Projection in barycentric coordinates

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At the beginning of this paper some elementary properties of the simplices, barycentric coordinates, convex polyhedra, affine ratio of three points and Euler's theorem are mentioned. The reader who wishes to know more about the geometrical background is reffered to [1]. A central projection in n-dimensional Euclidean space is modelled with the help of barycentric coordinates, the center of the projection is a vertex of the simplex, the opposite hyper-side is the hyper-plane of the projection (hyper-screen). The direction of the parallel projection is chosen as the direction of one edge of the simplex.

A simplex on a straight line has 2 vertices, then $v = 2$. If we join segments (without overlapping) we get again a segment with two vertices, $v = 2$.

A simplex in a plane has 3 vertices and 3 sides: $v - s = 0$. If we join the triangles in a two-dimensional space (without overlapping) we get a polygon with n vertices and n sides: $v - s = 0$.

A simplex in a three-dimensional space has 4 vertices, 6 edges and 4 faces: $v - e + f = 4 - 6 + 4 = 2$ If we join tetrahedra in a three-dimensional space (without overlapping) we get *a simply connected polyhedron* - it can be homeomorphicly mapped onto a ball. Convex polyhedra belong to such polyhedra. Two polyhedra joined along two identical k-polygons (faces) lose two faces, k vertices and k edges. We get again

$$v - e + f = v_1 - e_1 + f_1 + v_2 - e_2 + f_2 - k + k - 2 = 2\,,$$

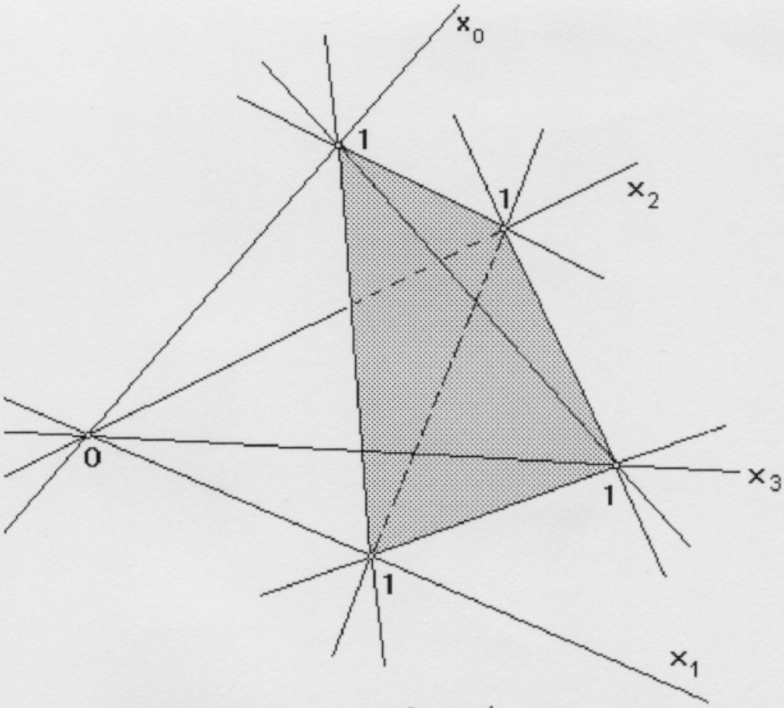
the so called *Euler's relation.*

To complete the proof we shall add the case of smoothing onto one face, where is in fact the previous plane case, and we shall deal also the case of a plane section adding m vertices and j edges to each of the both parts of the polyhedron. These added elements will be deleted after joining .

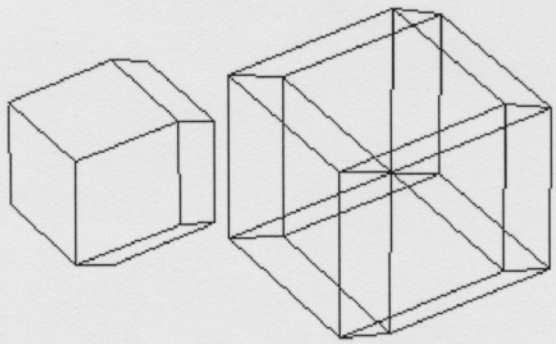
In a four-dimensional space we have analogously $v - e + s - p = 0$, where p is the number of boundary polytopes.

In an n-dimensional space we have: $v - e + s - p + \cdots = 1 - (-1)^n$.

This Euler's formula (a cradle of algebraic topology) is valid for simply connected polytopes.

