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Using the Neumann series expansion for assembling Reduced Order Models

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Abstract

An efficient method to remove the limitation in selecting the master degrees of freedom in a finite element model by means of a model order reduction is presented. A major difficulty of the Guyan reduction and IRS method (Improved Reduced System) is represented by the need of appropriately select the master and slave degrees of freedom for the rate of convergence to be high. This study approaches the above limitation by using a particular arrangement of the rows and columns of the assembled matrices $\boldsymbol{K}$ and $\boldsymbol{M}$ and employing a combination between the IRS method and a variant of the analytical selection of masters presented in (Shah, V. N., Raymund, M., Analytical selection of masters for the reduced eigenvalue problem, International Journal for Numerical Methods in Engineering 18 (1) 1982) in case first lowest frequencies had to be sought. One of the most significant characteristics of the approach is the use of the Neumann series expansion that motivates this particular arrangement of the matrices' entries. The method shows a higher rate of convergence when compared to the standard IRS and very accurate results for the lowest reduced frequencies. To show the effectiveness of the proposed method two testing structures and the human vocal tract model employed in (Vampola, T., Horacek, J., Svec, J. G., FE modeling of human vocal tract acoustics. Part I: Prodution of Czech vowels, Acta Acustica United with Acustica 94 (3) 2008) are presented.

Keywords: Guyan reduction, IRS method, Neumann series, analytical selection of masters, ill-conditioning, human vocal tract

1. Introduction

1.1. The importance of the eigenbehavior analysis in structural dynamics

In finite element analysis of structures — whose size spans between the medium size of 1 000 dofs to the large size of several millions of dofs — are used different methods for solving free vibration and dynamic response problems. The reason that motivates to effectively solve, for instance, dynamic response of structures to transient loads like earthquakes, windgusts relies on the need to assess the safety boundaries of a structure by determining its eigenbehavior. This entails determining a certain number of eigenfrequencies that, in some case, might be quite high. For example, the design of important buildings may require up to 200 lowest eigenpairs. Also, early model of the space station skylab had over 200 modes below 2 Hz that were of critical importance for design of the attitude control system. For such cases, the most suitable methods implemented in famous commercial packages like Ansys, Nastran and Adina are the subspace iteration method (SIM) [2] and the Block Lanczos algorithm. However, the implementation of such methods requires significant computational skills, e.g. in the subspace iteration method the use of appropriate shifting techniques to speed up the convergence rate without which would be impossible to calculate many low frequencies without losing the important properties of eigenvectors orthogonality.

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1.2. Models Order Reduction — A brief historical background on different procedures for the selection of the masters degrees of freedom

In order to reduce the time of computation of the approximating lowest frequencies a number of methods have been developed in the past decades and many of them are based on the concept of transformation matrix. Guyan's condensation [4] for instance assumes the inertia terms associated with the discarded degrees of freedom are neglected. In order to make some allowance for these missing inertia terms [9] added to the static reduction transformation an extra term which allows the modal vectors in the full model to be approximated more accurately even though it relies on a static transformation matrix. The knowledge of the reduced matrices plays also an important role in the validation of an experimental modal model [1] because the experimental modal vectors are used together with the analytically derived mass matrix in order to evaluate the orthogonality between the formers and the analytical modal vectors. Finally, the transformation used in a reduced order model may also be used to expand the measured mode shapes to the full size of the finite element model, and these mode shapes may then be used in the test analysis correlation or model updating exercises [3]. In order to preserve the values of the lower frequencies, both Guyan's reduction and IRS method require a suitable manipulation of their matrices' entries. This means that the first step in a reduction process must be focused on the selection of the masters or, equivalently, on the elimination of the slaves. Several qualitative and quantitative guidelines have been addressed in this direction. Levy [8] proposed to select as masters those degrees of freedom that have the largest entries in the mass matrix and to those which have the largest movements in the modes of interest. Popplewell et al. [10] focused their research on the energy concept and recommended to choose as secondary master degrees of freedom those which conserve the greatest possible the strain energy and to eliminate those degrees of freedom that do not contribute to the system's kinetic energy. These guidelines however were constructed *ad hoc* for simple structures having very few degrees of freedom and do not lend themselves to more complex analyses. Quantitative algorithms emerge there where the qualitative guidelines fail. Henshell and Ong [5] proposed an automatic procedure to select masters and it is based on the wavefront approach. Kidder [7] proposed a guideline for the selection of the masters based on the bandwidth approach. In such a way if the masters had to be simultaneously selected an iterative process would be required to find those degrees of freedom that satisfy the guideline, thus resulting extremely demanding from the computational standpoint. A very similar approach has been implemented by Shah and Raymund [11] who, instead of operating a simultaneous choice of masters, eliminate one slave at time thus not requiring any iterative process.

1.3. Scope of the article

This article describes a method for calculating with high accuracy the lowest frequencies of a reduced model by using a particular arrangement of rows and columns, which is motivated by an analysis of the Neumann series expansion. A possible alternative will be shown in case the first lowest frequencies had to be sought, by combining the IRS method with the elimination technique employed by [11]. Finally, three numerical examples will show the effectiveness of the proposed method.

