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Shape optimization of a Timoshenko beam together with an elastic foundation

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Abstract

In this article we are going first to aim at the variational formulation of the bending problem for the Timoshenko beam model. Afterwards we will extend this problem to the Timoshenko beam resting on the Winkler foundation, which is firmly connected with the beam. Hereafter a shape optimization for the aforementioned problems is presented. The state problem is here represented by the system of two ordinary differential equations of the second order. The optimization problem is given as a minimization of the so-called compliance functional on the set of all admissible design variables. For our purpose as the design variable we will select the beam thickness. Shape optimization problems have attracted the interest of many applied mathematicians and engineers. The objective of this article is to present a solution method for one of these problems and its demonstration by examples.

Keywords: Timoshenko beam, Winkler foundation, finite element method, shape optimization

1. Introduction

Nowadays it is well known that the classical Euler-Bernoulli beam theory is valid only for long span, equivalently thin, beams. In 1921 S. P. Timoshenko proposed a new beam theory that has been used for short, equivalently thick, beams. Unlike the Euler-Bernoulli hypothesis, the Timoshenko beam theory supposes that the plane section originally normal to the beam middle axis remains plane but not necessarily normal to the deformed axis, as in addition also transverse shear deformations can occur. Thus, using this theory it is possible to analyze thicker beams more accurately than by the classical beam theory.

A variational formulation of the bending problem and a finite element model will be interested us in the first part of this work. But foremost we want to present here some shape optimization of the Timoshenko beam with regard to the beam thickness. The criterion will be done by the compliance functional. Afterwards we are going to deal with the beam resting on a Winkler foundation. For this case it is possible to consider beside the beam thickness also the foundation stiffness optimization, but this opportunity will not be demonstrated in this article.

Several works have been done on this field but none of them is concerning the Timoshenko beam. The thickness optimization for the Euler–Bernoulli beam model was mostly examined, as it is from the theoretical and computational point of view a sort of fundamental and interesting case. Optimization of a beam with a subsoil of Winkler's type was studied in [7]. Object of optimization was the thickness and the subsoil stiffness. Especially it was focused on numerical modeling of the problem using ANSYS software system. Thickness optimization of a beam with a rigid obstacle was treated in [11], where an approach based on sensitivity analysis and

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nonlinear optimization methods was used. Optimal design of a beam on an unilateral elastic subsoil was presented in [12]. Existence of at least one solution was proved and conditions ensuring the solvability of the state problem were formulated.

2. Timoshenko beam with Winkler foundation

Let us consider a beam of the length L. The displacement field is

$$u_x(x,y) = u(x) - y\theta(x), \quad u_y(x,y) = w(x), \quad u_z(x,y) = 0, \tag{1}$$

where θ denotes the rotation of the cross-section plane about a normal to the middle axis x, u is the axial displacement of this axis and w is its transverse displacement. Little analyzing gives us

$$\theta(x) = w'(x) - \gamma(x), \quad \gamma(x) = \frac{Q(x)}{\kappa GA}. \tag{2}$$

Here Q is the transverse shear force, γ is the angle of shearing, G is the shear modulus, A is the cross-section area and κ is the shear correction factor. This factor is dependent on the cross-section and on the type of problem; one frequently used formula is $\kappa = \frac{10(1+\nu)}{12+11\nu}$ (ν is the Poisson's ratio) or sometimes $\kappa = \frac{5}{6}$ (see e.g. [3]).

Substituting (1) into the Green-St Venant strain tensor we obtain after some rearrangements (for the details see e.g. [3] or [6]) the system of two equations with the unknowns w and θ

$$(EI\theta')' + \kappa GA(w' - \theta) + m = 0, \tag{3}$$

$$(\kappa GA(w' - \theta))' + q = 0, \tag{4}$$

with E denoting the elasticity modulus, I the moment of inertia, $q(x)$ the applied transverse load and $m(x)$ the applied moment. The values of E and G are assumed constant, whereas I and A will be functions of the beam cross-section proportions b and t. Here b denotes the beam width and its height is considered in the interval $[-t, t]$. As we want later to optimize the beam thickness, t will be a function of x and for simplicity referred as the thickness although it is actually only a "half-thickness". For definiteness we will study the beam with a rectangular cross-section, hence we have

$$I(x) = \frac{2}{3}bt^3(x), \quad A(x) = 2bt(x). \tag{5}$$

Now we will deal with the variational formulation of the Timoshenko beam bending problem. First we must define suitable spaces for our unknowns. Let V be the space of kinematically admissible deflections v such that

$$H_0^1((0,\mathrm{L})) \subseteq V \subseteq H^1((0,\mathrm{L})). \tag{6}$$

Let us remember that the *Sobolev space* $H^1((0,\mathrm{L}))$ consists of those functions $v \in L^2((0,\mathrm{L}))$ for which derivatives v' (in the distribution sense) belong to the space $L^2((0,\mathrm{L}))$. The *Lebesgue space* $L^2((0,\mathrm{L}))$ is defined as the space of all measurable functions on $(0,\mathrm{L})$, the squares of which have a finite Lebesgue integral. Finally we define in (6) the space $H_0^1((0,\mathrm{L}))$ by

$$H_0^1((0,\mathrm{L})) = \{v \in H^1((0,\mathrm{L})) : v(0) = v(\mathrm{L}) = 0\}. \tag{7}$$

More information in relation to Sobolev spaces can be found e.g. in [1]. The same we can make also for kinematically admissible rotations η. The respective spaces will be distinguished as V_1 and V_2.