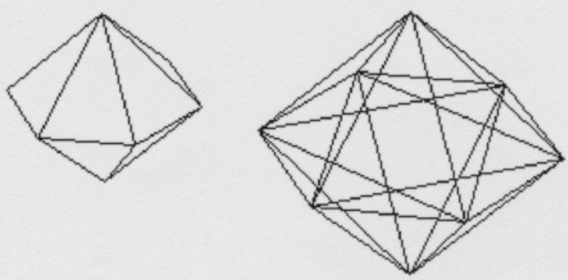


Barycentric coordinates in $E^3 \subset E^4$



Hypercube in E^4
(rhombic dodekahedron)

16 vertices
32 edges
24 squares
8 cubes
Schläfli's symbol (4,3,3)



Cocube in E^4
(two pyramids)

8 vertices
24 edges
32 triangles
16 tetrahedra
Schläfli's symbol (3,3,4)

In the space $\mathbf{E}^n$ points $P_0, P_1, \cdots, P_n$ form vertices of a simplex if and only if the vectors $(P_1 - P_0), (P_2 - P_0), \cdots, (P_n - P_0)$ are linearly independent. If the points $\{P_0, P_1, \cdots, P_n\}$ form the vertices of a simplex in $\mathbf{E}^n$ then we have for every point $L \in \mathbf{E}^n$

$$L = P_0 + \lambda_1(P_1 - P_0) + \lambda_2(P_2 - P_0) + \cdots + \lambda_n(P_n - P_0).$$

$$L = \sum_{i=0}^{n} a_i P_i, \qquad a_0 = 1 - \sum_{i=1}^{n} \lambda_i, \qquad a_j = \lambda_j,\ j = 1, \cdots, n, \qquad \Rightarrow \sum_{i=0}^{n} a_i = 1.$$

The numbers $[a_0, a_1, a_2, \cdots, a_n]$ are called *barycentric coordinates* of the point L with respect to the simplex $\{P_0, P_1, \cdots, P_n\}$.

In $\mathbf{E}^n$ a one-to-one correspondence exists between the points of the given simplex $\{P_0, P_1, \cdots, P_n\}$ and their barycentric coordinates $[a_0, a_1, a_2, \cdots, a_n]$ (with the condition that their sum equals one).

Let the vectors $(P_1 - P_0), (P_2 - P_0), \cdots, (P_n - P_0)$ be linearly independent. Thus for arbitrarily chosen j $(j = 1, \cdots, n)$ the vectors $(P_i - P_j)$, $i = 0, \cdots n, \imath \neq j$ are also linearly independent. For, if $\sum_{i=0, i\neq j}^{n} a_i(P_i - P_j) = \mathbf{0}$ and

$$\mathbf{0} = \sum_{i=0, i\neq j}^{n} a_i\,(P_i - P_0) \; - \; (P_j - P_0) \sum_{i=0, i\neq j}^{n} a_i\,, \text{ then } a_k = 0 \; (\forall k = 0, 1, \cdots, n).$$

Hence we can define *linearly independent points* with the help of linearly independent vectors $P_1 - P_0, P_2 - P_0, \cdots, P_n - P_0$. If the points $\{P_0, P_1, P_2, \cdots, P_n\}$ were linearly dependent, to each point L could belong many (n+1)-tuples $a_0, a_1, a_2, \cdots, a_n$. It is not convenient to call such groups of numbers the coordinates (barycentric).

The *affine ratio* $d(A; P_0, P_1)$ of the point $A = a_0 P_0 + a_1 P_1$, $A \not\equiv P_1$ with respect to the points P_0, P_1 is the ratio $-a_1/a_0$, where a_0, a_1 are the barycentric coordinates in the simplex $\{P_0, P_1\}$. The affine ratio is a real number not equal to one. Passing to projective extension of the straight line, we see that the improper point A_∞ of the straight line $P_0 P_1$ has the affine ratio equal to one.

In a metric space the barycentric coordinate a_0 of the point A with respect to the simplex $\{P_0, \cdots, P_n\}$ equals the ratio of the oriented distance of the point A and the hyperplane $\{P_1, \cdots, P_n\}$ and the hight (positive) of the simplex between P_0 and the opposite hyper-face $\{P_1, \cdots, P_n\}$. In the same way we visualize the barycentric coordinate a_i.

Let $S^{n+1} = \{P_0, P_1, P_2, \cdots, P_n\}$ in $\mathbf{E}^m$, $m \geq n$ be a simplex and $A \in \mathbf{E}^m$ the point. Let us seek the barycentric coordinates of the point A with respect to the simplex S^{n+1}.

$$A = \sum_{i=1}^{i=m} \mathrm{a}_i \mathbf{e}_i, \qquad A = \sum_{i=0}^{n} a_i P_i, \qquad \sum_{i=0}^{n} a_i = 1.$$

Since the vectors $(P_i - P_0)$, $i = 0, \cdots, n$ are linearly independent, it is possible to choose such n coordinates (suppose that n first ones) from m coordinates that the matrix

$$P = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \mathrm{p}_{01} & \mathrm{p}_{11} & \cdots & \mathrm{p}_{n1} \\ \mathrm{p}_{02} & \mathrm{p}_{12} & \cdots & \mathrm{p}_{n2} \\ \dots & \dots & \ddots & \dots \\ \mathrm{p}_{0n} & \mathrm{p}_{1n} & \cdots & \mathrm{p}_{nn} \end{pmatrix} \quad \text{is non-singular}.$$

