RATIONAL BIANGLE SURFACE PATCHES

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ABSTRACT

The concept of the rational biangle surface patch of the degree $2n$ is introduced. The construction is mostly close to (n, n) tensor product surface case because it has $(n+1)^2$ control points and the implicit degree $2n^2$ in general. The biangle has many similar properties: a convex hull property; boundary Bézier curves can be easy calculated; a subdivision and degree elevation algorithms are available. The quadratic biangle (when $n = 1$) is a patch on an oval quadric surface with four control points. In particular, one can realize any biangle with two cyclic arcs on the sphere.

Keywords: Rational surface patch, Quadric, Complex projective line.

1. INTRODUCTION

The most popular parametric surfaces in computer aided geometric design are two kinds of rational surfaces: tensor product surfaces and Bézier triangles. Though in theory every rational surface can be obtained using this technique, the actual realization often is complicated.

For example consider quadric surfaces. A tensor product surface of degree $(1, 1)$ with four noncomplanar control points gives us exactly a double ruled quadric. On the other hand till now there was no such simple realizations of oval quadrics including a sphere. Special Bézier triangles of degree 2 (resp. tensor product surfaces of degree $(2, 2)$) were used in the latter case [Dietz93]. In general they represent Steiner surfaces of implicit degree 4 [Seder85] (resp. degree 8 [Manch92]).

In this paper we fill this gap by introducing a four control point scheme for an oval quadrics (ellipsoid, elliptic paraboloid and two sheet hyperboloid). The scheme produces the patch rationally parameterized by some biangle region between two circular arcs in the plane of complex numbers **C**. Hence the boundary of the patch is formed by two Bézier curves (in fact, conics). We demonstrate main properties of this quadratic biangle and naturally generalize it to even degrees.

In Section 2 the definition of the quadratic biangle and its main properties are formulated. Section 3 explains how the idea of the biangle naturally follows from two sources: the "generalized stereographic projection" [Dietz93] and natural extensions of Bézier surfaces to projective domains [DeRos91]. Here we motivate the definition of the biangle and proof its properties formulated in Section 2. We extend the definition of the biangle patch with similar properties to higher degrees in Section 4. Some applications for a sphere are discussed in Section 5.

Remark. Throughout the paper the following notations will be used: Rez, Imz, and $\bar{z}$ denote a real, an imaginary part, and a complex conjugate of a complex number z, respectively. Also we underline homogeneous coordinates, i.e., $\underline{P} = (w, wP) \in \mathbf{R}^4$ is a homogeneous point corresponding to a point $P \in \mathbf{R}^3$ with a weight w.

2. QUADRATIC BIANGLE

2.1 Definition

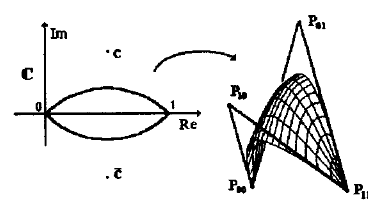

Fig. 1. The quadratic biangle patch.

Let D be a region in the plane of complex numbers **C** bounded by two circular arcs with the angle α between them (see Fig. 1)

$$(2.1)\quad D = \{z \in \mathbf{C} : |z - c|^2 \le |c|^2,\ |z - \bar{c}|^2 \le |c|^2\},\quad c = \frac{1}{2} + \frac{i}{2}\cot\frac{\alpha}{2}.$$

The *quadratic biangle* with the control net $P_{ij} \in \mathbf{R}^3$, weights w_{ij}, $i, j = 0, 1$, and the angle α is defined as a mapping $K: D \to \mathbf{R}^3$ given by a formula

$$K(z) = \frac{\sum_{i=0}^{1} \sum_{j=0}^{1} w_{ij} P_{ij} f_{ij}(z)}{\sum_{i=0}^{1} \sum_{j=0}^{1} w_{ij} f_{ij}(z)}, \tag{2.2}$$

where

$$\begin{aligned} f_{00}(z) &= |1 - z|^2, \; f_{01}(z) = |\bar{c}|^2 - |z - \bar{c}|^2, \\ f_{10}(z) &= |c|^2 - |z - c|^2, \; f_{11}(z) = |z|^2. \end{aligned} \tag{2.3}$$

The *patch* of the rational biangle is an image $K(D)$.