2. Proposed method

2.1. Theoretical background on Reduced Order Models. Guyan's and IRS methods

A frequently performed analysis in structural dynamics is the modal analysis the purpose of which is to estimate the vibratory properties of a structural system. In order to study the system the following generalized eigenproblem is employed:

$$\boldsymbol{K}\boldsymbol{\phi} = \lambda \boldsymbol{M}\boldsymbol{\phi}, \tag{1}$$

where $\boldsymbol{K}$ and $\boldsymbol{M}$ are the n-by-n stiffness and mass matrices of the finite element assemblage while the eigenvalues and eigenvectors λ_i and $\boldsymbol{\phi}_i$ are the free vibration frequencies squared, ω_i^2, and the mode shape vectors. The generalized eigenproblem (1) for modes and frequencies of a multi-degree-of-freedom system may be written in the partitioned form as:

$$\begin{bmatrix} \boldsymbol{K}_{mm} & \boldsymbol{K}_{ms} \\ \boldsymbol{K}_{sm} & \boldsymbol{K}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} - \lambda \begin{bmatrix} \boldsymbol{M}_{mm} & \boldsymbol{M}_{ms} \\ \boldsymbol{M}_{sm} & \boldsymbol{M}_{ss} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_m \\ \boldsymbol{x}_s \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{Bmatrix}. \tag{2}$$

Some authors [6] also refers $\boldsymbol{x}_m$ as to juncture coordinates and $\boldsymbol{x}_s$ as to interior coordinates. Manipulating the second row of the above system of equations one obtains the following relationship between slave and master coordinates:

$$\begin{aligned} (\boldsymbol{K}_{ss} - \omega^2 \boldsymbol{M}_{ss})\boldsymbol{x}_s &= -(\boldsymbol{K}_{sm} - \omega^2 \boldsymbol{M}_{sm})\boldsymbol{x}_m, \text{ that is} \\ \boldsymbol{x}_s &= -(\boldsymbol{K}_{ss} - \omega^2 \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \omega^2 \boldsymbol{M}_{sm})\boldsymbol{x}_m, \end{aligned} \tag{3}$$

where λ has been replaced by ω^2, i.e. the values of the natural frequencies of the idealized, discrete system that is possible to calculate by using the generalized eigenproblem. Eq. (3) is the main constituent of the transformation matrix:

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{I}_{mm} \\ -(\boldsymbol{K}_{ss} - \omega^2 \boldsymbol{M}_{ss})^{-1}(\boldsymbol{K}_{sm} - \omega^2 \boldsymbol{M}_{sm}) \end{bmatrix}. \tag{4}$$

The Guyan reduction assumes that the associated inertia terms for the slaves are negligible in which case the previous matrix (4) becomes:

$$\boldsymbol{T}_s = \begin{bmatrix} \boldsymbol{I}_{mm} \\ -(\boldsymbol{K}_{ss})^{-1}(\boldsymbol{K}_{sm}) \end{bmatrix}. \tag{5}$$

The reduced matrices obtained when doing this assumption are then:

$$\begin{aligned} \boldsymbol{K}_G &= \boldsymbol{T}_s^T \cdot \boldsymbol{K} \cdot \boldsymbol{T}_s, \\ \boldsymbol{M}_G &= \boldsymbol{T}_s^T \cdot \boldsymbol{M} \cdot \boldsymbol{T}_s. \end{aligned}$$

The IRS method [9] allows for an extra term that is added to the static transformation matrix. The relationship between the slaves and masters becomes therefore:

$$\boldsymbol{x}_s = \left[-\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm} + \boldsymbol{K}_{ss}^{-1}(\boldsymbol{M}_{sm} - \boldsymbol{M}_{ss}\boldsymbol{K}_{ss}^{-1}\boldsymbol{K}_{sm})\boldsymbol{M}_G^{-1}\boldsymbol{K}_G\right]\boldsymbol{x}_m. \tag{6}$$

The “enriched” transformation matrix takes then the following form:

$$\boldsymbol{T}_{IRS} = \boldsymbol{T}_s + \boldsymbol{S}\boldsymbol{M}\boldsymbol{T}_s\boldsymbol{M}_G^{-1}\boldsymbol{K}_G, \tag{7}$$

where $\boldsymbol{S} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_{ss}^{-1} \end{bmatrix}$ is a n-by-n matrix. The reduced matrices become

$$\begin{aligned} \boldsymbol{K}_{IRS} &= \boldsymbol{T}_{IRS}^T \cdot \boldsymbol{K} \cdot \boldsymbol{T}_{IRS}, \\ \boldsymbol{M}_{IRS} &= \boldsymbol{T}_{IRS}^T \cdot \boldsymbol{M} \cdot \boldsymbol{T}_{IRS} \end{aligned}$$

and are employed within the iterative process,

$$\boldsymbol{T}_{IRS,i+1} = \boldsymbol{T}_s + \boldsymbol{S}\boldsymbol{M}\boldsymbol{T}_{IRS,i}\boldsymbol{M}_{IRS,i}^{-1}\boldsymbol{K}_{IRS,i}.$$

In summary, in order to obtain better estimates in terms of reduced stiffness and mass matrices, equation (7) is first computed and then substituted into $\boldsymbol{K}_{IRS}$ and $\boldsymbol{M}_{IRS}$. Then the transformation, $\boldsymbol{T}_{IRS,i+1}$ is employed for subsequent iterations with the aim of improving the estimates of reduced matrices, $\boldsymbol{K}_{IRS}$ and $\boldsymbol{M}_{IRS}$.