For the given Cartesian coordinates $[\mathrm{a}_1, \cdots, \mathrm{a}_n]$ the barycentric coordinates to be found are the unique solution of the following system of linear equations

$$P \cdot (a_0, a_1, a_2, \cdots, a_n)^T = (1, \mathrm{a}_1, \mathrm{a}_2, \cdots, \mathrm{a}_n)^T, \quad \vec{a} = P^{-1} \cdot \vec{\mathrm{a}}.$$

The matrix row of all units is written on the first place, but this is not necessary.

In the space $\mathbf{E}^3$ the revolution of tetrahedron $\{P_0, P_1, P_2, P_3\}$ around the axis passing through the origin (the old axis z receives the geographic-spherical coordinates (ϕ, ψ)) and the following translation by vector $\vec{q}$ ($\vec{q} = (q_1, q_2, q_3)$) can be expressed using matrices

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathrm{p}^*{}_{01} & \mathrm{p}^*{}_{11} & \mathrm{p}^*{}_{21} & \mathrm{p}^*{}_{31} \\ \mathrm{p}^*{}_{02} & \mathrm{p}^*{}_{12} & \mathrm{p}^*{}_{22} & \mathrm{p}^*{}_{32} \\ \mathrm{p}^*{}_{03} & \mathrm{p}^*{}_{13} & \mathrm{p}^*{}_{23} & \mathrm{p}^*{}_{33} \end{pmatrix} = T(q) \cdot R(\phi, \psi) \cdot P =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ q_1 & 1 & 0 & 0 \\ q_2 & 0 & 1 & 0 \\ q_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi\sin\psi & \cos\phi\cos\psi \\ 0 & \cos\phi & -\sin\phi\sin\phi & \sin\phi\cos\psi \\ 0 & 0 & \cos\psi & \sin\psi \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathrm{p}_{01} & \mathrm{p}_{11} & \mathrm{p}_{21} & \mathrm{p}_{31} \\ \mathrm{p}_{02} & \mathrm{p}_{12} & \mathrm{p}_{22} & \mathrm{p}_{32} \\ \mathrm{p}_{03} & \mathrm{p}_{13} & \mathrm{p}_{23} & \mathrm{p}_{33} \end{pmatrix},$$

$$(T(q) \cdot R(\phi, \psi) \cdot P)^{-1} = P^{-1} \cdot R(\phi, \psi)^T \cdot T(-q).$$

In other words: instead of moving the projecting simplex, we can first translate the object conversely and then revolve it conversely - in fact this is a trivial conclusion.

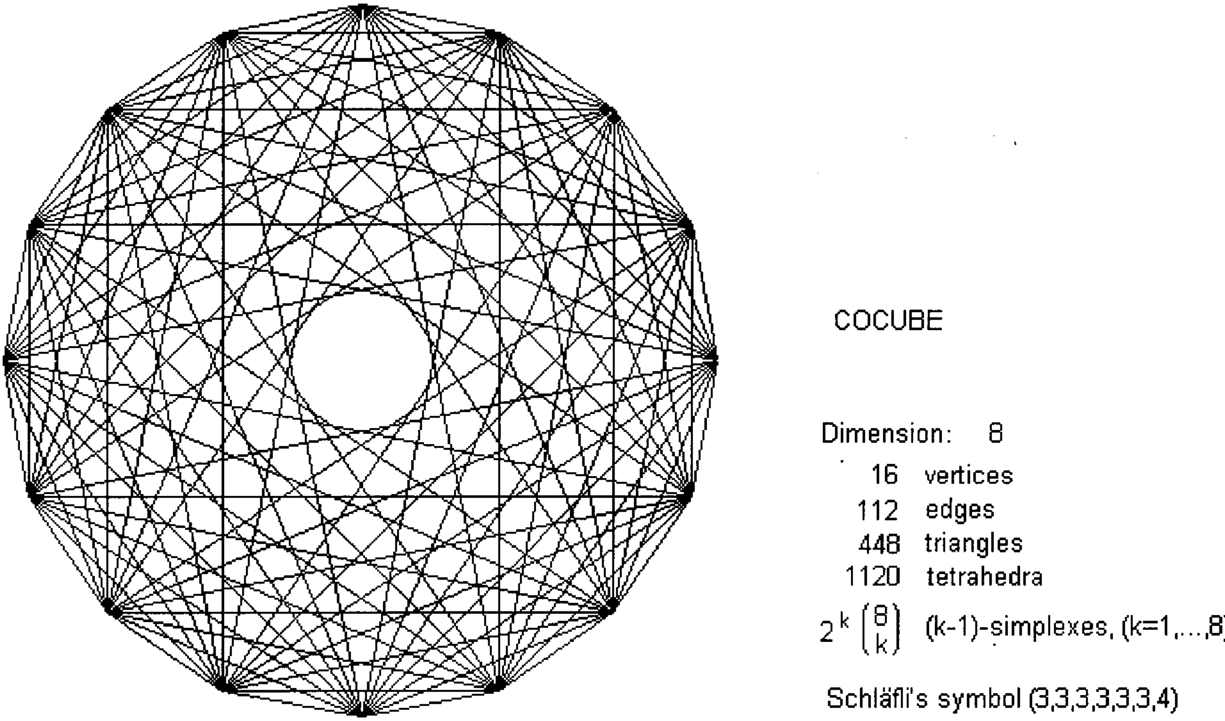
In the projective extension of the space $\mathbf{E}^n \to \mathbf{P}^n$ we shall add the 0-th coordinate instead of the usual (n+1)-th coordinate. Let us consider the point of $\mathbf{P}^n$ lying in the hyperplane $\rho: -x_0 + x_1 + x_2 + \cdots + x_n = 0$. The hyperplane ρ passes through n points

