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A time stepping method in analysis of nonlinear structural dynamics

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Abstract

In this paper a new method is proposed for the direct time integration method for structural dynamics problems. The proposed method assumes second order variations of the acceleration at each time step. Therefore more terms in the Taylor series expansion were used compared to other methods. Because of the increase in order of variations of acceleration, this method has higher accuracy than classical methods. The displacement function is a polynomial with five constants and they are calculated using: two equations for initial conditions (from the end of previous time step), two equations for satisfying the equilibrium at both ends of the time step, and one equation for the weighted residual integration. Proposed method has higher stability and order of accuracy than the other methods.

Keywords: direct time integration, weighted residual, nonlinear structural dynamics, higher accuracy, second order acceleration

1. Introduction

There are two main methods for the analysis of structural dynamics problem; modal superposition and direct time integration. While for the analysis of linear structures both methods are applicable, for nonlinear analysis, the latter method is the only option.

In the structural dynamics problems, governing equation is a second order differential equation [4, 12]. For solving differential equations of nonlinear systems, the numerical procedure can be used in the incremental step [4]. Among different methods, those related to Newmark's method are the most common methods in structural dynamics. The direct time integration of the equations provides the response of the system as discrete intervals of time which are usually equally spaced. Determination of the response involves the computation of three structural responses; displacement, velocity, and acceleration at each time step.

In nonlinear analysis, stiffness is calculated at the beginning of each time step and then response is calculated at the end of this time step with assuming that stiffness is constant through out the step. Therefore nonlinearity is considered with calculating stiffness again at the beginning of next time step. Calculated responses will be considered at the end of each time step as the initial conditions for next time step. Therefore system nonlinearity behavior is replaced with a series of consecutive approximate linear differential equations [4, 5, 9, 12].

In the explicit methods, in each time step, equation of motion is written at the beginning of the time step and the unknown values at the end of time step are calculated explicitly, but in the implicit methods, unknown values at the end of time step are calculated by writing the equation of motion at those points [1, 4–6, 8, 9, 11, 12, 14]. Because implicit methods require

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more calculation in each time step with a smaller number of time steps, in the past it has been shown that the implicit methods are more accurate than the explicit ones [7, 14].

Because of the approximation in the formulation and calculation of these methods, it is expected to have some error compared to exact solution that the error is usually a function of time step length, frequency content of the load and also degree of nonlinearity.

In conditionally stable methods, the instability occurs when time step size is more than a specific value (critical time step). While in unconditionally stable methods, instability never happens, regardless of the time step size [8, 9, 12, 14].

Because central difference method is very simple for implementation in the nonlinear systems, among explicit time integration methods, it is one of the most widely used methods [2, 4, 9]. Other known method for analysis of nonlinear structural dynamics is a family of Newmark's method that these methods assume a constant or linear behavior for the variation of acceleration at each time step [4, 8, 9, 12].

In this paper, a time integration method is proposed that is both implicit and explicit and it assumes a second order variation of the acceleration within each time step. The proposed method is shown to have higher accuracy compared to conventional methods.

2. Proposed Method

The differential equation describing a nonlinear system can have the general form:

$$\ddot{x} + f(\dot{x}, x, t) = 0. \tag{1}$$

Therefore the equation of motion for a nonlinear system (with nonlinear stiffness) is:

$$M\ddot{x} + C\dot{x} + K_i x = P, \tag{2}$$

where M and C are the mass and damping matrix; K_i is the stiffness matrix in the i-th time step; P is the vector of applied forces; x, $\dot{x}$ and $\ddot{x}$ are the displacement, velocity and acceleration vectors, respectively. The initial conditions are $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ where x_0 and $\dot{x}_0$ are the initial displacement and velocity vectors, respectively.

The acceleration in each time step is assumed to be a second order function which results in the displacement to be a fourth order complete polynomial in each time step. Therefore the displacement function contains five constants. Those constants are obtained from; two initial conditions from the end of previous time step, satisfying the equation of motion at both ends of the time step, and setting the weighted residual of the method in the step equal to zero.