It is well known that the finite element method distinguishes between *natural* and *essential boundary conditions*. The first ones are contained in the space V, the second ones are built into the variational formulation. Let us remark, that the beam fixed at the both ends requires working with the spaces (7). Since the concrete boundary conditions are not important for our next explanation, without a loss of generality we can simply assume that the beam is fixed at both ends, so that we will work with the spaces $V_1 = V_2 = H_0^1((0, \mathrm{L}))$.

Using test functions from the above defined spaces we can obtain from the system (3)–(4) after integration by parts

$$\int_0^{\mathrm{L}} EI\theta' \eta' \,\mathrm{d}x - \int_0^{\mathrm{L}} \kappa GA(w' - \theta)\eta \,\mathrm{d}x = \int_0^{\mathrm{L}} m\eta \,\mathrm{d}x \qquad \forall \eta \in V_2, \tag{8}$$

$$\int_0^{\mathrm{L}} \kappa GA(w' - \theta)v' \,\mathrm{d}x = \int_0^{\mathrm{L}} qv \,\mathrm{d}x \qquad \forall v \in V_1. \tag{9}$$

Summing these equations together leads to

$$\int_0^{\mathrm{L}} EI\theta' \eta' \,\mathrm{d}x + \int_0^{\mathrm{L}} \kappa GA(w' - \theta)(v' - \eta) \,\mathrm{d}x = \int_0^{\mathrm{L}} m\eta \,\mathrm{d}x + \int_0^{\mathrm{L}} qv \,\mathrm{d}x \quad \forall v \in V_1, \eta \in V_2 \tag{10}$$

and this can be interpreted as the equation for a stationary point of the potential energy of the Timoshenko beam and it is possible to write it as follows

$$J'_{TB}(w, \theta; v, \eta) = 0 \qquad \forall v \in V_1, \eta \in V_2. \tag{11}$$

$J'_{TB}(w, \theta; v, \eta)$ denotes the Gâteaux derivative of J_{TB} at the point $\{w, \theta\}$ in the directions v, η (see e.g. [1]). Equations (11) and (10) imply that the functional of potential energy has the form

$$J_{TB}(w, \theta) = \frac{1}{2}\int_0^{\mathrm{L}} EI\,(\theta')^2 \,\mathrm{d}x + \frac{1}{2}\int_0^{\mathrm{L}} \kappa GA\,(w' - \theta)^2 \mathrm{d}x - \int_0^{\mathrm{L}} m\theta \,\mathrm{d}x - \int_0^{\mathrm{L}} qw \,\mathrm{d}x. \tag{12}$$

It is easy to prove that this functional is strictly convex. Then the equation (11) can be consequently rewritten as

$$J_{TB}(w, \theta) = \min_{v \in V_1,\, \eta \in V_2} J_{TB}(v, \eta). \tag{13}$$

The problem of finding a pair $\{w, \theta\} \in V_1 \times V_2$ such that (13) holds we will call the *variational formulation* of the Timoshenko beam bending. The convexity implies the unique solution of the minimization problem (13) and also the fact that (13) can be equivalently represented by the pair of equations (8)–(9).

Now let us go forward to the problem with a Winkler foundation. If k_F is the foundation stiffness, then its potential energy reads as

$$J_{WF}(w) = \frac{1}{2}\int_0^{\mathrm{L}} k_F w^2 \,\mathrm{d}x. \tag{14}$$

From here we immediately obtain the functional of total energy for the system beam plus foundation

$$J(w,\theta) = J_{TB}(w,\theta) + J_{WF}(w). \tag{15}$$

The variational formulation of this problem is as follows:

$$\left\{ \begin{array}{l} \text{Find functions } \{w,\theta\} \in V_1 \times V_2 \text{ such that} \\ J(w,\theta) = \min\limits_{v\in V_1,\, \eta \in V_2} J(v,\eta). \end{array} \right. \tag{16}$$

The functional (15) obviously retains the strict convexity, hence (16) can be equivalently rewritten as

$$J^{'}(w,\theta;v,\eta) = 0 \qquad \forall v \in V_1, \eta \in V_2. \tag{17}$$

It gives us

$$\int_0^{\mathrm{L}} EI\theta^{'}\eta^{'}\mathrm{d}x - \int_0^{\mathrm{L}} \kappa GA(w^{'} - \theta)\eta\,\mathrm{d}x = \int_0^{\mathrm{L}} m\eta\,\mathrm{d}x \qquad \forall \eta \in V_2, \tag{18}$$

$$\int_0^{\mathrm{L}} \kappa GA(w^{'} - \theta)v^{'}\mathrm{d}x + \int_0^{\mathrm{L}} k_F w v\,\mathrm{d}x = \int_0^{\mathrm{L}} qv\,\mathrm{d}x \qquad \forall v \in V_1 \tag{19}$$

and from here we can deduce the following system of two equations

$$(EI\theta^{'})^{'} + \kappa GA(w^{'} - \theta) + m = 0, \tag{20}$$

$$(\kappa GA(w^{'} - \theta))^{'} + k_F w + q = 0 \tag{21}$$

presenting the extension of the original system (3)–(4) by the foundation term.