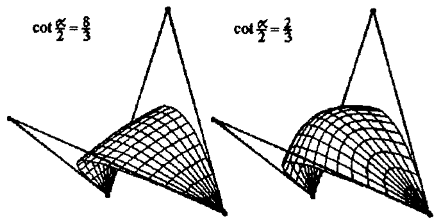

Fig. 2. The effect of changing α.

The angle α is useful for shape modeling: changing α we get all oval quadrics containing two given conics (Fig. 2). Nevertheless, for simplicity we fix the angle $\alpha = \pi/2$ (so $c = (1+\mathrm{i})/2$) in all our considerations bellow (except Section 5). One can generalize all formulas for an arbitrary $0 < \alpha < \pi$ almost straightforward.

2.2 Properties

The proofs of the following properties are postponed to Section 3.4.

(i) *Convex hull property.* The patch is contained in a tetrahedron with four vertices P_{ij} if all $w_{ij} > 0$.

(ii) *Boundary.* The patch has a boundary composed from two conic arcs which have corresponding three control points P_{00}, P_{01}, P_{11} and P_{00}, P_{10}, P_{11} but with a little bit different weights $\widetilde{w}_{ij}$

$$\widetilde{w}_{ij} = \begin{cases} w_{ij}, & \text{if } i = j \\ w_{ij}/\sqrt{2}, & \text{if } i \neq j. \end{cases} \tag{2.4}$$

(iii) *Subdivision.* The biangle K can be subdivided into one smaller biangle K' with control points

$$\underline{P}'_{ij} = (\underline{P}_{ij} + \sqrt{2}\underline{P}_{j,i+1} + \underline{P}_{ji})/(2 + \sqrt{2}),$$
$$i, j = 0, 1, \quad (i + 1 \text{ means } 0 \text{ if } i = 1)$$

and two quadratic Bézier triangles T^0 and T^1 with control points $\underline{T}^0_{ijk}$ and $\underline{T}^1_{ijk}$ $(i + j + k = 2)$

$$\underline{T}^0_{020} = \underline{P}_{00}, \; \underline{T}^0_{002} = \underline{P}'_{00}, \; \underline{T}^0_{200} = \underline{P}'_{11},$$
$$\underline{T}^0_{011} = (\underline{P}_{00} + \underline{P}_{01}/\sqrt{2})/2, \; \underline{T}^0_{101} = \underline{P}'_{10}/\sqrt{2},$$
$$\underline{T}^0_{110} = (\underline{P}_{00} + \underline{P}_{10}/\sqrt{2})/2.$$

The formulas for $\underline{T}^1_{ijk}$ are similar. The corresponding subdivision of the domain D is illustrated in Fig. 3.

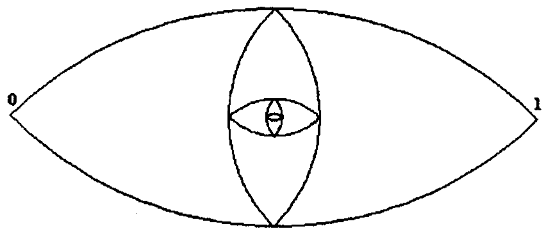

Fig. 3. Four steps of the domain subdivision.

Here left and right curved triangles are mapped to the quadratic Bézier triangles T^0, T^1 and the region between them is mapped to the biangle K'.

(iv) *Reparameterization.* Let the control points P_{ij} be fixed. The biangle patch (i.e., the image $K(D)$) does not changes if the weights are changed according to $\widehat{w}_{ij} = \lambda^{i+j} w_{ij}$, for arbitrary $\lambda > 0$.

(v) *Implicit equation.* The equation of the patch in barycentric coordinates associated with four control points is:

$$\frac{1}{w_{01}^2}\lambda_{01}^2 + \frac{1}{w_{10}^2}\lambda_{01}^2 - \frac{2}{w_{00}w_{11}}\lambda_{00}\lambda_{11} = 0. \tag{2.5}$$

The affine type of this quadric depends on the sign of $I = w_{01}^2 + w_{10}^2 - 2w_{00}w_{11}$:

(a) it is an ellipsoid if $I < 0$,

(b) it is an elliptic paraboloid if $I = 0$,

(c) it is two sheet hyperboloid if $I > 0$.