If the objective is to find the displacement in the time t_i, first time interval $[0, t]$ is divided to the i smaller sub-interval. In the beginning of the calculation, the displacement is determined in the time step Δt; and then in the second time step $2\Delta t$ to $i\Delta t$. In the i-th time step, displacement function in the $\delta \in [0, \Delta t]$ interval can be shown as:

$$x(\delta) = a_i\delta^4 + b_i\delta^3 + c_i\delta^2 + d_i\delta + e_i, \tag{3}$$

where a_i to e_i are the unknown coefficients that should be determined. Therefore, the velocity function is defined as:

$$\dot{x}(\delta) = 4a_i\delta^3 + 3b_i\delta^2 + 2c_i\delta + d_i \tag{4}$$

and the acceleration function is:

$$\ddot{x}(\delta) = 12a_i\delta^2 + 6b_i\delta + 2c_i. \tag{5}$$

That c_i, d_i and e_i are calculated using the above equations:

$$x(\delta = 0) = x_{i-1} \quad \rightarrow \quad e_i = x_{i-1}, \tag{6}$$
$$\dot{x}(\delta = 0) = \dot{x}_{i-1} \quad \rightarrow \quad d_i = \dot{x}_{i-1}. \tag{7}$$

By placing Eqs. (6) and (7) into the equation of motion at the beginning of this time step, we have:

$$M(2c_i) + C(d_i) + K_i(e_i) = P_{i-1}, \tag{8}$$

therefore c_i is:

$$c_i = (2M)^{-1} \cdot (P_{i-1} - C\dot{x}_{i-1} - K_i x_{i-1}). \tag{9}$$

Now by satisfying equation of motion at the end of the present time step, we have:

$$M\ddot{x}_i + C\dot{x}_i + K_i x_i = P_i, \tag{10}$$

which results in:

$$\begin{aligned} M(12a_i\Delta t^2 + 6b_i\Delta t + 2c_i) + C(4a_i\Delta t^3 + 3b_i\Delta t^2 + 2c_i\Delta t + d_i) + \\ K_i(a_i\Delta t^4 + b_i\Delta t^3 + c_i\Delta t^2 + d_i\Delta t + e_i) \;=\; P_i. \end{aligned} \tag{11}$$

The final equation is obtained from weighted residual integral. Because this method is approximate, it does not satisfy the equilibrium equation of motion in domain of $[0, \Delta t]$ interval. The residual of the method in satisfying the equation of motion is defined as:

$$R = M\ddot{x} + C\dot{x} + K_i x - P. \tag{12}$$

Then the residual is forced to be zero over the domain and using a unit weight function, we obtain:

$$\int_0^{\Delta t} 1 \times R\,\mathrm{d}t = 0. \tag{13}$$

Finally by solving Eqs. (11) and (13), the values of a_i and b_i can be determined. Therefore by calculating these five unknowns in i-th time step, displacement, velocity and acceleration vectors at the end of the i-th time step is calculated as follow:

$$x_i = a_i\Delta t^4 + b_i\Delta t^3 + c_i\Delta t^2 + d_i\Delta t + e_i, \tag{14}$$
$$\dot{x}_i = 4a_i\Delta t^3 + 3b_i\Delta t^2 + 2c_i\Delta t + d_i, \tag{15}$$
$$\ddot{x}_i = 12a_i\Delta t^2 + 6b_i\Delta t + 2c_i. \tag{16}$$

3. Stability, order of accuracy, and overshooting effect

For evaluation of stability of the present method, single degree of freedom system is considered and the magnification matrix is derived for calculating the eigenvalues of the matrix [3, 10, 13]. Absolute eigenvalues of the matrix must be smaller than or equal to one. For undamped systems, in softening conditions such as $K_i/K_0 = 0$ (stiffness at the end of time step to the stiffness at the beginning of time step i), proposed method has not any instability, but for $K_i/K_0 = 0.5$ has a small local instability at $\Delta t/T_0 = 0.72 - 0.76$ (T_0 is period at the beginning of first time step) which it can be resolved for damping ratio as $\xi = 3.4$ %. For $K_i/K_0 = 1$ has a small local instability too at $\Delta t/T_0 = 0.52 - 0.54$ which it can be resolved for damping ratio as

$\xi = 4.6$ %. The central difference and linear acceleration methods have smaller limitation of stability compared to proposed method.

By replacing the differential equation by the difference equation, the local truncation error is created in each time step which local truncation error is relative to the order of accuracy of one method. The order of the accuracy of the proposed method is three which is higher than the other methods.