3. Finite element model for the Timoshenko beam

Now we proceed to a finite element discretization of our problems. As the problem without the foundation (3)–(4) can be obtained from (20)–(21) by putting $k_F = 0$, we will refer to the last one. For this purpose we have to define some division of the interval $[0, \mathrm{L}]$ into subintervals $K_i = [x_{i-1}, x_i]$, where we have generated *nodes* $0 = x_0 < x_1 < \ldots < x_n = \mathrm{L}$. Without loss of generality, we will restrict ourself to an equidistant division, i.e. $x_i - x_{i-1} = h$ for all i. Formally, the *discrete problem* reads as follows:

$$\left\{ \begin{array}{l} \text{Find } \{w_h,\theta_h\} \in V_{1,h} \times V_{2,h} \text{ such that} \\ J(w_h,\theta_h) = \min\limits_{v_h\in V_{1,h},\, \eta_h \in V_{2,h}} J(v_h,\eta_h)\,, \end{array} \right. \tag{22}$$

which is equivalent to

$$\int_0^{\mathrm{L}} EI\theta_h^{'}\eta_h^{'}\mathrm{d}x - \int_0^{\mathrm{L}} \kappa GA(w_h^{'} - \theta_h)\eta_h\mathrm{d}x = \int_0^{\mathrm{L}} m\eta_h\mathrm{d}x \qquad \forall \eta_h \in V_{2,h}, \tag{23}$$

$$\int_0^{\mathrm{L}} \kappa GA(w_h^{'} - \theta_h)v_h^{'}\mathrm{d}x + \int_0^{\mathrm{L}} k_F w_h v_h\mathrm{d}x = \int_0^{\mathrm{L}} qv_h\mathrm{d}x \qquad \forall v_h \in V_{1,h}. \tag{24}$$

$V_{k,h}$ is a finite-dimensional subspace of the given space V_k, $k = 1, 2$. Because, in contrast to the Euler-Bernoulli beam, we need not C^1-continuity, it is quite natural to choose the simplest approximation method, i.e.

$$V_{k,h} = \{v_h \in V_k : v_h|_{K_i} \in P_1(K_i) \quad \forall i = 1, \dots, n\} \qquad k = 1, 2, \tag{25}$$

where $P_1(K_i)$ denotes the set of linear polynomials defined on K_i and hence $V_{k,h}$ contains continuous piecewise linear functions.

Now we can continue as it is usual for the standard finite element method. We define the *Lagrange basis functions* for our space (25) and afterwards the *shape functions* on a single element, which are beneficial from the practical computation point of view. Finally we obtain a system of linear equations (see e.g. [2] or [9]).

Unfortunately, there is a serious difficulty in this procedure — the phenomenon called the *shear locking* (see e.g. [10]). A brief explanation is as follows. Let us consider (2) after the finite element discretization. It results in

$$\gamma_h = w_h' - \theta_h. \tag{26}$$

We expect that γ_h converges to zero as the thickness $t \to 0$ (so-called Euler-Bernoulli limit). But according to (25) on the right side of (26) we have the difference of a constant and a linear function on every element and it will never give zero. Hence the shear strains, which are equal to γ_h, cannot be arbitrary small and in practice the computed deflections can be much smaller than the exact solution.

There are several possibilities how to handle this problem. We have chosen the way that is mathematically completely correct. Therefore we define a new approximation for the unknown w so that w_h' will have the same polynomial degree as θ_h. For this purpose we put

$$V_{1,h} = \{v_h \in V_1 : v_h|_{K_i} \in P_2(K_i) \quad \forall i = 1, \dots, n\} \tag{27}$$

and $P_2(K_i)$ denotes the set of quadratic polynomials defined on K_i. Of course, we must add an extra node in the middle of each interval K_i. The space $V_{2,h}$ remains the same as in (25).