3. MOTIVATION

3.1 Complex projective line

The reader is referred to [Berge77] for fundamentals of projective geometry. A short introduction can be found in [Patte85].

Let us remind the construction of generalized stereographic projection from [Dietz93]. Let S be the sphere given by equation $x_0^2 = x_1^2 + x_2^2 + x_3^2$ in the real projective space $\mathbf{RP}^3$. The generalized stereographic projection $\delta: \mathbf{RP}^3 \to S \subset \mathbf{RP}^3$ is obtained from the following mapping

$$\widetilde{\delta}: \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \longmapsto \begin{pmatrix} p_0^2 + p_1^2 + p_2^2 + p_3^2 \\ 2p_0p_1 - 2p_2p_3 \\ 2p_1p_3 + 2p_0p_2 \\ p_1^2 + p_2^2 - p_0^2 - p_3^2 \end{pmatrix}, \tag{3.1}$$

using its homogeneity

$$\widetilde{\delta}(\lambda p_0, \lambda p_1, \lambda p_2, \lambda p_3) = \lambda^2 \widetilde{\delta}(p_0, p_1, p_2, p_3), \quad \lambda \in \mathbf{R}.$$

Let 4-dimensional real space $\mathbf{R}^4$ (the domain of $\tilde{\delta}$) be identified with 2-dimensional complex space $\mathbf{C}^2$ by introducing new variables $z_0 = p_2 + ip_1$ and $z_1 = p_0 + ip_3$. Then we can rewrite (3.1) in terms of complex variables

$$(3.2) \qquad \tilde{\psi}\colon \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \longmapsto \begin{pmatrix} |z_0|^2 + |z_1|^2 \\ 2\,\mathrm{Im}(z_0\bar{z}_1) \\ 2\,\mathrm{Re}(z_0\bar{z}_1) \\ |z_0|^2 - |z_1|^2 \end{pmatrix}.$$

Note that $\tilde{\psi}$ is *complex* homogeneous, since

$$\tilde{\psi}(\lambda z_0, \lambda z_1) = |\lambda|^2 \tilde{\psi}(z_0, z_1), \quad \text{for every } \lambda \in \mathbf{C}.$$

Hence the formula (3.2) induces the mapping defined on a complex projective line $\psi\colon \mathbf{CP}^1 \to S \subset \mathbf{RP}^3$. In fact the mapping ψ is an isomorphism in the sense of real algebraic geometry. It is also closely related to the classical "Hopf fibration" construction [Berge77].

3.2 Polynomials on $\mathbf{CP}^1$

Now we turn to an idea of natural extensions of Bézier surfaces to projective domains. In [DeRos91] it was proved that a tensor product (resp. triangular) surface as a mapping defined on an affine quadrangle (resp. triangle) can be homogenized and extended to a product of two projective lines $\mathbf{RP}^1 \times \mathbf{RP}^1$ (resp. projective plain $\mathbf{RP}^2$).

These two cases have one general scheme: there is some projective domain and some suitable class of polynomial mappings.

Consider the first case in more details. The tensor product surface of degree (n, n) is represented by polynomials which are homogeneous of degree n separately on the first pair and the second pair of variables and can be written in the following matrix form

$$(3.3) \qquad \mathbf{B}(u_0, u_1) \cdot \mathbf{P} \cdot \mathbf{B}(v_0, v_1)^T,$$

where $\mathbf{B}^n = (B_0^n, \dots, B_n^n)$ is a row of homogeneous Bernstein polynomials $B_i^n(u_0, u_1) = \binom{n}{i} u_0^{n-i} u_1^i$ and $\mathbf{P} = (\underline{P}_{ij})$ is a square matrix with entries from $\mathbf{R}^4$.