$$(1,1,0,\cdots,0),\ (1,0,1,\cdots,0),\ \cdots,\ (1,0,0,\cdots,1).$$

If $x_0 \neq 0$ the homogeneous coordinates of our point divided by the number x_0, can be considered as barycentric coordinates of the point X, $\vec{\mathrm{x}} = \vec{OX}$, with respect to the (n-1)-dimensional simplex

$$\{(1,0,\cdots,0),\ (0,1,\cdots,0),\cdots,\ (0,0,\cdots,1)\}$$

in the hyperplane $\mathrm{x}_1 + \mathrm{x}_2 + \cdots + \mathrm{x}_n = 1$. This hyperplane naturally belongs to the space $\mathbf{E}^n$.



In the Euclidean plane $\mathbf{E}^2$ we have

$$\begin{pmatrix} 1 \\ \mathrm{a}_1 \\ \mathrm{a}_2 \end{pmatrix} = P \cdot \vec{\mathrm{a}} = \begin{pmatrix} 1 & 1 & 1 \\ \mathrm{p}_{01} & \mathrm{p}_{11} & \mathrm{p}_{21} \\ \mathrm{p}_{02} & \mathrm{p}_{12} & \mathrm{p}_{22} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix},$$

$$\det(P) = (\mathrm{p}_{11} - \mathrm{p}_{01})(\mathrm{p}_{22} - \mathrm{p}_{02}) - (\mathrm{p}_{21} - \mathrm{p}_{01})(\mathrm{p}_{12} - \mathrm{p}_{02}),$$

$$P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} \mathrm{p}_{11}\mathrm{p}_{22} - \mathrm{p}_{12}\mathrm{p}_{21} & \mathrm{p}_{12} - \mathrm{p}_{22} & \mathrm{p}_{21} - \mathrm{p}_{11} \\ \mathrm{p}_{02}\mathrm{p}_{21} - \mathrm{p}_{01}\mathrm{p}_{22} & \mathrm{p}_{22} - \mathrm{p}_{02} & \mathrm{p}_{01} - \mathrm{p}_{21} \\ \mathrm{p}_{01}\mathrm{p}_{12} - \mathrm{p}_{02}\mathrm{p}_{11} & \mathrm{p}_{02} - \mathrm{p}_{12} & \mathrm{p}_{11} - \mathrm{p}_{01} \end{pmatrix}.$$

We can write the *central projection* from the vertex P_0 onto the hyperplane $\{P_1, \cdots, P_n\}$ in the barycentric coordinates:

$$A^* = \sum_{j=1}^{n} a_j^* P_j\,, \quad a_j^* = \frac{a_j}{\sum\limits_{k=1}^{n} a_k} = \frac{a_j}{1 - a_0}, \quad j = 1, \cdots, n\,.$$

It is easy to verify that $\sum\limits_{i=1}^{n} a_i^* = 1, \quad (1 - a_0)A^* + a_0 P_0 = A.$

Points of a hyperplane passing through the center of projection parallel to the plane of projection, i.e. the coordinate a_0 of which is equal to one have no image.

In the linear perspective: $a_j > 0,\ j = 1, \cdots, n,\ -\infty < a_0 < 1,\ a_0 \nearrow 1\ \Rightarrow A$ will be visible.

The inner points of the simplex are the points between the centre of projection and the plane of projection. They are useful not only for the estimate of the visual angle, but also for visibility.