2.2. The Neumann series expansion

It can be observed that the accuracy by which the reduced matrices are calculated by means of either methods depends on how much accurate is the transformation matrix. In turns this matrix strongly depends on the inverse of a sum in (3). This inverse can be first arranged in the following form,

$$\left(\boldsymbol{K}_{ss} + (-\omega^2 \boldsymbol{M}_{ss})\right)^{-1}$$

and subsequently rewritten in a more simple notation, i.e. as $(\boldsymbol{A}+\boldsymbol{B})^{-1}$, where $\boldsymbol{A} = \boldsymbol{K}_{ss}$ and $\boldsymbol{B} = (-\omega^2 \boldsymbol{M}_{ss})$. Furthermore,

$$\left[\boldsymbol{K}_{ss}\left(\boldsymbol{I} - (\omega^2 \boldsymbol{K}_{ss}^{-1} \boldsymbol{M}_{ss})\right)\right]^{-1} = \left(\boldsymbol{I} - (\omega^2 \boldsymbol{K}_{ss}^{-1} \boldsymbol{M}_{ss})\right)^{-1} \boldsymbol{K}_{ss}^{-1},$$

which in the chosen notation becomes:

$$[\boldsymbol{I} - (-\boldsymbol{A}^{-1}\boldsymbol{B})]^{-1}\boldsymbol{A}^{-1} = [\boldsymbol{I} - \boldsymbol{D}]^{-1}\boldsymbol{A}^{-1}, \quad \boldsymbol{D} = (-\boldsymbol{A}^{-1}\boldsymbol{B}).$$

The factor $[\boldsymbol{I} - \boldsymbol{D}]^{-1}$ plays an important role in the accuracy by which it is possible to compute the frequencies of a reduced system. In the theory of matrix algebra [12] this inverse can be calculated by means of the Neumann series expansion according to which one can express the inverse in the form of a series expansion,

$$[\boldsymbol{I} - \boldsymbol{D}]^{-1} = \boldsymbol{I} + \boldsymbol{D} + \boldsymbol{D}^2 + \boldsymbol{D}^3 + \ldots = \sum_{k=0}^{\infty} \boldsymbol{D}^k.$$

In plane words, $[\boldsymbol{I} - \boldsymbol{D}]^{-1}$ can be approximated when $\boldsymbol{D}$ has entries of small magnitude. If entries of $\boldsymbol{B}$ are small enough to insure that

$$\lim_{k\to\infty} (\boldsymbol{A}^{-1}\boldsymbol{B})^k = \boldsymbol{0},$$

then because of the above series expansion it follows that:

$$(\boldsymbol{A}+\boldsymbol{B})^{-1} = \left(\sum_{k=0}^{\infty} \boldsymbol{D}^k\right) \boldsymbol{A}^{-1} \cong \left(\boldsymbol{I} - (-\boldsymbol{A}^{-1}\boldsymbol{B})\right) \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}.$$

This is a way for approximating the inverse of $(\boldsymbol{A} + \boldsymbol{B})$ by means of a first approximation of the Neumann series expansion. Since $\boldsymbol{D}$ encapsulates the entries of the matrices $\boldsymbol{K}_{ss}$ and $\boldsymbol{M}_{ss}$, in order for the inverse to accurately converge to the Neumann series it is necessary that also the entries of these matrices be small. If not, the above approximation would not be valid anymore, thus affecting the overall reduction process. This means that those degrees of freedom which poorly contribute to the kinetic energy and to the strain energy will be slaves. This fact was pointed out already in [10] yet achieved in a different way. To understand how practically know what degrees of freedom consider as slaves and hence eliminating them, consider again the relationship

$$\left(\boldsymbol{I} - (\omega^2 \boldsymbol{K}_{ss}^{-1} \boldsymbol{M}_{ss})\right)^{-1}.$$

If one slave at time is considered then the previous relation tells that if the ratio $\frac{\|\boldsymbol{K}_{ss}\|}{\|\boldsymbol{M}_{ss}\|}$ is selected as the highest possible this will minimize the term $\boldsymbol{D}$ thus allowing the Neumann series approximation to converge[1]. Developing further,

$$(\boldsymbol{A}+\boldsymbol{B})^{-1} \cong \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{A}^{-1}$$

[1] The convergence of the Neumann series is strongly related to the ill-conditioning of $\boldsymbol{K}_{ss}$.

important insights on the convergence of this series expansion can be achieved. Replacing again $\boldsymbol{A}$ and $\boldsymbol{B}$ with $\boldsymbol{K}_{ss}$ and $-\omega^2\boldsymbol{M}_{ss}$, moving the second term of the right-hand side of the previous equation to the left-hand side yields to:

$$\boldsymbol{K}_{ss}^{-1} - (\boldsymbol{K}_{ss} - \omega^2\boldsymbol{M}_{ss})^{-1} \cong \boldsymbol{K}_{ss}^{-1}(-\omega^2\boldsymbol{M}_{ss})\boldsymbol{K}_{ss}^{-1}. \tag{8}$$

If the infinite norm of both sides is imposed and the properties of norms are considered, then the previous equation, from a practical standpoint, can be written as:

$$\|\boldsymbol{K}_{ss}^{-1} - (\boldsymbol{K}_{ss} - \omega^2\boldsymbol{M}_{ss})^{-1}\| \lesssim \left\|\boldsymbol{K}_{ss}^{-1}\right\| \cdot \left\|-\omega^2\boldsymbol{M}_{ss}\right\| \cdot \left\|\boldsymbol{K}_{ss}^{-1}\right\|.$$