In particular when $n = 1$ we have

$$\begin{pmatrix} u_0 & u_1 \end{pmatrix} \begin{pmatrix} \underline{P}_{00} & \underline{P}_{01} \\ \underline{P}_{10} & \underline{P}_{11} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

On the other hand the formula of the generalized stereographic projection in our complex interpretation (3.2) can be expressed in the analogous form

$$(3.4) \qquad \begin{pmatrix} z_0 & z_1 \end{pmatrix} \begin{pmatrix} \underline{Q}_{00} & \underline{Q}_{01} \\ \underline{Q}_{10} & \underline{Q}_{11} \end{pmatrix} \begin{pmatrix} \bar{z}_0 \\ \bar{z}_1 \end{pmatrix}.$$

Indeed just put

$$\underline{Q}_{00} = (1, 0, 0, 1)^T, \quad \underline{Q}_{01} = (0, -i, 1, 1)^T,$$
$$\underline{Q}_{10} = (0, i, 1, 0)^T, \quad \underline{Q}_{11} = (1, 0, 0, -1)^T.$$

Note that our matrix $(\underline{Q}_{ij})$ is Hermitian symmetric not accidentally. This is a condition for the polynomial to have real values.

The obvious generalization of for arbitrary n is defined by the the formula similar to (3.3)

$$(3.5) \qquad \mathbf{B}(z_0, z_1) \cdot \mathbf{Q} \cdot \mathbf{B}(\bar{z}_0, \bar{z}_1)^T,$$

where $\mathbf{Q} = (\underline{Q}_{ij})$ is a Hermitian symmetric matrix with entries from $\mathbf{C}^4$. Hence we have obtained a class of polynomial mappings defined on a complex projective line $\mathbf{CP}^1$.

3.3 Justification of the biangle

We are going to employ the result of [DeRos91] towards the inverse direction. Consider the class of polynomial mappings (3.5). Our program is to restrict the domain $\mathbf{CP}^1$ of these mappings to a suitable affine domain and obtain a surface construction à la Bernstein–Bézier .

There are two main attributes of such constructions: some affine domain and some distinguished polynomial basis defined on it which satisfies two conditions:

(a) the sum of basic polynomials is equal to 1;

(b) basic polynomials are non-negative in the domain.

Consider the case $n = 1$ at first. Then our class of functions coincide with symmetric Hermitian forms on two variables

$$(3.6) \qquad \sum_{i=0}^{1} \sum_{j=0}^{1} q_{ij} z_i \bar{z}_j, \quad q_{ij} = \bar{q}_{ji} \in \mathbf{C}.$$

Fix an affine complex line $z_0 + z_1 = 1$ in $\mathbf{CP}^1$. Now at a glance $\{z_i \bar{z}_j : i, j = 0, 1\}$ is a good candidate for the base, since

$$1 = |z_0 + z_1|^2 = z_0\bar{z}_0 + z_0\bar{z}_1 + z_1\bar{z}_0 + z_1\bar{z}_1.$$

Note that $z_0\bar{z}_1$, $z_1\bar{z}_0$ in fact do not belong to our class (3.4). Hence we choose the other basis $f_{ij}(z_0, z_1)$

$$(3.7) \qquad \begin{aligned} f_{00} &= z_0\bar{z}_0, \; f_{01} = \bar{c} z_0\bar{z}_1 + c z_1\bar{z}_0, \\ f_{10} &= c z_0\bar{z}_1 + \bar{c} z_1\bar{z}_0, \; f_{11} = z_1\bar{z}_1. \end{aligned}$$

From the condition (a) follows $c + \bar{c} = 1$ and $c = (1 + ia)/2$ with some $a \in \mathbf{R}$. The formula (3.4) defines some mapping $\underline{K}\colon \mathbf{C}^2 \to \mathbf{R}^4$. It has the following form in the new basis

$$(3.8) \qquad \underline{K}(z_0, z_1) = \sum_{i=0}^{1} \sum_{j=0}^{1} \underline{P}_{ij} f_{ij}(z_0, z_1).$$

Now the justification of the biangle definition (Sec. 2) easy follows. Indeed, fix an affine part of the complex projective line $\mathbf{CP}^1$ defined by $z_0 + z_1 = 1$ and introduce coordinates $z = z_1$. Then $z_0 = 1 - z$ and we can write our basis (3.7) and the mapping (3.8) in an affine form (2.3) and (2.2), respectively. The condition (b) is equivalent to inequalities $f_{ij}(z) \geq 0$ which define exactly the region D in $\mathbf{C}$ (see (2.1)). It is clear from Fig. 1 that D is a biangle region bounded by two circles with centers c and $\bar{c}$. Hence, $a = \cot(\alpha/2)$.