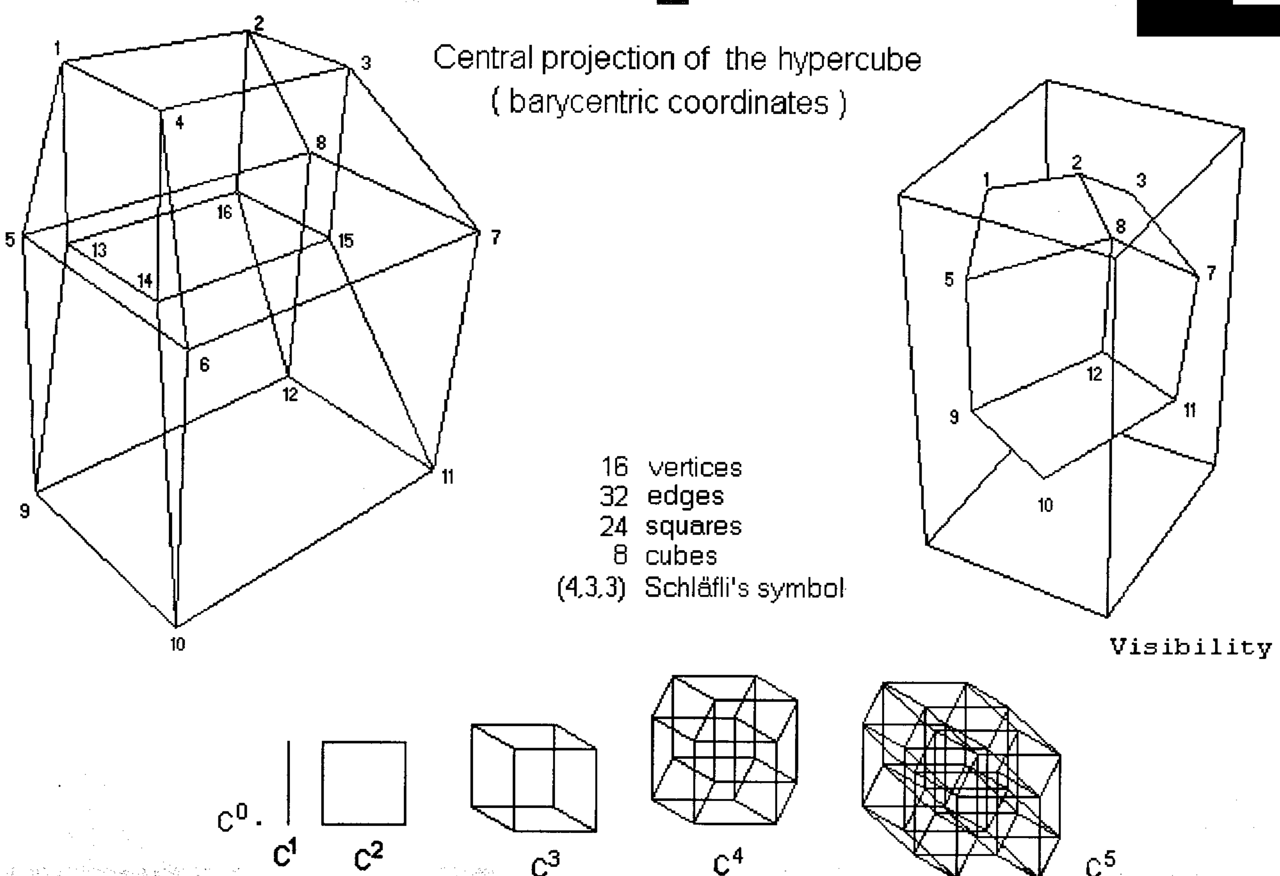
In photographing (onto a hyperplane $a_0 = 0$) the pre-image points fulfil: $a_j < 0,\ j = 1, \cdots, n,\ 1 < a_0 < +\infty,\ a_0 \searrow 1\ \Rightarrow A$ will be visible. The points of the space with the 0-th coordinate greater than one and other barycentric coordinates negative, lie in a projective pyramid outside the plane of projection. We can again estimate the visual angle and decide about visibility.

In the Euclidean space $\mathbf{E}^3$ we denote: $O \equiv \mathbf{0},\ P_0 \equiv S \equiv O + d \cdot \mathbf{e}_3,\ P_3 \equiv O \equiv H,\ P_1 \equiv O + \mathbf{e}_1,\ P_2 \equiv O + \mathbf{e}_2$. In the orthonormal basis we have

$$\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3\}, \quad A = H + a_1(P_1 - H) + a_2(P_2 - H) + \frac{a_3}{d}(P_0 - H), \quad a_0 = \frac{a_3}{d},$$

$$a_1^* = \frac{d \cdot a_1}{d - a_3}, \quad a_2^* = \frac{d \cdot a_2}{d - a_3}.$$

This is in accordance with usual expression of the central projection through homogeneous coordinates α_i



$$\begin{pmatrix} \alpha_1^+ \\ \alpha_2^+ \\ 0 \\ \alpha_4^+ \end{pmatrix} = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & d \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{pmatrix} .$$

If we project the point A from the vertex P_k of the simplex $\{P_0, P_1, \cdots, P_n\}$ onto the hyperplane determined by the simplex $\{P_0, P_1, \cdots, P_{k-1}, P_{k+1}, \cdots, P_n\}$, the barycentric coordinates $\{a_0^*, a_1^*, \cdots, a_{k-1}^*, a_{k+1}^*, \cdots, a_n^*\}$ of the image A^* will satisfy

$$a_j^* = \frac{a_j}{1 - a_k}, \quad j = 0, 1, \cdots, k-1,\ k+1, \cdots, n .$$

We will choose a chain of the central projections, such that the dimension of space diminishes by one and in practice we will end by the projection onto the plane.