Now we are able to evaluate the element matrix for an element of the length h. Let $x_i = 0$, $x_{i+1/2} = \frac{h}{2}$, $x_{i+1} = h$ and let us denote

$$w_i = w_h(x_i),\ w_{i+1/2} = w_h(x_{i+1/2}),\ w_{i+1} = w_h(x_{i+1}), \tag{28}$$

$$\theta_i = \theta_h(x_i),\ \theta_{i+1} = \theta_h(x_{i+1}). \tag{29}$$

Then we have for $x \in [0, h]$

$$w_h(x) = \frac{1}{h^2}[(2x^2 - 3hx + h^2)w_i + (2x^2 - hx)w_{i+1} + (-4x^2 + 4hx)w_{i+1/2}], \tag{30}$$

$$\theta_h(x) = \frac{1}{h}[(-x + h)\theta_i + x\theta_{i+1}]. \tag{31}$$

Substituting these relations into (23)–(24) gets after some integrations the beam element matrix

$$\begin{pmatrix} \frac{7\kappa GA}{3h} & \frac{5\kappa GA}{6} & \frac{\kappa GA}{3h} & \frac{\kappa GA}{6} & -\frac{8\kappa GA}{3h} \\ \frac{5\kappa GA}{6} & \frac{EI}{h} + \frac{\kappa GAh}{3} & -\frac{\kappa GA}{6} & -\frac{EI}{h} + \frac{\kappa GAh}{6} & -\frac{2\kappa GA}{3} \\ \frac{\kappa GA}{3h} & -\frac{\kappa GA}{6} & \frac{7\kappa GA}{3h} & -\frac{5\kappa GA}{6} & -\frac{8\kappa GA}{3h} \\ \frac{\kappa GA}{6} & -\frac{EI}{h} + \frac{\kappa GAh}{6} & -\frac{5\kappa GA}{6} & \frac{EI}{h} + \frac{\kappa GAh}{3} & \frac{2\kappa GA}{3} \\ -\frac{8\kappa GA}{3h} & -\frac{2\kappa GA}{3} & -\frac{8\kappa GA}{3h} & \frac{2\kappa GA}{3} & \frac{16\kappa GA}{3h} \end{pmatrix} \begin{pmatrix} w_i \\ \theta_i \\ w_{i+1} \\ \theta_{i+1} \\ w_{i+1/2} \end{pmatrix}. \tag{32}$$

As we have internal nodes in every element, we can use the *static condensation* technique (see e.g. [2]) to eliminate the unknowns associated with these nodes. After that we obtain

$$\begin{pmatrix} \frac{\kappa GA}{h} & \frac{\kappa GA}{2} & -\frac{\kappa GA}{h} & \frac{\kappa GA}{2} \\ \frac{\kappa GA}{2} & \frac{EI}{h}+\frac{\kappa GAh}{4} & -\frac{\kappa GA}{2} & -\frac{EI}{h}+\frac{\kappa GAh}{4} \\ -\frac{\kappa GA}{h} & -\frac{\kappa GA}{2} & \frac{\kappa GA}{h} & -\frac{\kappa GA}{2} \\ \frac{\kappa GA}{2} & -\frac{EI}{h}+\frac{\kappa GAh}{4} & -\frac{\kappa GA}{2} & \frac{EI}{h}+\frac{\kappa GAh}{4} \end{pmatrix} \begin{pmatrix} w_i \\ \theta_i \\ w_{i+1} \\ \theta_{i+1} \end{pmatrix}. \tag{33}$$

The same can be done also for the foundation element matrix. For example [13] contains more relevant details.

4. Optimization

In this section we shall formulate a shape optimization problem for the Timoshenko beam model presented in the previous sections. We will optimize the thickness of the beam with respect to a compliance cost functional. Let us note, that here can be used also the name *sizing optimization*, as only typical size of a structure is optimized.

Let the thickness t be a function depending on x and occurring in the Timoshenko beam model as a part of the cross-section area A and its inertia moment I as it is in (5). To define an optimization problem we have to specify the class U_{ad} of *admissible thicknesses*

$$U_{ad} = \Bigg\{ t \in C^{0,1}([0,\mathrm{L}]) \ : \ 0 < T_0 \le t(x) \le T_1 \quad x \in [0,\mathrm{L}], \\ \int_0^{\mathrm{L}} t(x)\,\mathrm{d}x = T_2, \quad |t'(x)| \le T_3 \quad \forall x \in [0,\mathrm{L}] \Bigg\}, \tag{34}$$

where the positive constants T_0, T_1, T_2 and T_3 are chosen in such a way that U_{ad} is nonempty. The set U_{ad} consists of all Lipschitz continuous functions that are uniformly bounded together with the absolute value of their first derivatives in $[0,\mathrm{L}]$. Moreover the volume of the beam is preserved and fixed during the optimization.