3.4 Proof of the properties from Section 2.2

(i) The convex hull property directly follows from inequalities $f_{ij} \geq 0$, for all $i, j = 0, 1$, $z \in D$.

(ii) Here it is convenient to use a homogeneous variant of the biangle

$$\underline{K}(z_0, z_1) = \sum_{i=0}^{1}\sum_{j=0}^{1} \underline{P}_{ij} f_{ij}(z_0, z_1), \tag{3.9}$$

where $\underline{P}_{ij} = (w_{ij}, w_{ij}P_{ij})$. Consider a line segment in $\mathbf{C}^2$ between two points $(1,0)$ and $(0,d)$ (with a real parameter t), i.e., $z_0 = 1 - t$, $z_1 = dt$. After substituting these expressions into (3.9) we have

$$\underline{K}(1-t, \mu t) = \underline{P}_{00}(1-t)^2 + (cd + \bar{c}\bar{d})\underline{P}_{01}(1-t)t + (c\bar{d} + \bar{c}d)\underline{P}_{10}(1-t)t + |d|^2\underline{P}_{11}t^2.$$

If we choose $d = \mathrm{i}\bar{c}/|c|$ then it is a boundary Bézier curve with control points P_{00}, P_{10}, P_{11} and weights w_{00}, kw_{10}, w_{11}, where $k = (c\bar{d} + \bar{c}d)/2 = 1/\sqrt{2}$.

(iii) We define the smaller biangle K' as a composition of K and an appropriate parameter domain transformation which moves D to D' (see Fig. 3). In fact this is the unique linear-fractional transformation which takes points 0, 1/2, 1 to points $1/2 + \mathrm{i}(\sqrt{2} - 1)/2$, $1/2$, $1/2 - \mathrm{i}(\sqrt{2} - 1)/2$.

(iv) Here we use the unique linear-fractional transformation which takes points 0, 1/2, 1 to points 0, $\lambda/(1+\lambda)$, 1 (cf. [Patte85]).

4. BIANGLE OF DEGREE $2n$

4.1 Basic polynomials

Here we are going to generalize the definition of the biangle to higher even degrees $2n$. In fact we already have the general formula (3.5) with complex control points. So we need to choose such a basis f^n_{ij}, $i, j = 0, \ldots, n$, for that class of polynomials of degree $2n$ that

$$\sum_{i=0}^{n}\sum_{j=0}^{n} f^n_{ij}(z) = 1, \quad f^n_{ij}(z) \geq 0, \quad \text{if } z \in D. \tag{4.1}$$

The idea is to use homogeneous polynomials with positive coefficients k^n_{ij} of the basis f_{00}, f_{01}, f_{10}, f_{11}

$$f^n_{ij} = \begin{cases} k^n_{ij}(f_{00})^{n-i-j}(f_{01})^j(f_{10})^i, & i+j \leq n, \\ k^n_{ij}(f_{11})^{i+j-n}(f_{01})^{n-i}(f_{10})^{n-j}, & i+j \geq n. \end{cases}$$

The coefficients k^n_{ij}, $i, j = 0, \ldots, n$, are defined recurrently. For $n = 1$ let $k^1_{00} = k^1_{00} = k^1_{00} = k^1_{00} = 1$. When $n > 1$ define

$$\begin{aligned} k^n_{ij} &= k^{n-1}_{ij} + k^{n-1}_{i-1,j} + k^{n-1}_{i,j-1} \\ &\quad + (k^{n-1}_{i-2,j} + k^{n-1}_{i,j-2})/2, \quad i+j \leq n, \\ k^n_{ij} &= k^{n-1}_{i-1,j} + k^{n-1}_{i,j-1} + k^{n-1}_{i-2,j} + k^{n-1}_{i,j-2}, \quad i+j = n, \\ k^n_{ij} &= k^n_{n-j,n-i}, \quad i+j > n. \end{aligned}$$

Here we suppose $k^n_{ij} = 0$ if i or $j \notin \{0, \ldots, n\}$. Note that $f^1_{ij} = f_{ij}$ when $n = 1$. In fact these recurrent expressions are calculated by expanding the left side of an identity

$$(f_{00} + f_{01} + f_{10} + f_{11})\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f^{n-1}_{ij} = 1 \tag{4.2}$$

and substituting $f_{00}f_{11} = (f^2_{01} + f^2_{10})/2$.