The *parallel projection* $A \to A^*$ in the direction $\overrightarrow{P_0 P_n}$ onto the hyperplane $\{P_1, P_2, \cdots, P_n\}$ (we assume the simplex $\{P_0, P_1, P_2, \cdots, P_n\}$) is written in barycentric coordinates in the following way:

$$A = [a_0, a_1, a_2, \cdot, a_n],\ A^* = [a_1^*, a_2^*, \cdots, a_n^*], \quad a_1^* = a_1,\ a_2^* = a_2,\ a_{n-1}^* = a_{n-1},\ a_n^* = a_n + a_0 .$$

We have again $\sum_{i=1}^{n} a_i^* = 1, \quad A^* - A = a_0(P_n - P_0)$. To determine visibility the point with a greater coordinate a_0 has priority.

If we choose the simplex with the first vertex in origin, the second vertex on axis x, the third vertex in the plane $x\,y$ and the fourth vertex outside the plane $x\,y$, we get the matrix of the transformation P from the barycentric coordinates $\{a_i\}$ to the Cartesian coordinates $\{\mathrm{a}_k\}$; the columns will be the Cartesian coordinates of the vertices of the simplex

$$\begin{pmatrix} 1 \\ \mathrm{a}_1 \\ \mathrm{a}_2 \\ \mathrm{a}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & p_{21} & p_{31} & p_{41} \\ 0 & 0 & p_{32} & p_{42} \\ 0 & 0 & 0 & p_{43} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{\mathrm{a}} = P\,\vec{a} \qquad \text{and conversely}$$

$$P^{-1} = \frac{1}{\det P} \begin{pmatrix} \det P & -p_{32}p_{43} & (p_{31} - p_{21})p_{43} & p_{32}(p_{41} - p_{21}) - p_{42}(p_{31} - p_{21}) \\ 0 & p_{32}p_{43} & -p_{31}p_{43} & p_{31}p_{42} - p_{32}p_{41} \\ 0 & 0 & p_{21}p_{43} & -p_{21}p_{42} \\ 0 & 0 & 0 & p_{21}p_{32} \end{pmatrix} .$$

If we choose the regular tetrahedron with the height d, the center of gravity in the origin O, one vertex on the axis z, the next vertex under the axis $-x$, then the matrix of the transformation from the Cartesian coordinates to the barycentric coordinates will satisfy

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{d}{\sqrt{2}} & \frac{-d}{\sqrt{8}} & \frac{-d}{\sqrt{8}} & 0 \\ 0 & -d\sqrt{\frac{3}{8}} & d\sqrt{\frac{3}{8}} & 0 \\ \frac{-d}{4} & \frac{-d}{4} & \frac{-d}{4} & \frac{3\,d}{4} \end{pmatrix}^{-1} = \frac{1}{12d}\begin{pmatrix} 3d & 8\sqrt{2} & 0 & -4 \\ 3d & -4\sqrt{2} & -4\sqrt{6} & -4 \\ 3d & -4\sqrt{2} & 4\sqrt{6} & -4 \\ 3d & 0 & 0 & 12 \end{pmatrix}.$$

The visual angle can be limited by the interior of a tetrahedron.

We can consider *a convex hull* of the (n+1) points in $\mathbf{E}^k$, $k \leq n$, as an image of the parallel projection of the simplex in $\mathbf{E}^n$ onto $\mathbf{E}^k$. For $k < n$ the images of the vertices are not linearly independent, but each point L of the convex hull (and also of the pre-image, i.e. simplex in $\mathbf{E}^n$) can by written as $L^* = \sum_{i=0}^{n} \lambda_i P_i^*$ (not uniquely), $L = \sum_{i=0}^{n} \lambda_i P_i$ (uniquely), $\sum_{i=0}^{n} \lambda_i = 1, \qquad 0 \leq \lambda_i \leq 1 \quad \forall i, i = 0, \cdots, n\,.$

