New implicit method for analysis of problems in nonlinear structural dynamics

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Abstract

In this paper a new method is proposed for direct time integration of nonlinear structural dynamics problems. In the proposed method the order of time integration scheme is higher than the conventional Newmark's family of methods. This method assumes second order variation of the acceleration at each time step. Two variable parameters are used to increase the stability and accuracy of the method. The result obtained from this new higher order method is compared with two implicit methods; namely the Wilson-θ and the Newmark's average acceleration methods.
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1. Introduction

Problems in the theory of vibration are divided in two categories; wave propagation problems and inertia problems, which the latter is called structural dynamics. In these problems, governing field equation is a second order differential equation [1, 2]. For nonlinear systems, it is usually expected to solve equations of motion numerically [2]. In the time integration methods, time is divided to several time steps and an algorithm is used to predict the values of displacement, velocity or acceleration at each time based on previous value. The algorithm is based on an assumption for variation of displacement in each time step and satisfying the equation of motion in selected discrete times. In fact it is a form of finite difference solution for differential equations [2–11].

In nonlinear analysis, stiffness is calculated at the beginning of each time step and then response is calculated at the end of this time step with assuming that stiffness is constant throughout the step. Therefore nonlinearity is considered by continuously updating the stiffness. Calculated responses will be considered at the end of each time step as the initial conditions for next time step. Therefore system nonlinearity behavior is replaced with a series of consecutive approximate linear characteristics [1, 2, 5, 8].

In some of algorithms, in each time step, equation of motion is written at the beginning of the time step and the unknown values at the end of time step is explicitly calculated, these methods are called explicit methods. In some other methods, to calculate the unknown values at the end of time step it is required to write the equation of motion at this point, these methods are called implicit methods [2–9]. A method is called convergent if its error for a specific time is decreased, by decreasing time step length. Also, a method is consistent if the upper bound of

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its residue (error in satisfying the equation of motion), is a constant power of time step length. In accuracy evaluation of the time integration methods, usually two quantities are determined, numerical damping (dissipation) and periodic error (dispersion).

In unconditionally stable methods, instability never happens, no matter how the long time step is [1–5]. Newmark, [12], presented a one-step algorithm with two parameters and he noted that γ should be taken as 0.5, because for values more than 0.5 positive numerical damping will exist and for values less than that, it will have negative numerical damping (numerical instability). The average acceleration form appears to be the most popular one. After him, lots of researches have worked on his idea. Wilson presented a modified form of linear acceleration method, called Wilson-θ method [13], and improved it to an unconditionally stable method. He also proposed the concept of collocation to develop dissipative algorithms, which were further generalized in [13]. The Wilson-θ method is unconditionally stable for $\theta = 1.37$. This method is subject to both phase and amplitude errors depending on the time step used.

Classical methods such as the Newmark's method [12] or the Wilson-θ method [13] assume a constant or linear expression for the variation of acceleration at each time step. In conditionally stable methods, the time step must be smaller than a critical time step as a constant times the smallest period of the system, consequently often entails using time steps that are much smaller than those needed for accuracy [7]. In this paper, we illustrate how to derive equations of proposed method from the Taylor series expansion in which algorithmic parameters are inserted. In this new implicit method, it is assumed that the acceleration varies quadratically within each time step. Considering this assumption and employing the two parameters δ and α, the proposed method is derived.

2. Proposed Method

The governing nonlinear equation of motion is expressed as:

$$M\ddot{x} + C\dot{x} + K(x)x = P, \tag{1}$$

where M is a constant mass matrix, C is a constant damping matrix, and $K(x)$ is a nonlinear stiffness matrix; P is vector of applied forces; x, $\dot{x}$ and $\ddot{x}$ are the displacement, velocity and acceleration vectors respectively.

By Applying the Taylor series expansions of $x_{t+\Delta t}$ and $\dot{x}_{t+\Delta t}$ about time t and truncating the equations, the following forms of equations are obtained:

$$x_{t+\Delta t} = x_t + \Delta t \dot{x}_t + \frac{\Delta t^2}{2}\ddot{x}_t + \frac{\Delta t^3}{6}\dddot{x}_t + \alpha \Delta t^4 \ddddot{x}_t, \tag{2}$$

$$\dot{x}_{t+\Delta t} = \dot{x}_t + \Delta t \ddot{x}_t + \frac{\Delta t^2}{2}\dddot{x}_t + \delta \Delta t^3 \ddddot{x}_t. \tag{3}$$