Finally it follows from the whole construction of the basic polynomials f^n_{ij}, that conditions (4.1) are satisfied.

Remark. As a reflection of 1-dimensional complex nature of the biangle construction we have the following relation

$$\sum_{i+j=s} k^n_{ij} = \binom{2n}{s}.$$

4.2 Definition

Let the region $D \subset \mathbf{C}$ in the plane of complex numbers is defined by (2.1).

The *rational biangle* of degree $2n$ with the control net $P_{ij} \in \mathbf{R}^3$, $i, j = 0, \ldots, n$, and weights w_{ij}, $i, j = 0, \ldots, n$, is a mapping $K\colon D \to \mathbf{R}^3$ given by a formula (see Fig. 4)

$$K(z) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} w_{ij}P_{ij}f^n_{ij}(z)}{\sum_{i=0}^{n}\sum_{j=0}^{n} w_{ij}f^n_{ij}(z)}. \tag{4.3}$$

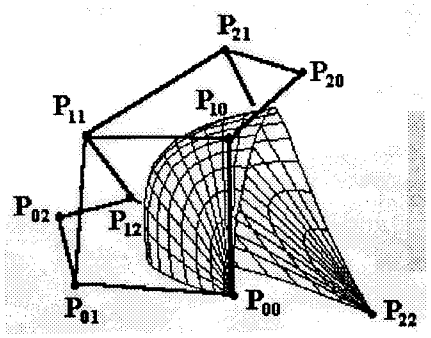

Fig. 4. The biangle patch of the degree 4.

4.2 Properties of the biangle of degree $2n$

(i) *Convex hull property.* The patch is contained in a convex hull of the control points P_{ij}, $i, j = 0, \ldots, n$, if all weights w_{ij} are positive.

(ii) *Boundary.* The patch has a boundary composed from two rational Bézier curves of degree $2n$ with control points coinciding with boundary points of the control net

$$P_{00}, P_{01}, \ldots, P_{0n}, P_{1n}, P_{2n}, \ldots, P_{nn} \text{ and}$$
$$P_{00}, P_{10}, \ldots, P_{n0}, P_{n1}, P_{n2}, \ldots, P_{nn}$$

but with a little bit different weights $\widetilde{w}_{ij}$

$$\widetilde{w}_{ij} = \frac{(\sqrt{2})^{|n-|n-i-j||} k_{ij}}{\binom{2n}{i+j}} w_{ij}.$$

(iii) *Subdivision.* The biangle of degree $2n$ can be subdivided into one smaller biangle and two Bézier triangles of the same degree $2n$ using the subdivision of the domain D (see Fig. 3) as in the quadratic case. Here we skip explicit formulas.

(iv) *Reparameterization.* The the patch will be the same if the weights are changed according to $\widehat{w}_{ij} = \lambda^{i+j} w_{ij}$ for arbitrary $\lambda > 0$.

(v) *Implicit degree.* Here we prove that implicit degree does not exceed $2n^2$. At first note that the parameterization of the biangle (4.3) is of type (3.5) (just it is written in a different basis). Then we can treat complex variables $z_j = x_j + \mathrm{i}y_j \in \mathbf{C}$ as pairs of real variables $(x_j, y_j) \in \mathbf{R}^2$, $j = 0, 1$. After the following complex substitution

$$x_j = \frac{1}{2}(u_j + v_j), \; y_j = \frac{\mathrm{i}}{2}(u_j - v_j), \quad j = 0, 1,$$

we get an expression of type (3.3). Therefore, the biangle of degree $2n$ is equivalent to some *complex* tensor product surface of bidegree (n, n) and has the same implicit degree $2n^2$ if control points are in general position [Manch92]. The implicit degree may be less than $2n^2$ if the patch is degree elevated or there are additional base points. For instance, there are some quartic biangles on a spindle and 2-horn cyclides (this follows from special parameterizations of spindle torus [Kras97]) which have implicit degree 4.