The tendency to overshoot from exact solution is significantly important factor which should be considered in an evaluation of numerical solutions. Proposed method from the displacement responses has a tendency to overshoot linearly in the displacement term and from the velocity responses, has a tendency to overshoot quadratically in the displacement term and linearly in velocity term.

4. Examples

In order to see the results of the proposed method and to see its advantages over the other existing methods, two examples are considered which the results obtained from the proposed method are compared with the central difference and linear acceleration (Newmark's) methods.

Example 1 [12]: Consider a single degree of freedom with the frame as shown in Fig. 1.

This system has an elastoplastic behavior as shown in Fig. 2.

Exciting force is applied on the spring damping system as shown in Fig. 3.

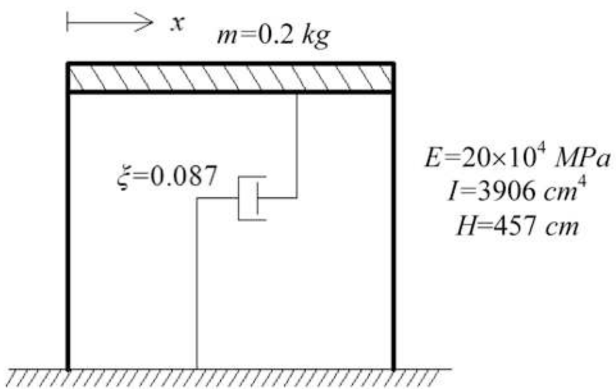

Fig. 1. Frame of structure

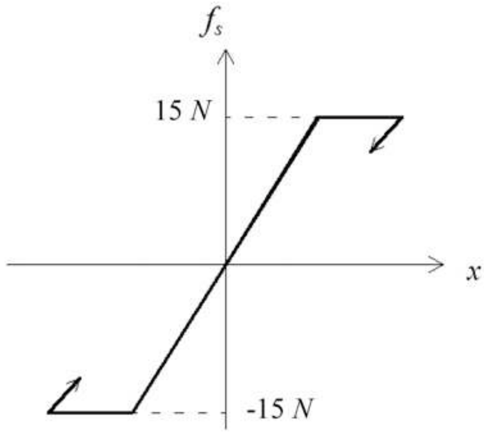

Fig. 2. Force-displacement relationship

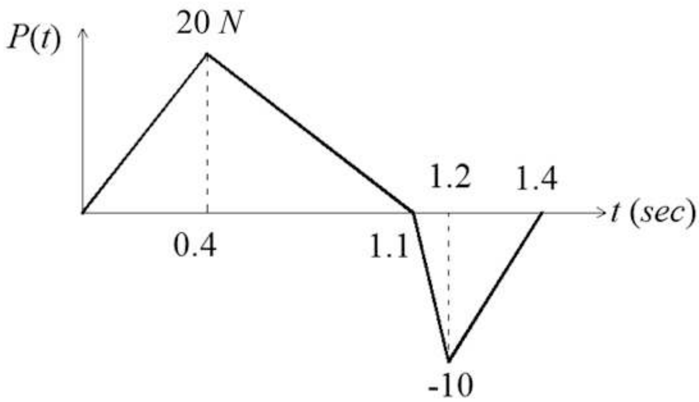

Fig. 3. Exciting force

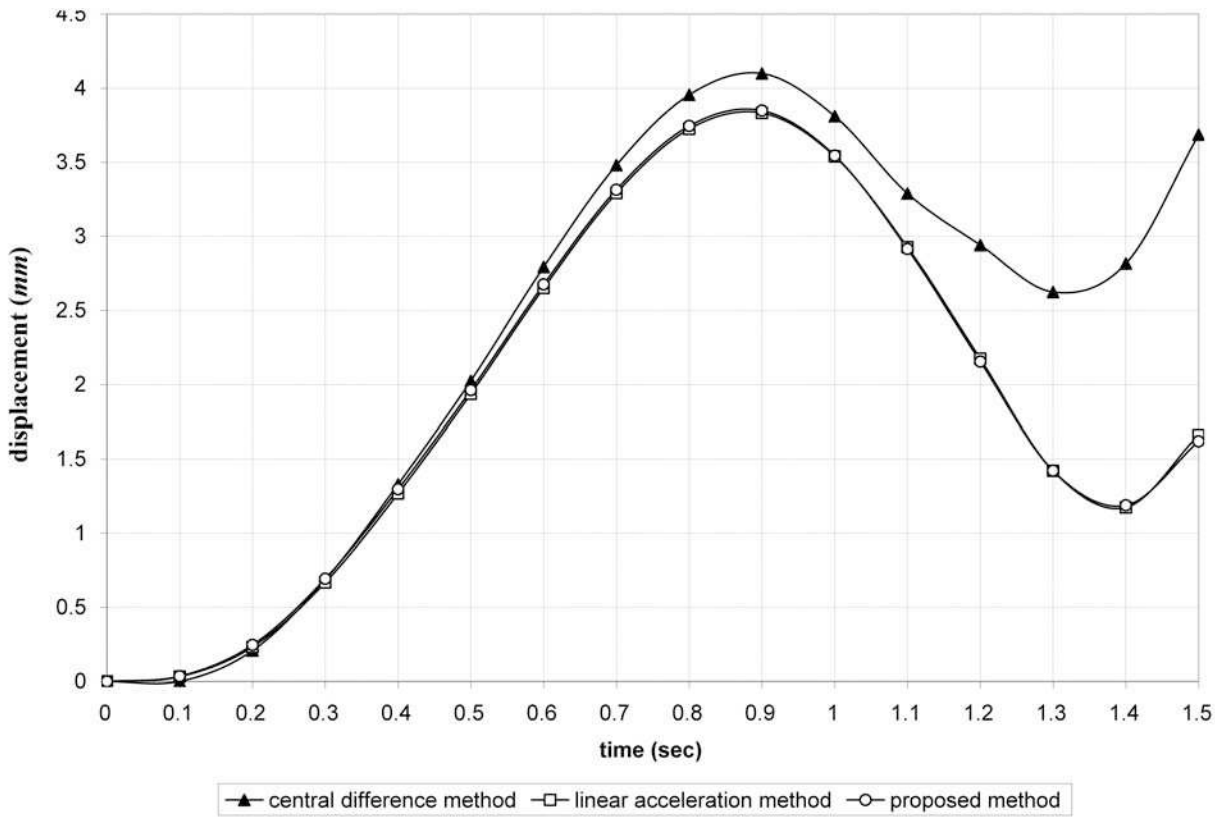

Fig. 4. Displacement responses versus time diagram for example 1

In Figs. 1, 2, and 3 the results of displacement, velocity, and acceleration of this system due to the applied loading $P(t)$ are shown for time step duration 0.1 sec. The results compare from central difference method, linear acceleration method, and the proposed methods.

