Creating Convex Hulls in E^2 Using Dual Representation

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Abstract

The dual representation of points, lines and polygons introduced in [Gun88] can also be used for computing convex hulls of a set of points in E^2. The main principles of the dual representation and a sketch of the algorithm for convex hull computation are given in this paper. Algorithm can be used both for statical and semi-dynamical case. More details can be seen in [Kol94].

1. Introduction

While working on acceleration of the line - polyhedron intersection computation by the means of the dual representation, some other application areas for this type of representation appeared. One of them is the problem of convex hulls construction. It is one of the most frequently solved problems in computer graphics, many algorithms were proposed to it, the complexity of the worst case proved ([Yao81], [Ben83] - the convex hull of a set of points can be found in time O(N log N) in the worst case) and no substantial efficiency improvement can be done. But, maybe, the solution by the means of the dual representation, as not so frequently used access, could be inspiring for other problems solutions.

We concentrate on construction of the convex hull of a set of points in E^2 in two cases : 1. all data are available at one time (statical problem)
2. the points are comming one after another (semi-dynamical problem)

Probably the best already known statical algorithm was done by [KiS86] and is output-sensitive; that means, that its expected time complexity is closer to O(N log K) than to O(N log N), where

N is the total number of points and K is the total number of points in convex hull.

The content of the paper is as follows. Chapter 2 gives a short survey of mathematical background. Chapter 3 shows the principles of tests used in the algorithm and the differences between statical and semi-dynamical version. Chapter 4 lists the algorithm. Chapter 5 concludes the paper. Chapter 6 brings references.

2. Mathematical background

The dual representation scheme used in this work was proposed in [Gun88].

Let's denote

$p = [p_1, p_2]^T$ a point in E^2

and

$L(a,c) = \{ x \in E^2 : x^T a = c \}, \ a \in E^2 - \{0\}, \ c \in E^1, \ a_2 \neq 0$

a non-vertical line in E^2.

Then the dual image D(p) of the point $p \in E^2$ is the line. Its equation can be written as

$x'_2 = - p_1 x'_1 + p_2$,

that means that the coordinates of a point are used as coefficients in line equation in a dual space representation.

The dual image D(L) of the line L is the point with coordinates

$$x'_1 = -a_1 / a_2$$
$$x'_2 = c / a_2$$

The dual representation of a convex polygon P in E^2 are two functions named TOP^P and BOT^P : $D(E^2) \rightarrow D(E^1)$. These functions can be proved to be piecewise linear, continuous and convex.

More precisely,

$$TOP^P(x'_1) = \max_{x_i} D(x_i)(x'_1)$$
$$BOT^P(x'_1) = \min_{x_i} D(x_i)(x'_1)$$

where $D(x_i)$, i=1,2,...,N are dual images of the polygon vertices x_i (line equations).

See also examples in Fig.1,3.

The dual representation defined in this way has two substantial properties : it doesn't change coincidence and vertical distances (that means, the distances on the vertical axis). These properties can be utilized for computer graphics problems solution.

For the purposes of our algorithm, we need to decide whether the given point **x** lies inside the given polygon P. In dual representation, from the point **x** we obtain a line D(**x**) and from P functions TOP^P and BOT^P. The character of these functions and their mutual position imply that D(**x**) can have maximally two intersections with them (special cases are not considered, in more detailes see [Kol94]).

The following situations can appear (see Fig.2) :

1. **x** lies inside P ⇔ D(**x**) has no intersections with either TOP^P or BOT^P
2. **x** lies outside P ⇔ D(**x**) has two intersections with TOP^P or two intersections with BOT^P or one intersection with both.

With this basical knowledge about dual representation we can advance to the convex hull problem.

3. The main principles used for convex hull construction

The convex hull CH(S) of a set of points S = { x_k, k=1,2,...,N } is in fact a polygon the vertices of which are some points from the given set. All the other points stay inside, no point can be outside the area of the polygon CH(S).

If the convex hull of some of the points (we will denote it $CH(S_{i-1})$ for the points { x_k, k=1,2,...,i - 1 }) is given, new point can be inserted or denied according to its position inside/outside $CH(S_{i-1})$. This is the problem of point in polygon test which can be done in dual representation.