Finally, dividing both members by $\|\boldsymbol{K}_{ss}^{-1}\|$ the following relation is obtained:

$$\frac{\|\boldsymbol{K}_{ss}^{-1} - (\boldsymbol{K}_{ss} - \omega^2\boldsymbol{M}_{ss})^{-1}\|}{\|\boldsymbol{K}_{ss}^{-1}\|} \lesssim \|\boldsymbol{K}_{ss}^{-1}\| \cdot \|\boldsymbol{K}_{ss}\| \cdot \frac{|-\omega^2|}{\frac{\|\boldsymbol{K}_{ss}\|}{\|\boldsymbol{M}_{ss}\|}},$$

where the right-hand side has been multiplied by $\frac{\|\boldsymbol{K}_{ss}\|}{\|\boldsymbol{K}_{ss}\|}$. The final inequality is:

$$\frac{\|\boldsymbol{K}_{ss}^{-1} - (\boldsymbol{K}_{ss} - \omega^2\boldsymbol{M}_{ss})^{-1}\|}{\|\boldsymbol{K}_{ss}^{-1}\|} \lesssim \|\boldsymbol{K}_{ss}^{-1}\| \cdot \|\boldsymbol{K}_{ss}\| \cdot \frac{\omega^2}{\frac{\|\boldsymbol{K}_{ss}\|}{\|\boldsymbol{M}_{ss}\|}}. \tag{9}$$

The initial equation (8) has purposely been developed further to yield equation (9) in order to obtain a mathematical insight capable to support the original problem, namely, the fact that the accuracy by which the reduced matrices are calculated by means of either methods depends on how much accurate is the transformation matrix. The left-hand side of equation (9) represents the relative change in the inverse of $\boldsymbol{K}_{ss}$ while the right-hand side is formed by two factors, $\|\boldsymbol{K}_{ss}^{-1}\| \cdot \|\boldsymbol{K}_{ss}\|$ and the ratio $\frac{\omega^2}{\frac{\|\boldsymbol{K}_{ss}\|}{\|\boldsymbol{M}_{ss}\|}}$ which expresses the relative change in $\boldsymbol{K}_{ss}$. In practice, an appropriate selection of masters capable of improving the accuracy of Guyan's or IRS method is characterized by convergence of the series expansion representing $(\boldsymbol{K}_{ss} - \omega^2\boldsymbol{M}_{ss})^{-1}$. This can be attained for those frequencies smaller than the smallest frequency of the eigenproblem given by $(\boldsymbol{I} - (\omega_s^2\boldsymbol{K}_{ss}^{-1}\boldsymbol{M}_{ss}))\,\boldsymbol{\phi}_s = \boldsymbol{0}$. Furthermore, if one slave at time is eliminated then the previous eigenproblem generates the highest eigenvalue provided that the ratio $\frac{\|\boldsymbol{K}_{ss}\|}{\|\boldsymbol{M}_{ss}\|}$ is the highest. This fact is strictly connected to what displayed in the right-hand side of equation (9) thus showing the role of the ratio $\frac{\|\boldsymbol{K}_{ss}\|}{\|\boldsymbol{M}_{ss}\|}$ which, after being maximized, insures that $\|\boldsymbol{K}_{ss}^{-1}\| \cdot \|\boldsymbol{K}_{ss}\|$ — known as condition number — does not magnifies too much the right-hand side of equation (9). This condition guarantees the static transformation matrix is accurate enough. This important fact that establishes a bridge between the Neumann series convergence and the condition necessary for its achievement was not highlighted for instance in [11].

2.3. Main feature of the proposed method

The fact that the ratio $\frac{\|\boldsymbol{K}_{ss}\|}{\|\boldsymbol{M}_{ss}\|}$ has to be maximized suggests the idea of arranging the rows and columns of both $\boldsymbol{K}$ and $\boldsymbol{M}$ based on the order of magnitude of the n-ratios $\frac{K_{ii}}{M_{ii}}$, $i = 1, 2, \ldots, n^\circ$ of degrees of freedom. First of all, 1) the ratios will be ordered from the smallest to the highest value and subsequently 2) the rows and columns will be arranged accordingly. This approach is very easy to program and generates very good improvements in terms of reduced eigenvalues as shown in the section 3 and represents the main feature of the new approach. In addition to this achievement, there exists also a possible alternative to improve the first lowest frequencies.

Shah and Raymund [11] used an analytical selection of masters based on the elimination of 1 slave at time accounting for 1) initial elimination of the dof having the highest ratio $\frac{K_{ii}}{M_{ii}}$, 2) reducing this matrix by applying Guyan reduction and 3) repeatedly applying this procedure as long as the required frequencies would accurately converge to the actual values. Their aim was to use as groundwork the approach by Kidder [7] by claiming that this method would substantially decrease the time of computation. As a matter of fact, for higher number of degrees of freedom than those used by the authors this method results to be computationally very expensive. In plane words, if the slaves are eliminated one by one then one avoids calculating very large $\boldsymbol{K}_{ss}^{-1}$ but the time of computation required to apply continuously guyan reduction is too high. This contradiction can be solved partially by applying two steps: the IRS method is used after appropriately arranging the rows and columns and subsequently the elimination method employing the IRS method itself will complete the reduction process. This allows to obtain even smaller error for the first frequencies and to reduce the time of computation and can be used as possible alternative.

3. Numerical examples

3.1. L-shaped cantilever and airplane wing

The L-shaped cantilever containing two stress risers and an airplane wing are analysed to verify the improvements of the proposed method over the elimination method by [11] and the standard IRS with no master degrees of freedom selection.