If the acceleration variation is assumed to be second order within time $t - \Delta t$ to $t + \Delta t$, the Eqs. (2) and (3) can be written as:

$$x_{t+\Delta t} = x_t + \Delta t \dot{x}_t + \left[\left(\alpha - \frac{1}{12}\right)\ddot{x}_{t-\Delta t} + \left(\frac{1}{2} - 2\alpha\right)\ddot{x}_t + \left(\alpha + \frac{1}{12}\right)\ddot{x}_{t+\Delta t}\right]\Delta t^2, \tag{4}$$

$$\dot{x}_{t+\Delta t} = \dot{x}_t + \left[\left(\delta - \frac{1}{4}\right)\ddot{x}_{t-\Delta t} + (1 - 2\delta)\ddot{x}_t + \left(\delta + \frac{1}{4}\right)\ddot{x}_{t+\Delta t}\right]\Delta t. \tag{5}$$

Eqs. (4) and (5) can be used to approximate the displacement and velocity at time $t + \Delta t$ respectively. It can be proven that this strategy guarantees the second-order accuracy for any

choice of δ and α. The parameters δ and α are introduced in order to improve accuracy and stability. Special case $\delta = 1/4$, $\alpha = 1/12$ leads to the linear acceleration method.

Consider equation of motion in time $t + \Delta t$ as following:

$$M\ddot{x}_{t+\Delta t} + C\dot{x}_{t+\Delta t} + K_t x_{t+\Delta t} = P_{t+\Delta t}. \tag{6}$$

By substituting Eqs. (4) and (5) into the equation of motion Eq. (6), $\ddot{x}_{t+\Delta t}$ is calculated. Note that x_0 and $\dot{x}_0$ are known and $\ddot{x}_0$ can be calculated using Eq. (1) at time $t = 0$. We need the solution at time Δt before we can begin to apply Eqs. (4) and (5). It can be computed by using any one step methods such as the linear acceleration or the average acceleration methods. Now, we can obtain $x_{2\Delta t}$ from Eqs. (4) and (5), then $x_{3\Delta t}$, and so on.

3. Examples

In order to see the result of the proposed method and to see its advantages over the other implicit existing methods, let's consider two examples which the results obtained from the proposed method are compared with the Wilson-θ and average acceleration (Newmark's) methods.

Example 1 [2]: Consider a single degree of freedom system in Fig. 1. Fig. 2 shows the equivalent spring force (f_s) versus displacement diagram with elastoplastic behavior. k_{el} from Fig. 1 is the slope of the linear part of the Fig. 2. This structure is under acting force (p) as Fig. 3. The initial conditions are $x(0) = \dot{x}(0) = 0$ that $0 \le t \le 9$ sec and $\Delta t = 0.1$.
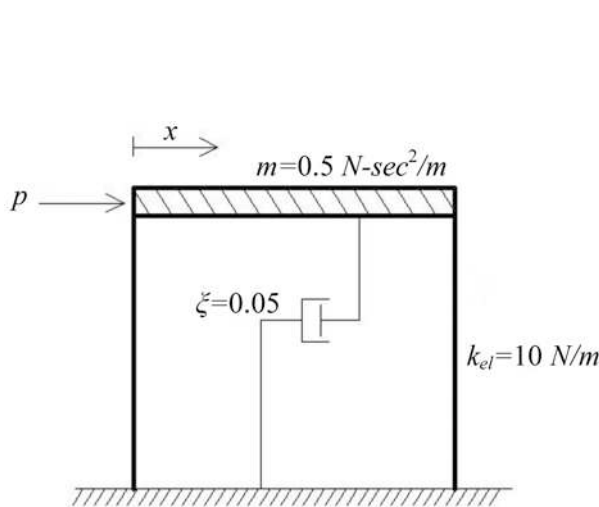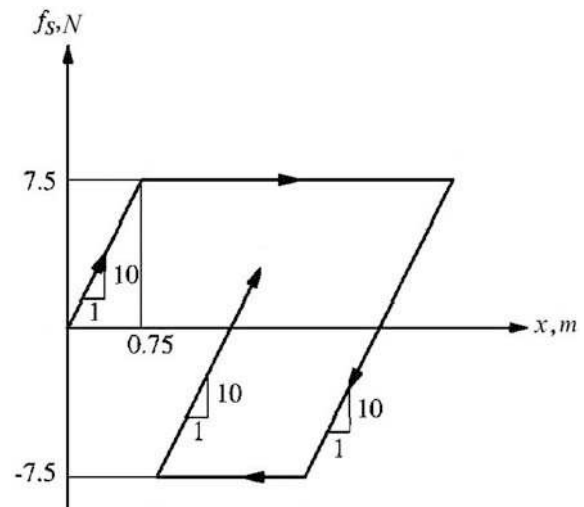