5. SPHERICAL PATCHES

Necessary and sufficient conditions for a quadratic biangle patch with an arbitrary angle $0 < \alpha < \pi/2$ to be a sphere are: $\angle P_{01}P_{00}P_{10} = \alpha$ (i.e., α is a "true" angle), $|P_{00}P_{ij}| = |P_{11}P_{ij}|$ and

$$\frac{w_{ij}}{\sqrt{w_{00}w_{11}}} = \frac{\cos \angle P_{11}P_{00}P_{ij}}{\cos(\alpha/2)},$$

where $(i, j) = (0, 1), (1, 0)$.

We will sketch the proof of the sufficiency as follows. Substitute a real argument $t = z$ in the basic functions f_{ij} and get $f_{00} = (1-t)^2$, $f_{01} = f_{10} = (1-t)t$, $f_{11} = t^2$. This means that a "middle conic" corresponding to real values of z has control points $P_0 = P_{00}$, $P_1 = (w_{01}P_{01} + w_{10}P_{10})/(w_{01} + w_{10})$, $P_2 = P_{11}$ and weights $w_0 = w_{00}$, $w_1 = (w_{01} + w_{10})/2$, $w_2 = w_{11}$. Then it is easy to check that it is a circular arc. Hence we get three circles on the quadric going through two points. Therefore it is a sphere.

Consider the octant of a sphere. It can be realized as the special quartic Bézier triangle [Farin88] (see Fig. 5).

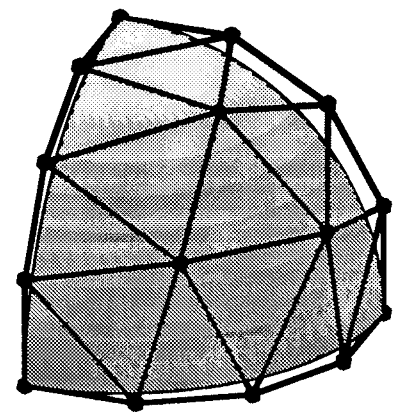

Fig. 5. The quartic Bézier triangle.

Using our approach we combine the octant from two pieces: the quadratic Bézier triangle and the biangle with $\alpha = \pi/4$ (Fig. 6).

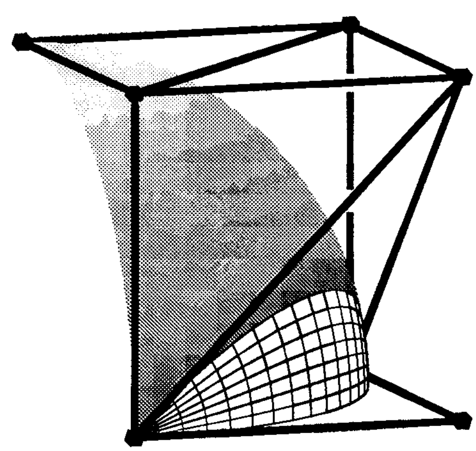

Fig. 6. The quadratic triangle and the biangle.

Notice that we use 7 control points in the new construction instead of 15 in the old one.

6. CONCLUSIONS

We have introduced the new kind of rational surface constructions—the biangle patch of degree $2n$. It is very similar to a well-known tensor product scheme of bidegree (n, n). We focused on the quadratic case when the biangle represents oval quadric surfaces.

However, it can be expected that higher degree cases are also interesting. For example, the biangle can be naturally extended to a projective domain $\mathbf{CP}^1$ which has a spherical topology. Hence, it is a different topology type in comparison with tensor product surfaces (resp. Bézier triangles) having a torus topology domain $\mathbf{RP}^1 \times \mathbf{RP}^1$ (resp. non-oriented domain $\mathbf{RP}^2$) [DeRos91].

Also it would be interesting to investigate whether our construction is useful for filling holes. At least in the quadratic case we control a shape of the patch with control points and weights more efficiently than using implicit equations (cf. [Holst87]).

A variety of convex closed surfaces can be constructed joining together only biangle patches. A simple example is shown in Fig. 7, where four quadratic biangles are used.

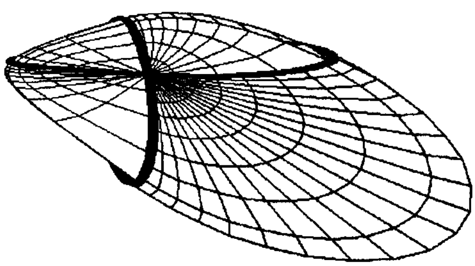

Fig. 7. The composition of 4 biangle patches.

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