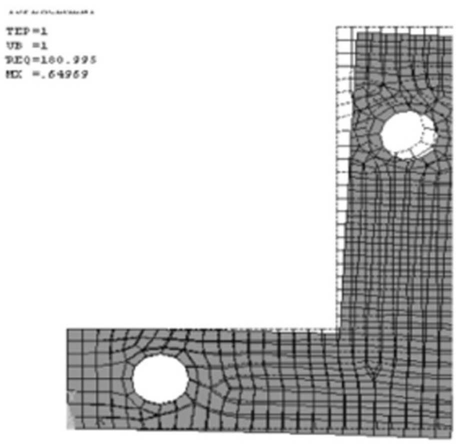

Fig. 1. L-shaped cantilever: 1st frequency, 180.99 Hz. $\rho = 7\,800\ \frac{\text{kg}}{\text{m}^2}$, $E = 200$ GPa, $\nu = 0.33$ N° of dof: $n = 854$

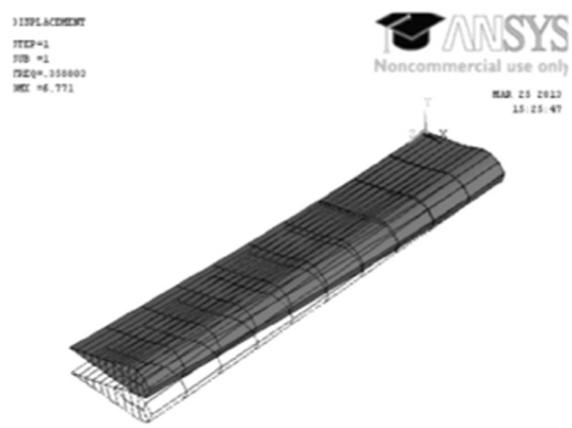

Fig. 2. Airplane wing: 1st frequency, 0.358 8 Hz. $\rho = 925\ \frac{\text{kg}}{\text{m}^2}$, $E = 200$ MPa, $\nu = 0.3$ N° of dof: $n = 780$

The Figs. 1 and 2 give some information regarding the material properties and the number of nodes of the structures. For both structures a reduction of 2 % has been carried out, i.e. the number of selected master degrees of freedom is 2 % of the overall number of dofs indicated in the figures. When the IRS method is applied after arranging the rows and column the rate of convergence of IRS method is higher than the value achieved when no arrangement is performed. This is shown by a comparison below. More specifically, when the proposed approach is adopted the value of the approximating, third frequency is already behind 0.2 %, Fig. 4. Conversely, for this same mode number the value of the error is about 1 % (Fig. 3) when the degrees of freedom have not been manipulated.

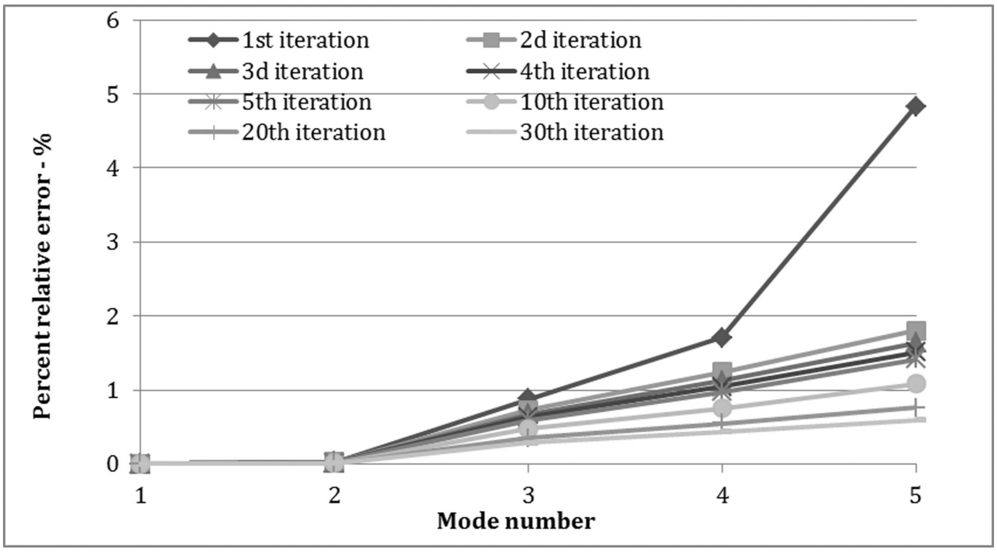

Fig. 3. Rate of convergence of the IRS method when no arrangement of the rows and columns is performed

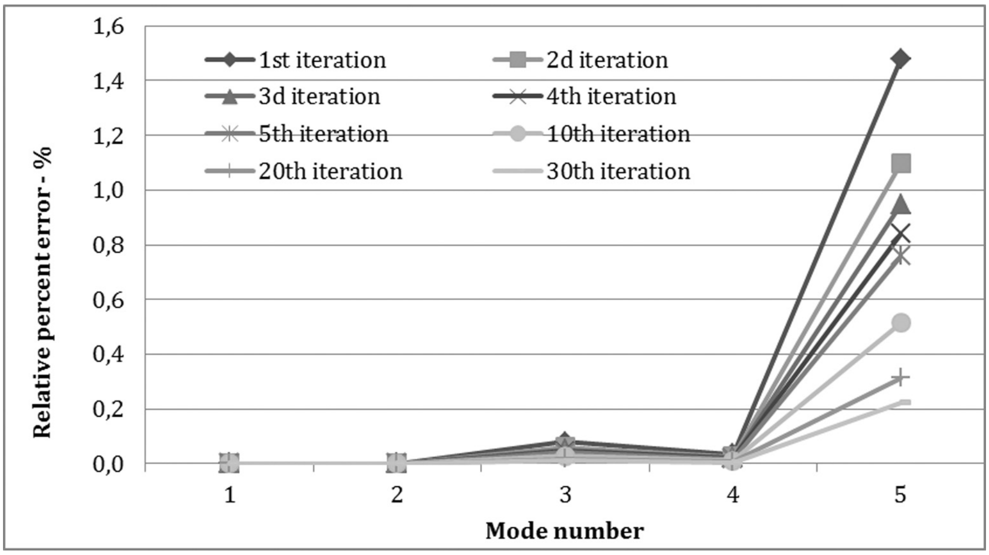

Fig. 4. Rate of convergence of the IRS method when rows and columns are arranged according to the proposed guideline. In a few iterations this new method allows to quickly reach the desired tolerance