The method described above was tested using a program, which generates vertices $\{P_1, P_2, P_3, P_4, P_5\}$ of a simplex in E^4 by randomizing all coordinates uniformly in the interval $[-10, 10]$. The first four points $\{P_1, P_2, P_3, P_4\}$ determine the hyperplane of projection ρ, the last point P_5 determines the center of projection S. To verify linear independence of the vertices the program computes the volume v of the simplex $\{P_1, P_2, P_3, P_4, P_5\}$:

$$P_i = (p_{1i}, p_{2i}, p_{3i}, p_{4i}),\ (i = 1, \cdots, 5), \quad d = \frac{1}{4!}\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ p_{11} & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} \end{vmatrix}, \quad v = |d|\,.$$

Let us choose the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in the hyperplane ρ:

$$\mathbf{f}_1 = P_2 - P_1,\ \mathbf{e}_1 = \frac{\mathbf{f}_1}{|\mathbf{f}_1|}, \quad \mathbf{f}_2 = P_3 - ((P_3 - P_1)\,\mathbf{e}_1)\,\mathbf{e}_1,\ \mathbf{e}_2 = \frac{\mathbf{f}_2}{|\mathbf{f}_2|},$$

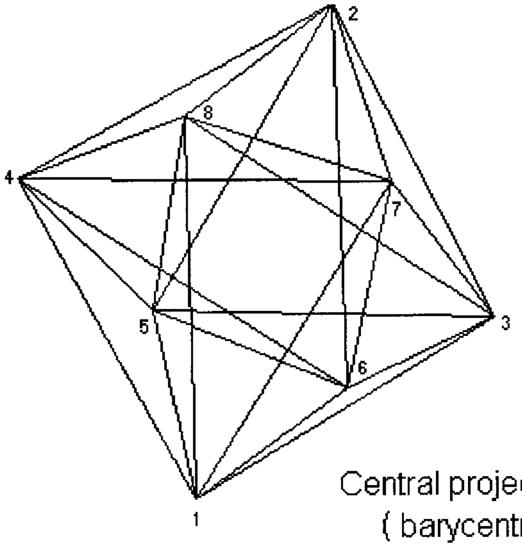
$$\mathbf{f}_3 = P_4 - ((P_4 - P_1)\,\mathbf{e}_1)\,\mathbf{e}_1 - ((P_4 - P_2)\,\mathbf{e}_2)\,\mathbf{e}_2, \quad \mathbf{e}_3 = \frac{\mathbf{f}_3}{|\mathbf{f}_3|}.$$

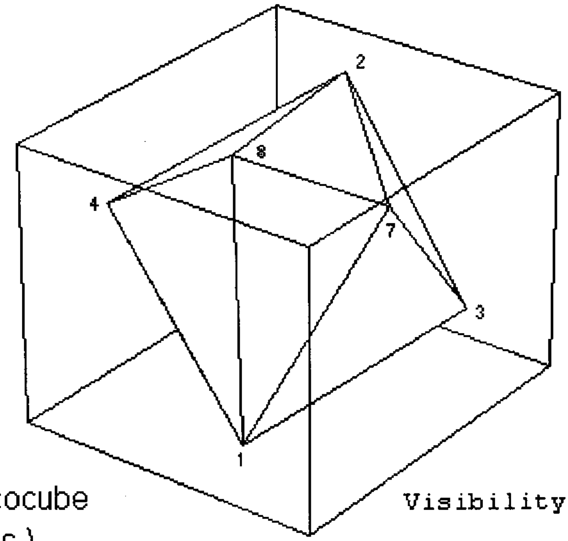
We construct also the principal point H as the orthogonal projection of S onto the hyperplane ρ

$$H = P_1 + ((P_5 - P_1)\,\mathbf{e}_1)\,\mathbf{e}_1 + ((P_5 - P_1)\,\mathbf{e}_2)\,\mathbf{e}_2 + ((P_5 - P_1)\,\mathbf{e}_3)\,\mathbf{e}_3$$

and focal distance $f = \sqrt{|P_5 - P_1|^2 - |H - P_1|^2}$.

By increasing the distance $|H - P_1|$ the central projection of the solid will be prolonged.