For an arbitrary but fixed $t \in U_{ad}$ the state problem is defined by (8), (9) and (18), (19), respectively. It can be proved that there is a continuous dependence between the design variable t and the state problem solution $\{w,\, \theta\}$. Further let us define the *compliance cost functional*

$$\mathcal{J}(t,\, w,\, \theta) = \int_0^{\mathrm{L}} q(x)\, w(x)\,\mathrm{d}x. \tag{35}$$

Functional $\mathcal{J}$ corresponds to the compliance of the transversally loaded beam. The compliance cost functional is not explicitly dependent on the design variable t, but generally the cost functional is a mapping $\mathcal{J} : U_{ad} \times V_1 \times V_2 \to \mathbb{R}$, see e.g. [4, 8]. Now we are ready to formulate the *shape optimization problem*:

$$\begin{cases} \text{Find } t^* \in U_{ad} \text{ such that} \\ \mathcal{J}(t^*,\, w^*,\, \theta^*) \le \mathcal{J}(t,\, w,\, \theta) \quad \forall t \in U_{ad}, \end{cases} \tag{36}$$

where $\{w,\, \theta\} = \{w(t),\, \theta(t)\}$ is a solution of the state problem for corresponding $t \in U_{ad}$. It can be proved that the optimization problem has at least one solution. We can describe the optimization problem by the following scheme:

$$t \longmapsto \{w,\, \theta\} \longmapsto \mathcal{J}(t,\, w,\, \theta). \tag{37}$$

Optimization problem in this form is not suitable for a numerical realization. Now we proceed to a discretization of our problem. The problem will be transformed to a new one defined by finite number of parameters. We start with the discretization of U_{ad}. Instead of general thickness $t(x)$ we will consider only those functions from U_{ad} that are Lipschitz continuous and piecewise linear on the partition $0 = x_0 < x_1 < \ldots < x_n = \mathrm{L}$, i.e., we define

$$U_{ad}^h = \left\{ t_h \in C^{0,1}([0,\mathrm{L}]) :\ t_h|_{K_i} \in P_1(K_i),\ \forall i = 1,\ldots,n \right\} \cap U_{ad}. \tag{38}$$

There is also an option to consider a stepped beam. It means to use piecewise constant thickness distribution instead of piecewise linear, see e.g. [4]. We can associate the design variable $t_h \in U_{ad}^h$ with an $(n+1)$-dimensional vector. Components of this vector are nodal values of t_h; i.e., $t_i = t_h(x_i)$, $i = 0,\ldots,n$. Then it is easy to see that U_{ad}^h can be identified with the finite dimensional set

$$\mathcal{U}^h = \Bigg\{ \boldsymbol{t} = (t_0,\, t_1,\, \ldots,\, t_n) \in \mathbb{R}^{n+1} :\ T_0 \le t_i \le T_1,\ i = 0,\ldots,n,$$
$$\sum_{i=1}^{n} \frac{h}{2}(t_{i-1} + t_i) = T_2,\ \ |t_{i-1} - t_i| \le h\,T_3,\ i = 1,\ldots,n \Bigg\}. \tag{39}$$

Using the finite element approach presented in the previous section the state problem transforms into a system of linear algebraic equations

$$\boldsymbol{K}(\boldsymbol{t})\,\boldsymbol{w}(\boldsymbol{t}) = \boldsymbol{F}, \tag{40}$$

where $\boldsymbol{K} = \boldsymbol{K}_b + \boldsymbol{K}_s$. Stiffness matrices $\boldsymbol{K}_b$, $\boldsymbol{K}_s \in \mathbb{R}^{(2n+2)\times(2n+2)}$ correspond to the beam and its foundation, respectively. These matrices are assembled from the element matrices (33). The vector $\boldsymbol{w} \in \mathbb{R}^{2n+2}$ consists of two parts. Coefficients w_i and θ_i, $i = 0,\ldots,n$, correspond to the transversal displacement and rotation of the cross section, respectively. These values are arranged as follows

$$\boldsymbol{w} = (w_0,\ \theta_0,\ w_1,\ \theta_1,\ \ldots,\ w_n,\ \theta_n) \in \mathbb{R}^{2n+2}. \tag{41}$$

Finally we can approach to a discretization of the cost functional. We use the trapezoid formula for numerical integration:

$$\mathcal{J}_h(\boldsymbol{t},\, \boldsymbol{w}) = \sum_{i=1}^{n} \frac{h}{2}\left(w_{i-1}\, q(x_{i-1}) + w_i\, q(x_i)\right) = \boldsymbol{w}^T \boldsymbol{B}\, \boldsymbol{q}, \tag{42}$$

where

$$\boldsymbol{B} = h\,\mathrm{diag}\,(1/2, 0,\ 1,\ 0,\ 1,\ 0,\ \ldots,\ 1,\ 0,\ 1/2,\ 0) \in \mathbb{R}^{(2n+2)\times(2n+2)},$$
$$\boldsymbol{q} = (q(x_0),\ 0,\ q(x_1),\ 0,\ \ldots,\ q(x_n),\ 0) \in \mathbb{R}^{2n+2}.$$

Therefore the discrete optimization problem leads to the following *nonlinear programming problem*:

$$\left\{ \begin{array}{l} \text{Find } \boldsymbol{t}^* \in \mathcal{U}^h \text{ such that} \\ \mathcal{J}_h(\boldsymbol{t}^*,\, \boldsymbol{w}^*) \le \mathcal{J}_h(\boldsymbol{t},\, \boldsymbol{w}) \quad \forall \boldsymbol{t} \in \mathcal{U}^h, \end{array} \right. \tag{43}$$

where $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{t})$ is a solution of the linear system (40) for corresponding $\boldsymbol{t} \in \mathcal{U}^h$. It can be proved that if we let $h \to 0+$ then solutions $(t_h,\, \{w_h,\, \theta_h\})$ of the approximate optimization problem will converge to the solution of the original problem (36).