Finally, the proposed method it is compared to other approaches: standard IRS, and analytical selection of masters for the two structures above. The performance is measured in terms of percent relative error and CPU. Note that the CPU values presented here have the purpose of illustrating the order of magnitude of the time of computation required by each approach so to enable a direct comparison. Two aspects emerge from this comparison:

1. The new method exhibits the best degree of accuracy and stability throughout the range of the computed reduced frequencies.
2. When is required a small number of frequencies, e.g. between 5 or 10 the incorporation of the elimination method within the IRS with rows and columns arrangement can be a good alternative because of the accurate values.

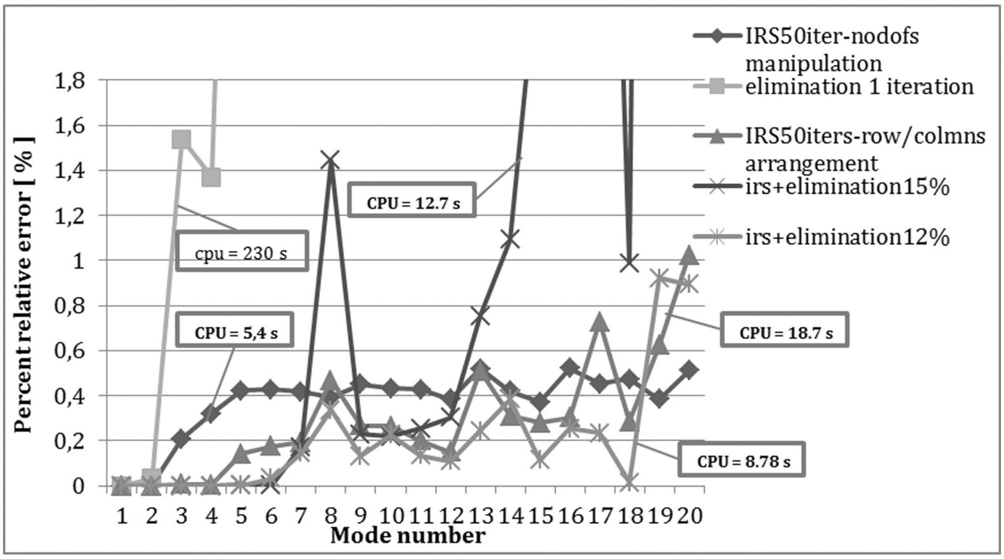

Fig. 5. L-shaped cantilever: comparison among different approaches: standard IRS with no manipulation (IRS50-no dofs manipulation); elimination technique (elimination 1 iteration); proposed method (IRS50iters-row/colmns arrangement); combination IRS and elimination technique (irs + elimination 15 % and irs + elimination 12 %)

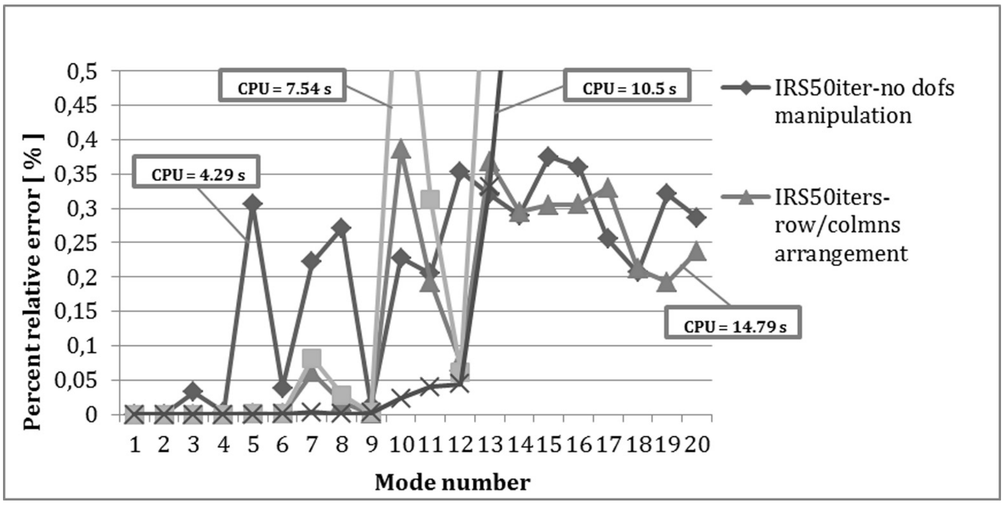

Fig. 6. Airplane wing: comparison among different approaches: standard IRS with no manipulation (IRS50-no dofs manipulation); proposed method (IRS50iters-row/colmns arrangement); combination IRS and elimination technique (irs + elimination 13 % and irs + elimination 16 %)

As predicted, the analytical selection of masters shows high time of computation compared to the other approaches and higher error. Finally, when this approach is combined with the IRS method, the best results are obtained when the process of elimination begins at around 12 %/16 % (and depending on the size of the matrices this threshold can be lowered, e.g. up to 5 % or 9 %) of the overall number of degrees of freedom. This allows to exploit the speed and accuracy of the IRS in the first phase and to execute a subsequent elimination process with much smaller matrices.