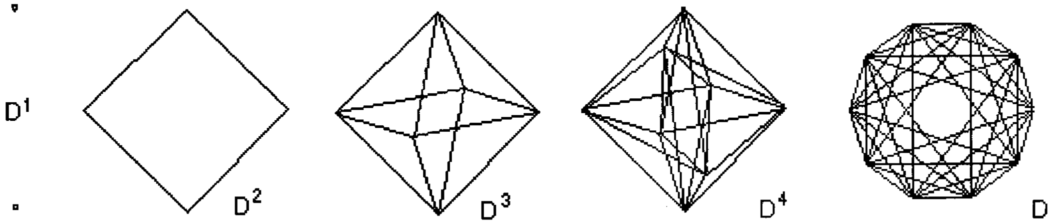




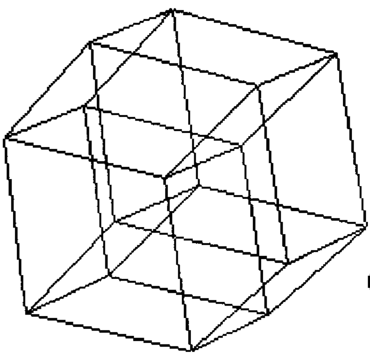
Visibility

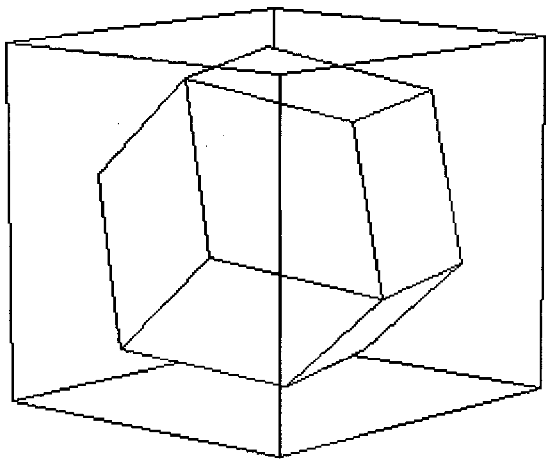
Central projection of the cocube
(barycentric coordinates)

8 vertices, 32 triangles, 24 edges, 16 tetrahedra, (3,3,4) Schläfli's symbol

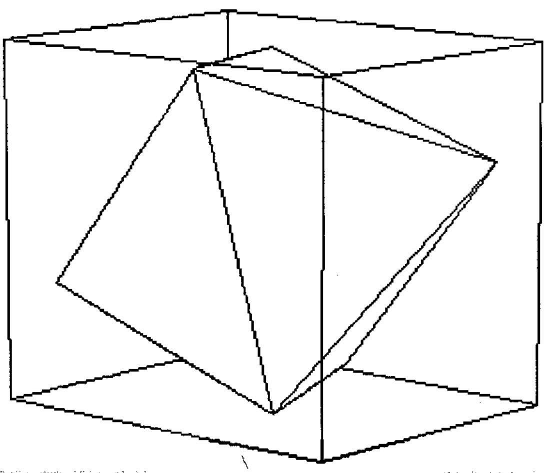


Hypercube

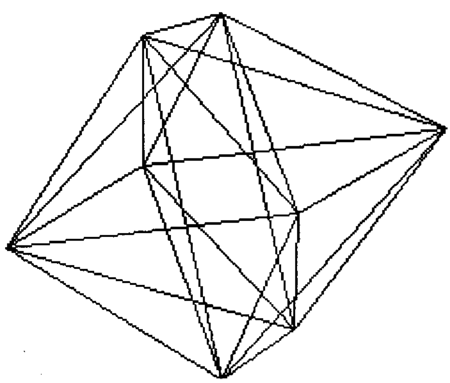




Orthogonal projection
$E^4 \rightarrow E^3$
in barycentric coordinates



Cocube



If the origin of the Cartesian coordinate system in the hyperplane ρ is put to the principal point H, then the tetrahedron $\{P_1, P_2, P_3, P_4\}$ of the hyperplane ρ will be expressed in the new coordinate system as

$$P_i^{\vdash} = (P_i - P_1)\,\mathbf{e}_i - (H - P_1)\,, \quad (i = 1, \cdots, 4)\,.$$

The image B^* of the point $B \in \mathbf{E}^4$ in the hyperplane ρ, which is described by barycentric coordinates $[b_1, b_2, b_3, b_4]$, is modelled in the hyperplane ρ as

$$B^{\Delta} = \sum_{i=1}^{4} b_i\, P_i^{\vdash}\,.$$

In the case of the orthogonal projection the plane of projection ρ passes through the origin $P_1 \equiv O$ of the coordinate system in $\mathbf{E}^4$ and through next three points P_2, P_3, P_4, the coordinates of which are again randomized in the interval $[-10, 10]$. The normal of the hyperplane ρ determines the direction of projection $\mathbf{s} = \overrightarrow{P_5 P_1}$,

$$\mathbf{n} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ p_{12} & p_{22} & p_{32} & p_{42} \\ p_{13} & p_{23} & p_{33} & p_{43} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{vmatrix}, \qquad P_5 = 10\,\frac{\mathbf{n}}{|\mathbf{n}|}\,.$$

Visibility is determined by the list of two-dimensional faces of the represented polyhedron. This problem was implemented using Mathematica system.

We have shown that the usual definition of the convex hull can be explained with the help of the parallel projection of the n-dimensional simplex. For Bézier curves and surfaces, and for general splines as well, we require that their parallel or "weak perspective" images are included in the convex hull of images of their control polygons or nets. In problems of this type the barycentric coordinates are sometimes used intuitively. The paper shows that projections in Euclidean space, necessary for visualization purposes, can be modelled in barycentric coordinates very easily.

References [1] M.Berger: Géométrie 3, Paris 1977