tookey, R. M., Ball, J. V. and Ball, A. A. The generalized cornu spiral and its application to span generation. Journal of Computational and Applied Mathematics, 102(1):37-47, 1999.

[Kan03] Ichiroh Kanaya, Yuya Nakano & Kasuke Sato. Classification of Aesthetic Surfaces. IN Proc. VSMM 2003, 289-296, 2003.

[Mee92] Meek, D.S. & Walton, D.J. Clothoid spline transition spirals; Math. Comp., 59:117-133, 1992.

[Miu06] Kenjiro T. Miura. A General Equation of Aesthetic Curves and Its Self-Affinity, Computer-Aided Design & Applications, Vol. 3, No. 1-4, 457-464, 2006.

[Nut72] Nutbourne, A.W., McLellan, P.M. and Kensit, R.M.L. Curvature profile for plane curves. Computer- Aided Design, 4:30-34, 1972.

[Pa177] Pal, T.K. & Nutbourne A.W. Two dimensional curve synthesis using linear curvature elements. Computer- Aided Design, 9:121-134, 1977.

[Pat98] Patrikalakis, N.M. & Maekawa, T. Shape Interrogation for Computer Aided Design And Manufacturing: Springer-Verlag, Berlin, 1998.

[Pug91] Pugh, S. Total design; Addison-Wesley Publishing Company, Great Britain, 1991.

[Sap90] Sapidis, N. & Farin, G. Automatic fairing algorithm for B-spline curves. Computer- Aided Design, 22:121-129, 1990.

[Sap94] Sapidis, N. Designing Fair Curves and Surfaces; Society for Industrial and Applied Mathematics, Philadelphia. 1994.

[Sch78] Schecter, A. Synthesis of 2D curves by blending piecewise linear curvature profiles; Computer- Aided Design, 10:8-18, 1978.

[Yos08] Yoshida, N., Hiraiwa, T. & Saito, T. Interactive control of planar class A Bezier curves using Logarithmic Curvature Graphs. Computer-Aided Design & Applications, Vol. 5, No. 1-4, 121-130, 2008.

[Yos06] Yoshida, N. & Saito, T. Interactive aesthetic curve segments. Visual Computer, 22, 896-905, 2006.

[Yos07] Yoshida, N. & Saito, T. Quasi-Aesthetic curves in rational cubic Bezier forms. Computer-Aided Design & Applications, Vol. 4, No. 1-4, 477-486, 2007.

[Yosh02] Yoshimoto F., & Harada,T. Analysis of the characteristics of curves in natural and factory products. IN Proceedings of the Second IASTED International Conference Visualization, Imaging and Image Processing (VIIP2002), 276-281, 2002.