3.2. Finite element model of the human vocal tract

The third numerical example analyzed is the human vocal tract modelled by Vampola et al. [13]. The purpose of the authors' study was to develop a finite element model that was able to accurately represent the 3-dimensional wave propagation instead of using a simple 1-dimensional model, thus representing the acoustic spaces of real male vocal tract for the Czech vowels /a:/,/e:/,/i:/,/o:/,/u:/. In this article the finite element model of the vocal tract is introduced with the aim of testing the effectiveness of the proposed approach thus showing the validity of the method independently from both the type of structure modelled and the size. Figs. 7 and 8 show the geometric configuration of the model. The structure has been discretized using 14 909 tetra elements and reduced up to 3 505 degrees of freedom.

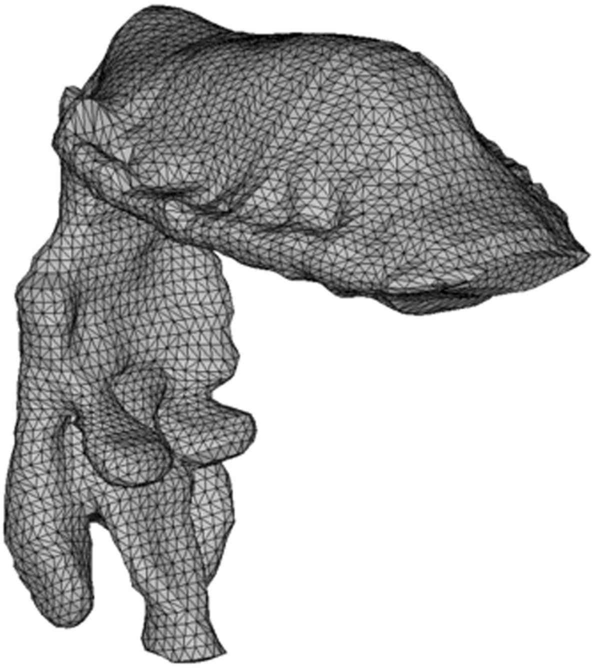

Fig. 7. Complete meshed model of the human vocal tract (vowel "a")

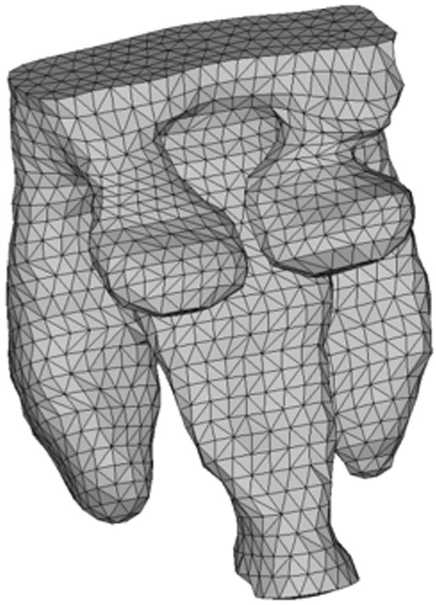

Fig. 8. Part of the model of the human vocal tract, n° of dof 3 505

Fig. 7 shows the complete meshed model of the human vocal tract (vowel "a"), while Fig. 8 shows a selected part of the complete model. For testing the proposed method only the selected part shown in Fig. 8 has been considered because it introduces more complex geometry and a higher number of degrees of freedom with respect to the structures used in the first 2 numerical examples. In order to test the performance of the new approach the IRS method has been used for extracting the first 200 frequencies. The results are summarized in the following Figs. 9 and 10.

Figs. 9 and 10 highlight the same features emerged when the proposed method has been applied to the L-shaped cantilever and to the airplane wing. The new method exhibits dramatic improvement: it allows to obtain in 1 only iteration a percent relative error of 10^{-6} % (Fig. 11) while when no manipulation of the matrices entries is performed even 5 iterations are not sufficient to achieve the same degree of accuracy. When the IRS method is combined with the elimination technique, the error committed during the computation of the first 20 frequencies is very similar to the error committed when employing the IRS with manipulation of the entries of $\boldsymbol{K}$ and $\boldsymbol{M}$. This behaviour is summarized in the Fig. 11.

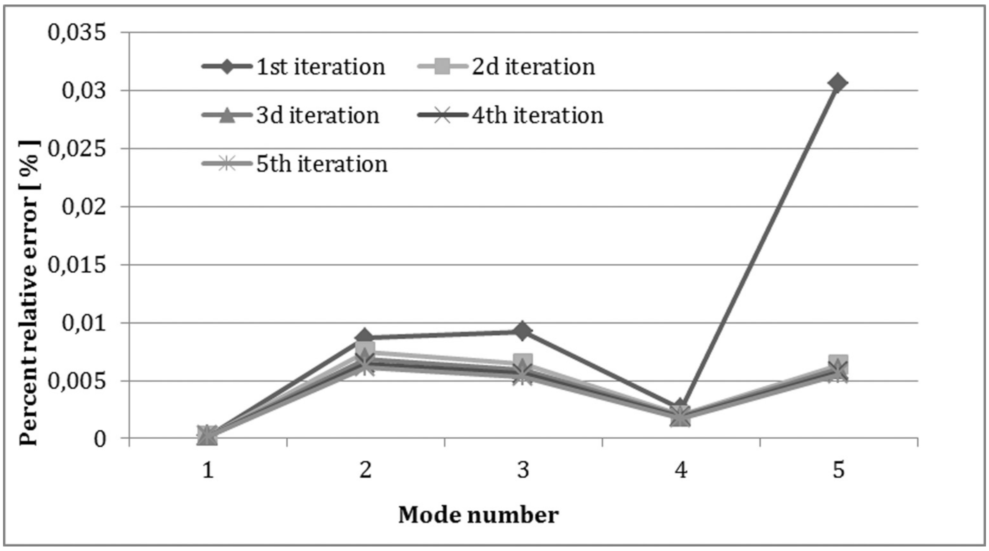

Fig. 9. Rate of convergence of the first 5 reduced frequencies for the Human Vocal Tract calculated by IRS method when no arrangement of the rows and columns is performed