The results obviously show that proposed method has a better responses in comparison with the other methods.

Example 2 [3]: Consider a two story shear building with initial conditions as; $x_0 = \dot{x} = [0, 0]^T$ in which has flexurally rigid floor beams and slabs. Nonlinear story stiffness for each story is defined as $k = k_0[1 + \lambda(\Delta x)^2]$, where Δx and k_0 are story drift and initial stiffness, respectively. Bottom and top stories have $k_0 = 10^7$ N/m and $\lambda = -100$, and $k_0 = 10^4$ N/m and $\lambda = -0.001$, respectively. Lumped masses are considered to be $1\,000$ kg and this system has been excited by a ground acceleration of $50 \cdot \sin(\omega \cdot t)$, $\omega = 1$ rad/sec, at the base of building. Natural frequencies of system are found to be 3.16 and 100.05 rad/sec, respectively. Displacement responses obtained from linear acceleration method (Newmark's method) by a time step duration of 0.001 sec resulted in exact solution. Figs. 7 and 8 show the comparison of displacement responses with $\Delta t = 0.02$ sec to exact solution for bottom and top stories, respectively.

According to the Fig. 7, central difference and linear acceleration methods have small jumps from the exact solution but the proposed method is on the exact solution line. On the other hand, according to the Fig. 8, all methods have similar response respect to the exact solution.

In this example, we presented only displacement responses, whereas the velocity and acceleration responses calculated using the proposed method are also more accurate than the other methods.

