Development of “surface” shape functions on the basis of invariant approximations technique

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Abstract **Shape functions with embedded boundary conditions have been developed on the basis of invariant approximations technique. Their application reduces the order of a system of equations and, subsequently, the computational error.**

Keywords **finite element, invariant approximation, boundary condition, elliptic differential equation.**

I. Introduction

The fundamental idea of the finite elements method [1] is to subdivide a domain under consideration into small sub-domains called finite elements (FE). Sought scalar and vector functions are approximated within each finite element by simple functions called shape functions. A shape function is a continuous function defined over a single FE. The shape functions of individual FE’s are combined into global shape functions, also called basis functions. Since first mathematical analysis of the method in the 1960’s [2] it has been developed by introducing new shape functions [3] and rigorous analysis of their stability, accuracy, reliability, and adaptability. Nodal and edge finite elements are widely used but their properties do not provide the possibility to “embed” the discontinuities of electromagnetic field variables, caused by abrupt changes in electric conductivity and magnetic permeability, in their configuration. Vector field variables have a physical and mathematical identity that goes beyond their representation in any particular coordinate frame. By dividing the vector into three Cartesian parts, node-based elements fail to take this into account. For example, boundary conditions in electromagnetics often take the form of a specification of only the part of the vector function that is tangential to the boundary. With node-based elements, this physical constraint must be transformed into linear relationships between the Cartesian components what increases the number of equations and, consequently, the computational error.

The aim of our research is to develop such shape functions that automatically take into account boundary conditions on the edges of two-dimensional FE or on the faces of three-dimensional FE. Those shape functions are called “surface” functions because they are used for modeling the sub-domains adjacent to boundaries between regions with different electromagnetic properties.

II. General Methodology of Surface Shape Function Construction

Let Ω be a bounded open set in $\Re^n$, $\mathbf{x} \in \Omega$, $k \in \Im$, and assume that $\mathrm{u} \in C^k(\Omega)$ can be extended from Ω to a continuous function on $\overline{\Omega}$, the closure of the set Ω, where u is the sought function described by partial differential equations supplemented by one of the following boundary conditions, with g denoting a given function defined on the boundary $\partial\Omega$: u = g on $\partial\Omega$ (Dirichlet boundary condition); ∂u/∂n = g on $\partial\Omega$, where n denotes the unit outward normal vector to $\partial\Omega$ (Neumann boundary condition); ∂u/∂n + σu = g on $\partial\Omega$, where $\sigma(\mathbf{x}) \geq 0$ on $\partial\Omega$ (Robin boundary condition).

In accordance with invariant approximations technique [4] the sought function and its derivative can be represented within m-th finite element in the form

$$\mathrm{u}[\mathbf{x}] = \vec{\mathbf{T}}[\mathbf{x}]\mathbf{T}_m^{-1}\vec{\mathbf{U}}_{m*}; \quad (1)$$

$$\partial\mathrm{u}/\partial n = \vec{\mathbf{T}}[\mathbf{x}]\mathbf{n}[\mathbf{x}]\overline{\mathbf{N}}\mathbf{T}_m^{-1}\vec{\mathbf{U}}_{m*}, \quad (2)$$

where $\vec{\mathbf{T}}[\mathbf{x}]$ is Taylor’s vector; $\mathbf{T}_m^{-1}$ is inverse Taylor’s matrix for m-th FE; $\vec{\mathbf{U}}_{m*}$ is the column of nodal values for m-th FE; $\mathbf{n}[\mathbf{x}]$ consists of cosines of the angles between unit outward normal vector to $\partial\Omega$ and a corresponding axis.

In case of, for example, Robin boundary condition given on one or more faces of a finite element constructed in the form of an invariant polyhedron [4] we receive a set of equations

$$\vec{\mathbf{T}}[\mathbf{x}_i]\mathbf{n}[\mathbf{x}_i]\overline{\mathbf{N}}\mathbf{T}_m^{-1}\vec{\mathbf{U}}_{m*} + \vec{\mathbf{T}}[\mathbf{x}_i]\mathbf{T}_m^{-1}\vec{\mathbf{U}}_{m*} = g_i, \ i = \overline{1,\ldots,s} \quad (3)$$

where s is the number of nodes located on the appropriate faces of the m-th finite element. The solution of (3) gives us the sought shape function comprising both nodal values in inside nodes and boundary conditions in surface nodes.

III. Conclusion

New shape functions that are invariant with respect to linear transformations of local and global coordinate frames and automatically satisfy boundary conditions on edges of two-dimensional and faces of three-dimensional FE’s have been developed. Their utilization reduces the order of equation system and, subsequently, the computational error.

IV. References

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