The evaluation of the cost functional involves a solving of the linear state problem. Consequently, the optimization algorithm should use as few function evaluations as possible. Thus some gradient information is needed. In what follows we shall evaluate the gradient of $\mathcal{J}_h$ with respect to $\boldsymbol{t}$. By using the classical chain rule of differentiation we can compute the derivative of the cost functional at point $\boldsymbol{t} \in \mathcal{U}^h$ and in direction $\boldsymbol{s} \in \mathbb{R}^{n+1}$:

$$\mathcal{J}_h'(\boldsymbol{t};\, \boldsymbol{s}) = \nabla_{\boldsymbol{w}}^T \mathcal{J}_h(\boldsymbol{t},\, \boldsymbol{w})\, \boldsymbol{w}'(\boldsymbol{t};\, \boldsymbol{s}). \tag{44}$$

By differentiating (40) we obtain

$$\boldsymbol{K}(\boldsymbol{t})\, \boldsymbol{w}'(\boldsymbol{t};\, \boldsymbol{s}) = -\boldsymbol{K}'(\boldsymbol{t};\, \boldsymbol{s})\, \boldsymbol{w}(\boldsymbol{t}). \tag{45}$$

To get full information on the gradient $\nabla_{\boldsymbol{t}} \mathcal{J}_h$, we need to compute the directional derivative (44) in $n+1$ linearly independent directions. We use the adjoint state technique to overcome this difficulty. Let us define the adjoint state problem

$$\boldsymbol{K}(\boldsymbol{t})\, \boldsymbol{p}(\boldsymbol{t}) = \nabla_{\boldsymbol{w}} \mathcal{J}_h(\boldsymbol{t},\, \boldsymbol{w}) = \boldsymbol{B}\, \boldsymbol{q}. \tag{46}$$

Then multiplying (46) by $\boldsymbol{w}'(\boldsymbol{t};\, \boldsymbol{s})$ we have

$$-\boldsymbol{p}^T(\boldsymbol{t})\, \boldsymbol{K}'(\boldsymbol{t};\, \boldsymbol{s})\, \boldsymbol{w}(\boldsymbol{t}) = \boldsymbol{p}^T(\boldsymbol{t})\, \boldsymbol{K}(\boldsymbol{t})\, \boldsymbol{w}'(\boldsymbol{t};\, \boldsymbol{s}) = \nabla_{\boldsymbol{w}}^T \mathcal{J}_h(\boldsymbol{t},\, \boldsymbol{w})\, \boldsymbol{w}'(\boldsymbol{t};\, \boldsymbol{s}), \tag{47}$$

where we used (45). Making use of (44) and (47) we obtain the final form of the directional derivative:

$$\mathcal{J}_h'(\boldsymbol{t};\, \boldsymbol{s}) = \nabla_{\boldsymbol{w}}^T \mathcal{J}_h(\boldsymbol{t},\, \boldsymbol{w})\, \boldsymbol{w}'(\boldsymbol{t};\, \boldsymbol{s}) = -\boldsymbol{p}^T(\boldsymbol{t})\, \boldsymbol{K}'(\boldsymbol{t};\, \boldsymbol{s})\, \boldsymbol{w}(\boldsymbol{t}). \tag{48}$$

For more detailed treatment of the sensitivity analysis approach we refer to [4, 5].

5. Computational examples

In the *first example* we consider the beam of length $\mathrm{L} = 10$. The load function q is piecewise constant and given by

$$q(x) = \begin{cases} 100 & x, < 5, \\ 1\,000 & x \geq 5. \end{cases} \tag{49}$$

The parameters related to the material properties and the cross sectional area of the beam are defined as follows: $b = 0.2$, $E = 2.19\,e6$, $G = E/[2(1+\nu)]$, $\nu = 0.3$. The beam is not supported by a foundation, thus $k_F = 0$. Let the set $\mathcal{U}^h$ be defined by the following parameters: $T_0 = 0.5$, $T_1 = 1$, $T_2 = 7.5$, $T_3 = 0.2$ and let the initial guess be $t_i^0 = 0.75$ for $i = 0, \ldots, n$. We used 32 finite elements in discretization; i.e., $n = 32$ and $h = 10/32$. The following boundary conditions are prescribed: $w(0) = 0$, $\theta(0) = 0$ and $w(\mathrm{L}) = 0$. The beam is clamped at the left end and simply supported at the right end. No moments are applied.

The nonlinear mathematical programming problem (43) has been solved by the *sequential quadratic programming method* implemented in Matlab function *fmincon*. The state problems were solved by the Cholesky method.

In fig. 1 the optimal thickness of the beam is shown. We can compare the optimal shapes when both the Timoshenko model and the Euler-Bernoulli model for the state problem were used. We can also compare the deflection of the beam for both models. The results are shown in fig. 2. The cost functional values are summarized in tab. 1.