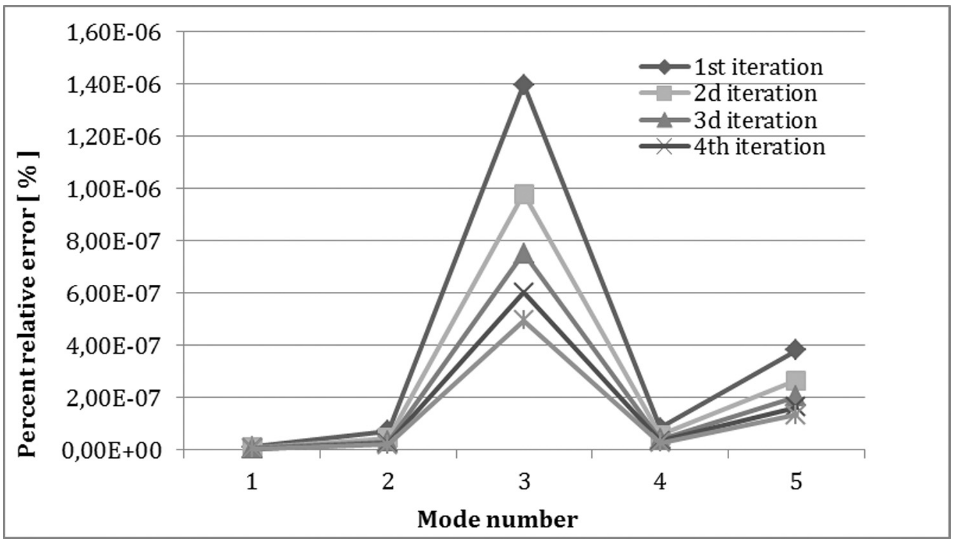

Fig. 10. Rate of convergence of the first 5 reduced frequencies for the Human Vocal Tract calculated by the proposed guideline

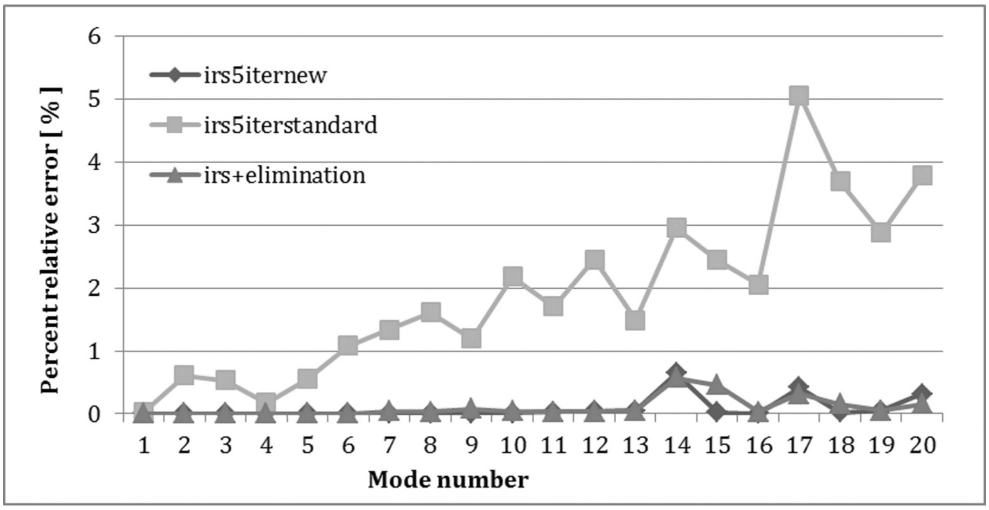

Fig. 11. Percent relative error committed using three different approaches: the new approach (irs5iternew), the IRS method with no manipulation (irs5iterstandard) and the combination of IRS method and the analytical selection scheme (irs + elimination)

Here, the percent relative error is computed with respect to the i-th, exact eigenfrequency obtained by a detailed finite element analysis and is measured by:

$$\text{error} = \left| \frac{\text{exact frequency} - \text{approximating frequency}}{\text{exact frequency}} \right| \cdot 100\,\%.$$

4. Conclusion

This paper proposes an effective method for the computation of eigenfrequencies of reduced models by using a particular arrangement of the rows and columns of the stiffness and mass matrices obtained by a finite element analysis. The study shows also a possible alternative to obtain for the first lowest frequencies accurate results when this method is combined with the analytical selection of masters implemented in [11].

The characteristics of the proposed method identified by the numerical results from numerical examples are summarized as follows:

1. The accuracy of the computed eigenfrequencies by IRS method is improved after arranging the rows and columns based on the highest ratios $\frac{K_{ii}}{M_{ii}}$, with respect to the standard IRS and the classical analytical selection procedure.

2. The rate of convergence by which the IRS method computes the reduced frequencies is outstanding when the new method is employed. Inasmuch as fewer iterations are necessary to obtain more accurate results, this allows the analyst to reduce the time of computation of the reduced stiffness, mass matrices and related frequencies.

3. A simple code is required and the method can be easily implemented.

Acknowledgements

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