In order to insert a point $\mathbf{x}_i$ into $CH(S_{i-1})$, we must find for it the right place in $CH(S_{i-1})$. Point insertion can cause some other point deletion, too. In the dual representation it is not a difficult task as the results of point-inside test can be utilized as follows :

According to the last chapter, point outside a polygon has two intersections with functions TOP/BOT. Let's suppose that $D(\mathbf{x}_i)$ has two intersections with $TOP^{CH(S_{i-1})}$ in line segments i_1 and i_2, see Fig.4. As $TOP^{CH(S_{i-1})}$ is convex, its part between both intersections always lies bellow $D(\mathbf{x}_i)$. That means that this part doesn't satisfy the definition of TOP (maxima) and has to be replaced by the line $D(\mathbf{x}_i)$, see Fig.4. After this reconstruction,

the function $TOP^{CH(S_{i-1})}$ together with unchanged $BOT^{CH(S_{i-1})}$ correspond to the convex hull $CH(S_i)$.

If the line $D(\mathbf{x}_i)$ has one intersection with TOP and one with BOT, it is necessary to find which side of TOP and BOT lies bellow $D(\mathbf{x}_i)$ and is to be replaced. The decision can be made according to the position of vertices of TOP and BOT to the line $D(\mathbf{x}_i)$ (by substitution to the line equation).

The last question to be solved is to find the initial convex hull. For this purpose the points which are extremal in both coordinates can be used. These points are always members of the convex hull and so it is advantageous to start with them, see Fig.5.

Now to the differences between statical and semi-dynamical algorithm. If we start the solution with the convex hull of extremal points, as was recommended in the last chapter, the points outside $CH(S_{i-1})$ can have only two intersections with $TOP^{CH(S_{i-1})}$ or two intersections with $BOT^{CH(S_{i-1})}$ but not one and one. That means that the statical variant of algorithm that starts the construction with extremes doesn't need the branch with one intersection with TOP and one with BOT functions. And, on the other side, if we consider this case, we are able to decide about points that needn't be inside the original box (given by extremes). See also Fig.6 for examples of the mutual position of $D(\mathbf{x}_i)$ and $TOP^{CH(S_{i-1})}$ and $BOT^{CH(S_{i-1})}$.

The whole algorithm is shown in the next chapter.

4. Algorithm for computation convex hull of a set of points in E^2

1. Find in the given set S the points with extremal coordinates in axis x_1 or x_2. If there exist more points with the same and minimal, resp. maximal coordinate, take only one for each direction (that means, 2-4 points; let's denote the total number of selected points p).
2. From these points construct polygon $CH(S_p)$ with p vertices and its dual representation $TOP^{CH(S_p)}$ and $BOT^{CH(S_p)}$ (from the definition; more effective constructions see in [Kol94]).
3. i := p + 1;
4. while i <= N do
   begin
5. Compute $D(x_i)$ and find whether $D(\mathbf{x}_i)$ intersects $TOP^{CH(S_{i-1})}$ and $BOT^{CH(S_{i-1})}$ and at which linear parts;

6. if no intersection then goto 12
7. if exist two intersections with $TOP^{CH(S_{i-1})}$ in linear parts no. k,l then
   begin
     Insert $D(x_i)$ into $TOP^{CH(S_{i-1})}$ between $D(x_k)$ and $D(x_l)$;
     if linear parts k,l don't neighbour then
       Delete from $TOP^{CH(S_{i-1})}$ parts from $D(x_{k+1})$ up to $D(x_{l-1})$
   end;
8. the same as step 7 but with BOT;
9. if exists one intersection with $TOP^{CH(S_{i-1})}$ in linear part k then
   { $[p_1', p_2']$ is the endpoint of the linear part k , $x_i = [p_1, p_2]$ }
   begin
     if the part k is not the last one (opened) then
       left := $p_1p_1' + p_2' - p_2 > 0$
     else
       left := $p_1p_1' + p_2' - p_2 < 0$;
     if left then
       Delete from $TOP^{CH(S_{i-1})}$ the left side (up to the part k-1) if it exists
     else
       Delete from $TOP^{CH(S_{i-1})}$ the right side (from the part k+1) if it exists
   end;
10. the same as step 9, but with BOT; in computation of the flag "left", the relations ">" and "<" have to be reversed;
11. p := p + 1; { the total number of the points in the convex hull }
12. $TOP^{CH(S_i)} := TOP^{CH(S_{i-1})}$; $BOT^{CH(S_i)} := BOT^{CH(S_{i-1})}$;
13. i := i + 1
end;

{ After finishing the algorithm, the convex hull is given by functions $TOP^{CH(S_N)}$ and $BOT^{CH(S_N)}$. The total number of points in the convex hull is p. }

Let's stop at the complexity of the proposed method. For N points in the set S, the time complexity will be O(N*f(N)); f(N) is given by the complexity of searching polygon-line intersection. The solution with f(N) = K is obvious. Memory demands are O(N) (for dual representation of CH(S)). No preprocessing is necessary.