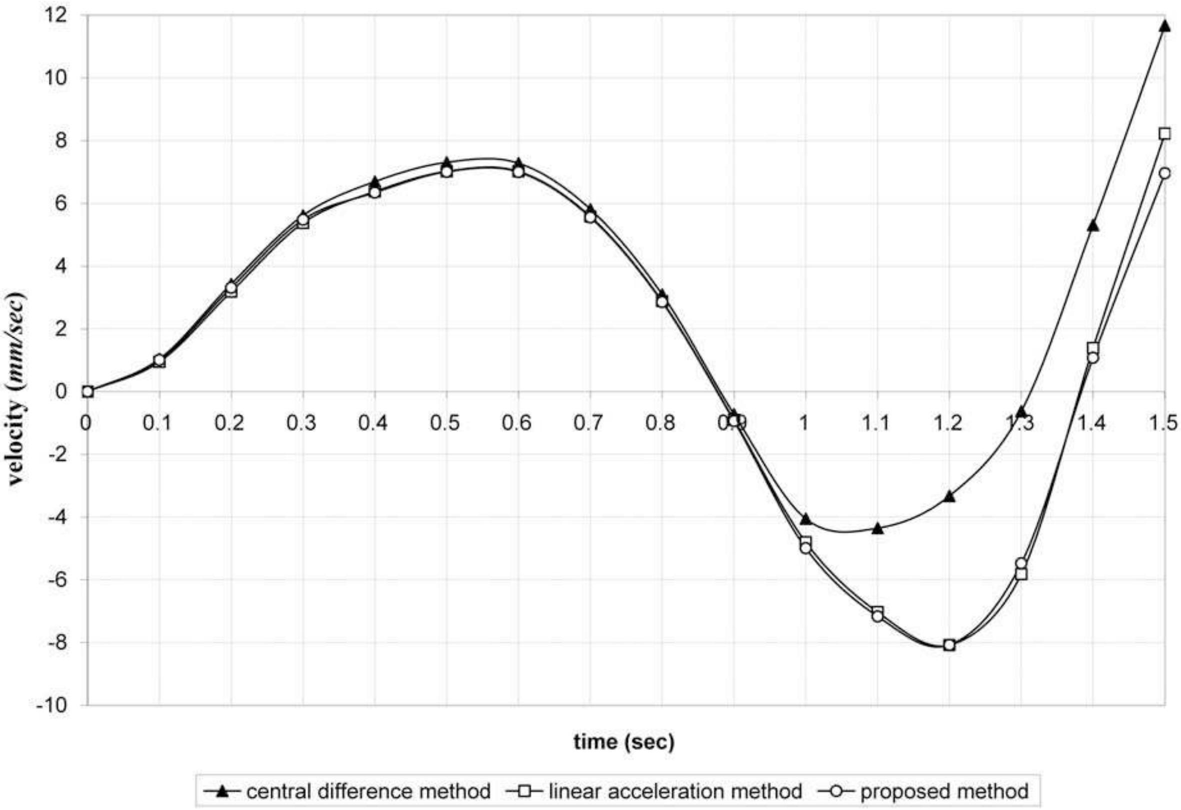

Fig. 5. Velocity responses versus time diagram for example 1

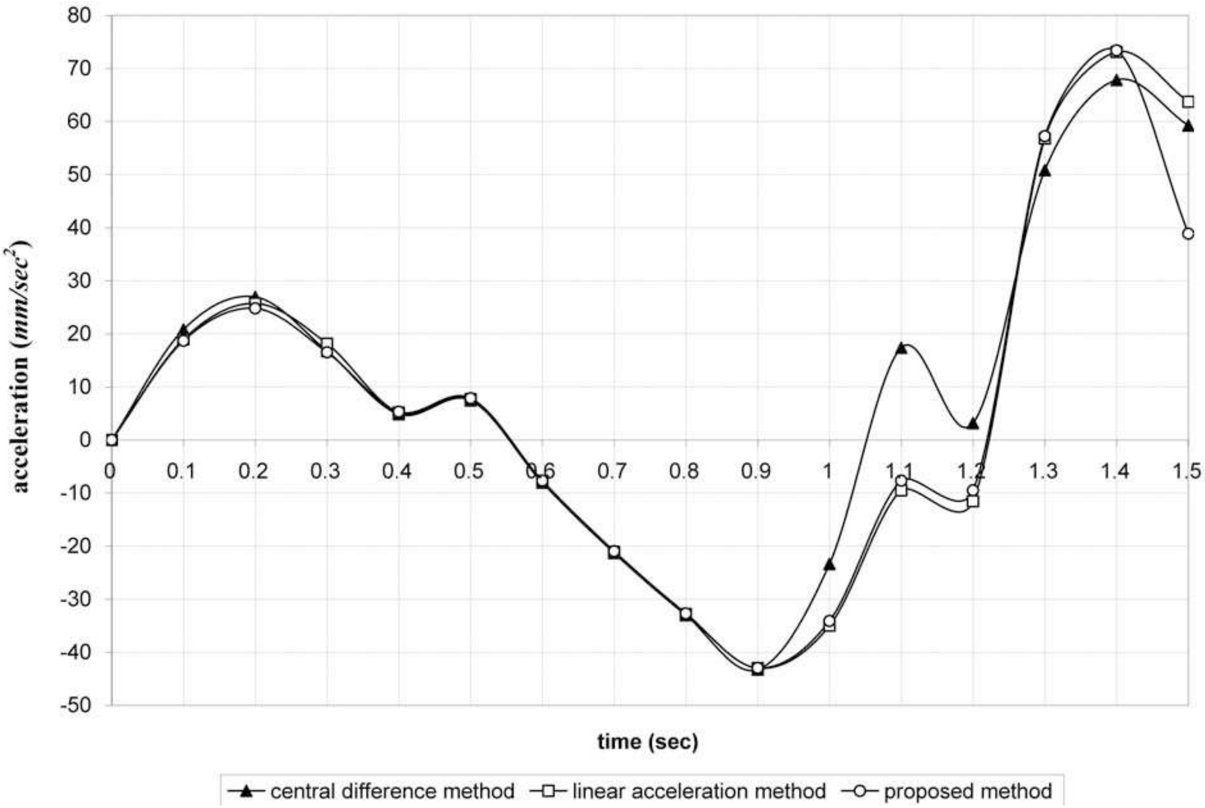

Fig. 6. Acceleration responses versus time diagram for example 1

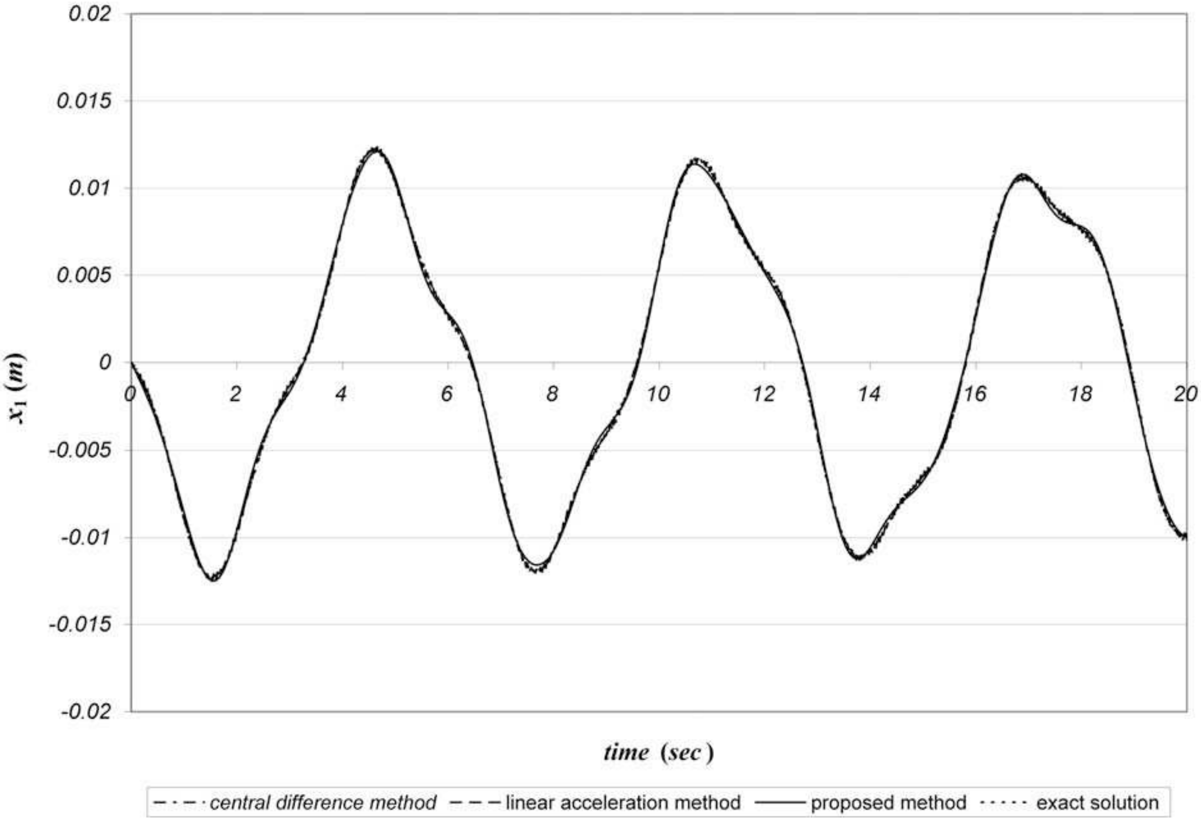

Fig. 7. Displacement responses versus time diagram for first story for example 2

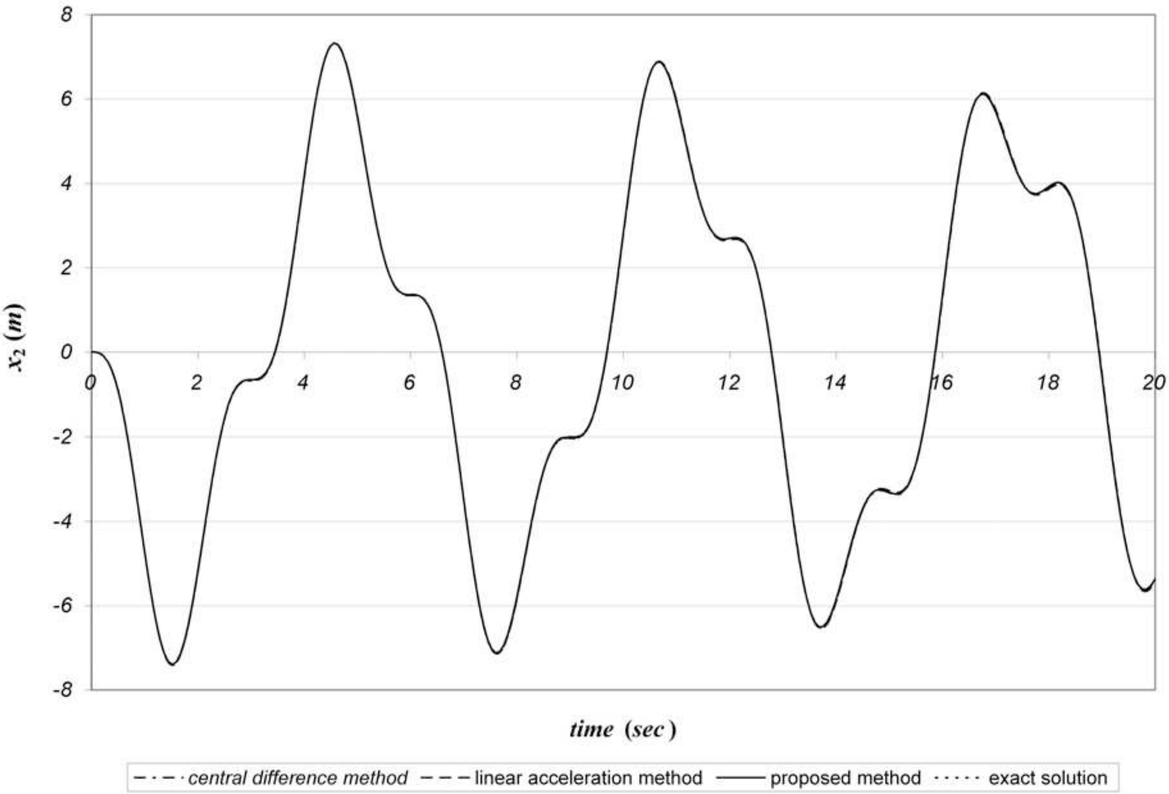

Fig. 8. Displacement responses versus time diagram for second story for example 2

5. Conclusion

A new method of time integration technique for problems in nonlinear structural dynamics was illustrated. To show the accuracy and response of the method, two examples were presented. A quadratic polynomial as a function of time was used in order to approximate the variation of acceleration in each time step. The displacement function had five constants that were calculated using: two initial conditions, from the end of previous time step, two equations from satisfying the equilibrium at both ends of the time step, and one equation for the weighted residual integration where the weight function is assumed to be unit function. The proposed method had a small local instability which it can be resolved by increasing the damping ratio that however had higher stability than the other methods. Also order of accuracy of the method was three.

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