Fig. 1. Frame of structure [2]
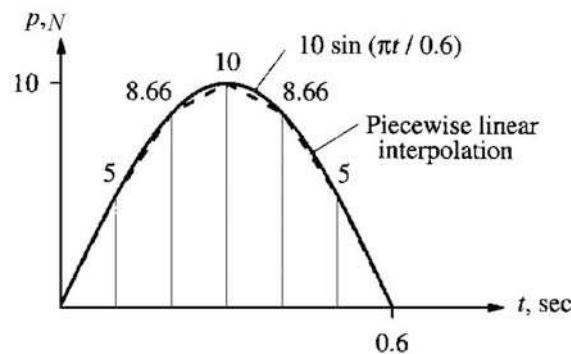

Fig. 2. Force-displacement relationship [2]

Fig. 3. Exciting force [2]

Table 1. Numerical responses using the Wilson-θ, average acceleration, and proposed methods

Time [sec]	Displacement Responses [m]		
	Wilson-θ ($\theta = 1.4$)	Average acceleration method	Proposed method ($\delta = 1/3, \alpha = 1/6$)
0 0	0 000 0	0 000 0	0 000 0
0.1	0.036 8	0.043 7	0.043 7
0.2	0.183 3	0.232 6	0.219 5
0.3	0.483 0	0.612 1	0.590 9
0.4	0.900 7	1.082 5	1.061 6
0.5	1.322 6	1.527 9	1.482 2
0.6	1.682 8	1.837 7	1.739 4
0.7	1.978 3	1.889 3	1.742 2
0.8	1.862 3	1.671 6	1.482 6
0.9	1.301 1	1.280 1	1.065 6

In Table 1 displacement results of this system due to the applied loading $P(t)$ (see Fig. 3) are given. The results obtained using of Wilson-θ, average acceleration, and proposed methods are compared.

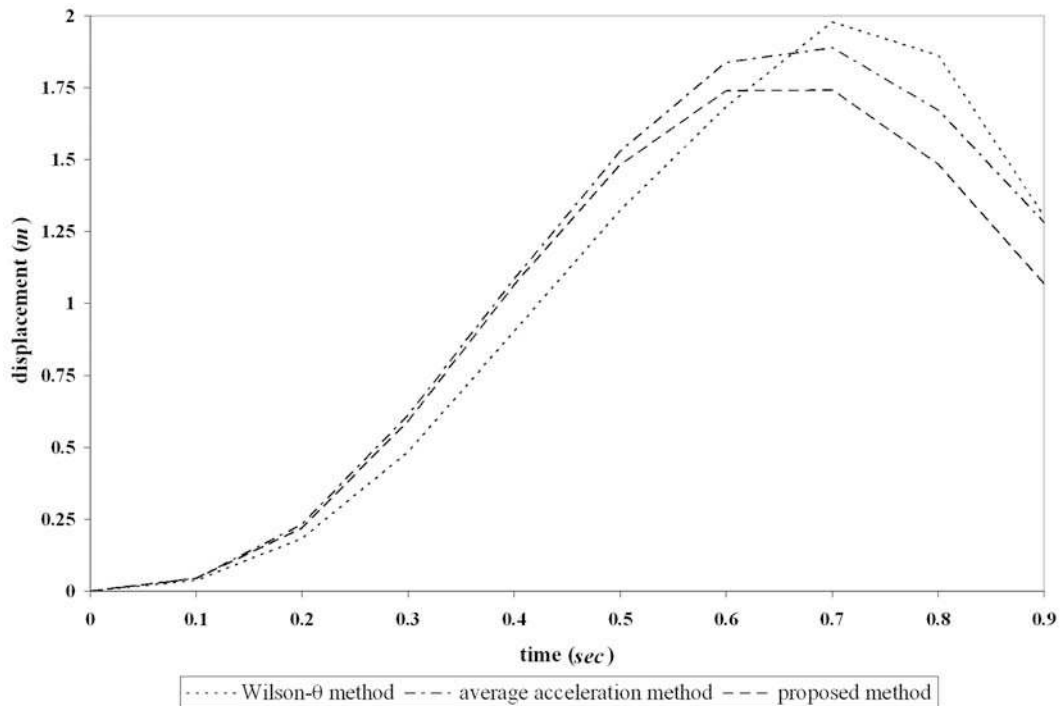Also the displacement responses versus time are shown in Fig. 4.

Fig. 4. Displacement responses versus time diagram for example 1

Example 2 [14]: Consider the second order nonlinear differential equation as following:

$$\ddot{x} + \sin x = 0, \tag{7}$$

with initial conditions $x(0) = \pi/2$ and $\dot{x}(0) = 0$ that $0 \leq t \leq 20$. Let's select $\Delta t = 0.1$ and define the error at time t as following:

$$e_t = |x_t - x_{t(exact)}|, \tag{8}$$

in which $x_{t(exact)}$ is the exact solution and x_t is the numerical solution (angle (degree)) at time t. The values obtained by the Wilson-θ, average acceleration, and proposed methods can be compared to each other in Fig. 5.
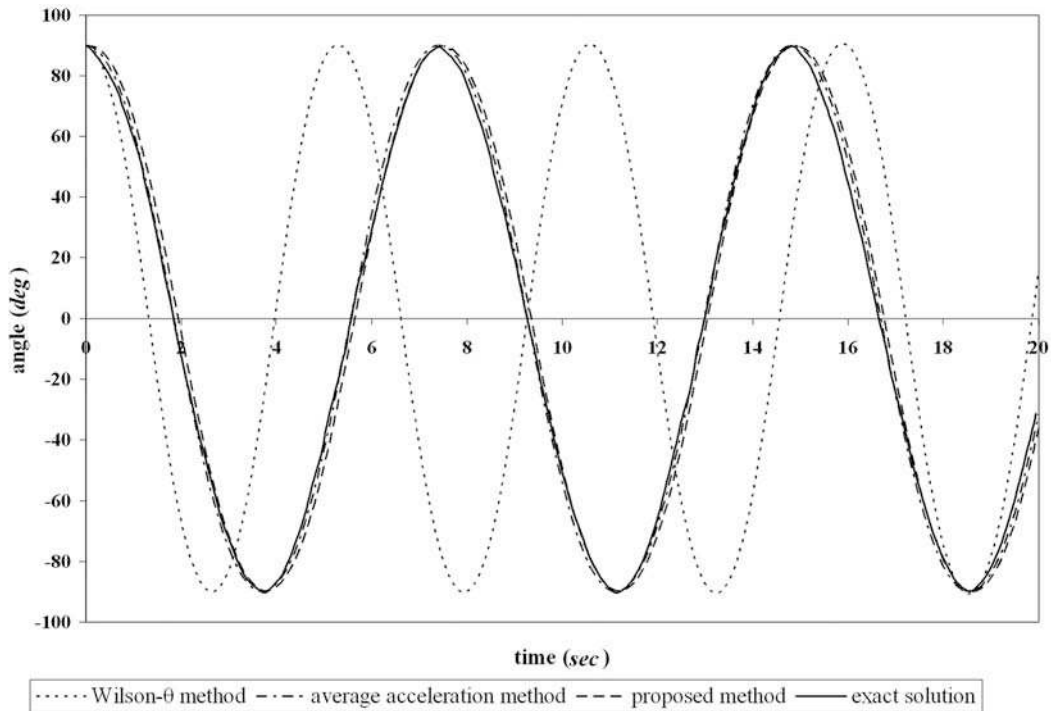

Fig. 5. Angle responses versus time diagram from example 2

The numerical solution calculated from mentioned methods and their error respect to the exact solution of Eq. (7) have been shown in Table 2 for $t = 6$ sec to $t = 7$ sec.

Table 2. Angle responses using the Wilson-θ, average acceleration, and proposed methods and their error respect to the exact solution

Time [sec]	Wilson-θ ($\theta = 1.4$)		Average acceleration method		Proposed method ($\delta = 1/3, \alpha = 1/6$)	
	x_t	e_t	x_t	e_t	x_t	e_t
6	62.160 2	33.331 4	34.366 0	5.537 2	28.138 0	0.690 8
6.1	53.881 0	18.297 2	41.556 6	5.972 8	35.592 2	0.008 4
6.2	44.690 7	1.939 0	48.369 1	5.617 4	42.714 0	0.037 6
6.3	34.704 1	14.532 9	54.751 8	5.514 9	49.446 3	0.209 3
6.4	24.070 0	31.296 3	60.670 5	5.304 2	55.743 1	0.376 8
6.5	12.971 8	47.445 4	66.090 7	5.673 6	61.564 4	1.147 2
6.6	1.621 5	64.631 9	70.983 8	4.730 4	66.881 4	0.628 0
6.7	−9.768 9	81.310 5	75.338 3	3.796 7	71.671 3	0.129 8
6.8	−20.964 5	96.228 3	79.142 7	3.878 9	75.917 0	0.653 1
6.9	−31.759 1	111.718	82.379 9	2.421 4	79.606 8	0.351 7
7	−41.963 5	125.158	85.049 9	1.855 7	82.729 4	0.464 7

Table 2 shows that the numerical values of x_t calculated using the proposed method are more accurate than those for the Wilson-θ and average acceleration methods. In this example, we presented only angle responses, whereas the angular velocity and angular acceleration responses calculated using the proposed method are also more accurate than the other methods.

4. Conclusion

A new implicit step by step integration technique for problems in structural dynamics was illustrated. A second order polynomial as a function of time was used in order to approximate the variation of acceleration during the time steps. Therefore the proposed method was shown more accurate values than the Wilson-θ and average acceleration methods. This method was a two parameter method (δ and α). Proposed method allows numerical damping while retaining second order accuracy. The new method can be used for either linear or nonlinear problems, although in this paper, we have discussed only nonlinear problems.

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