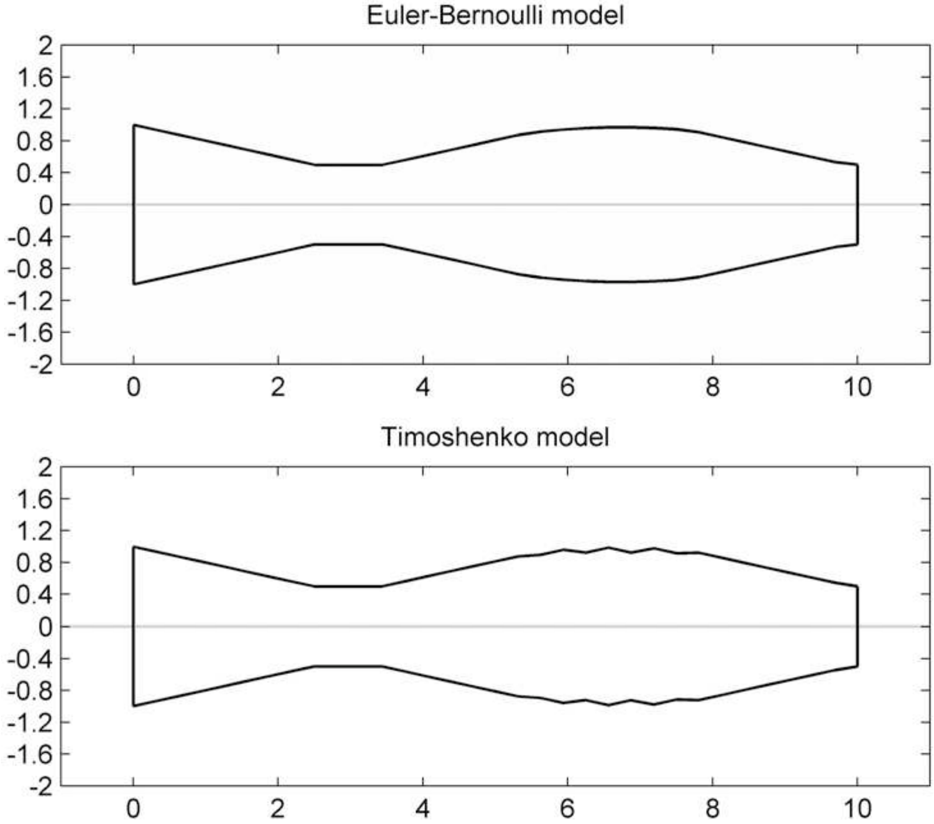

Fig. 1. An optimal thickness with respect to the compliance cost functional

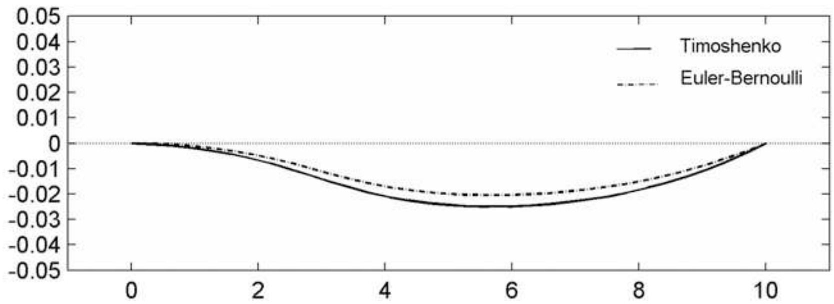

Fig. 2. A deflection of the optimal beam

Table 1. Cost functional values and number of iterations

Model	Initial	Final	Iter
Euler-Bernoulli	109.993 961 88	78.278 316 31	11
Timoshenko	125.507 121 96	96.189 756 66	16

The following abbreviations are used: *Model* = mathematical model used for the state problem, *Initial* = initial value of the cost functional, *Final* = final value of the cost functional, *Iter* = number of iterations.

In the *second example* we consider a beam of length $\mathrm{L} = 10$ that is supported by a foundation with the piecewise constant stiffness coefficient given by:

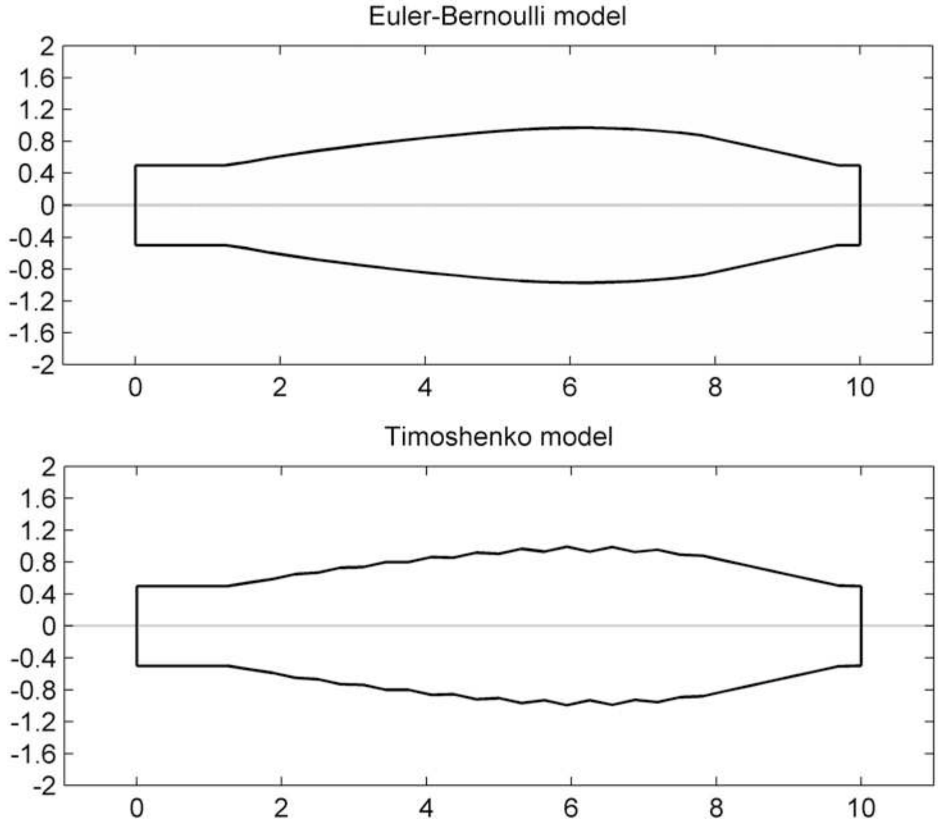

Fig. 3. A beam minimizing the compliance

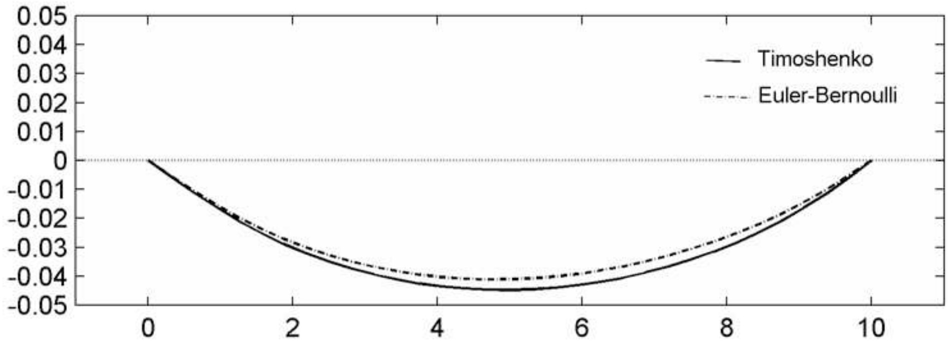

Fig. 4. A deflection of the beam with optimal thickness

$$k_F(x) = \begin{cases} 1\,000 & x < \frac{50}{16}, \\ 100 & x \geq \frac{50}{16}. \end{cases}$$

Let the beam be loaded by a piecewise constant load $q(x)$ given by (49). The parameters in the definition of $\mathcal{U}^h$ and the parameters related to the material and the cross section of the beam are the same as in the first example. We used 32 finite elements in discretization; i.e., $n = 32$ and $h = 10/32$. Boundary conditions defining a simply supported beam are prescribed as $w(0) = 0$ and $w(\mathrm{L}) = 0$. No moments are applied.

Optimal shapes and beam deflections reached for the second example are presented in fig. 3 and fig. 4. Decrease of the cost functional for both models is shown in tab. 2. From both

Table 2. Cost functional values and number of iterations

Model	Initial	Final	Iter
Euler-Bernoulli	209.187 411 08	150.600 022 27	18
Timoshenko	222.556 647 52	166.877 166 35	21

examples it follows that the Timoshenko model is more sensitive to changes of parameter T_3. In practice the constraint $|t'(x)| \leq T_3$ prevents thickness oscillation. If we drop this constraint or set $T_3 > 1$, the shape can oscillate wildly. For the Timoshenko model this phenomena starting to be apparent for $T_3 > 0.2$. Further, it can be seen that the optimal solutions for both models produce a significant decrease of the compliance in comparison with a reference design.

6. Conclusion

We presented the shape optimization problem of the transversally loaded elastic beam with a foundation of Winkler's type. The Timoshenko beam theory was used for modeling of the state problem. The variational formulation and the finite element approximation of the beam bending problem were demonstrated. The objective of the optimization was the thickness of the beam. The optimization problem was formulated as a minimization of the compliance cost functional over a set of admissible thicknesses. Several numerical experiments were done, where optimal shapes for a beam with and without foundation were shown and compared with results attained for the Euler-Bernoulli beam model. An apparent decrease of the compliance for optimal thickness in comparison to the reference design has been obtained in all examples. From these results it follows that the Timoshenko model is more sensitive to changes of the parameter T_3 defining the bounds of the thickness first derivative.

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