It is possible to reduce this complexity, if we reduce the complexity of two critical points : of TOP/BOT update and of polygon-line intersection computing. The first problem can be solved simply if we use for TOP/BOT double chained list. Then TOP and BOT update can be done in O(1). As to the second point, we need some intersection algorithm with logarithmic time complexity. The solution from [Meh84] with (2,4)-tree can be used.

With such improvements, we could construct the convex hull of a set of points in E^2 in time O(N log K) with O(N) preprocessing (construction of the tree) and with O(N log N) memory requirements (tree, chained list).

5. Conclusion

A new algorithm for computation of the convex hull of a set of points in E^2 for statical and semi-dynamical data on the basis of the dual representation is proposed.

The algorithm can be optimal in time if an algorithm of computing line-polygon intersection with logarithmical time complexity is used. It is sensitive to output as its complexity depends on the number of points which are members of the convex hull.

6. References

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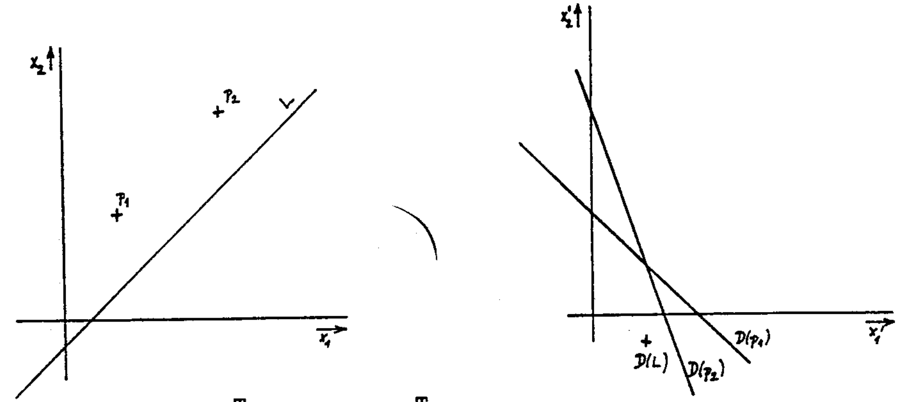
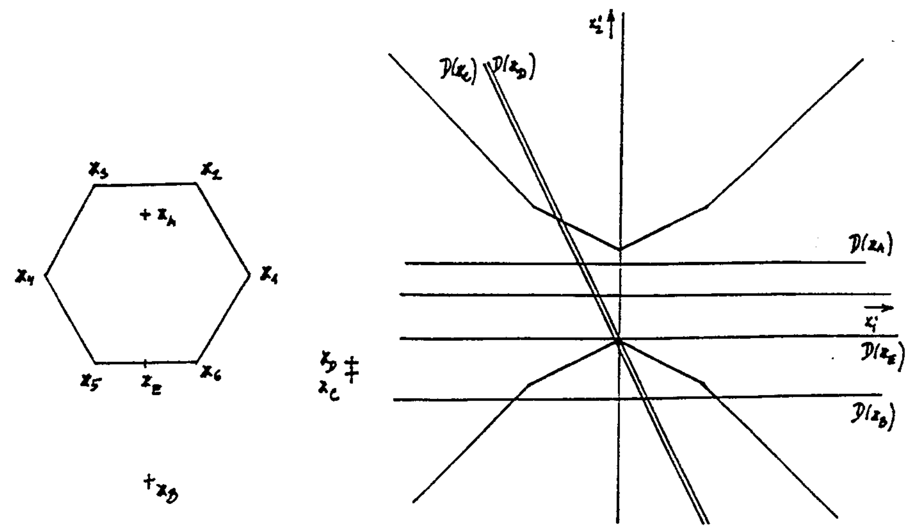


$p_1 = [1,2]^T$, $p_2 = [3,4]^T$, $L : x_2 - x_1 = -0.5$

$D(p_1)$: $x_2' = -x_1' + 2$, $D(p2)$: $x_2' = -3x_1' + 4$, $D(L) = [1,-0.5]^T$

Two points and a line segment and their dual representation

Fig.1



The possible situations that can appear during point-in-polygon test

Fig.2

Vertices :

$x_1 = [1,0]$
$x_2 = [0.5, 0.866]$
$x_3 = [-0.5, 0.866]$
$x_4 = [-1,0]$
$x_5 = [-0.5, -0.866]$
$x_6 = [0.5, -0.866]$

Edges :

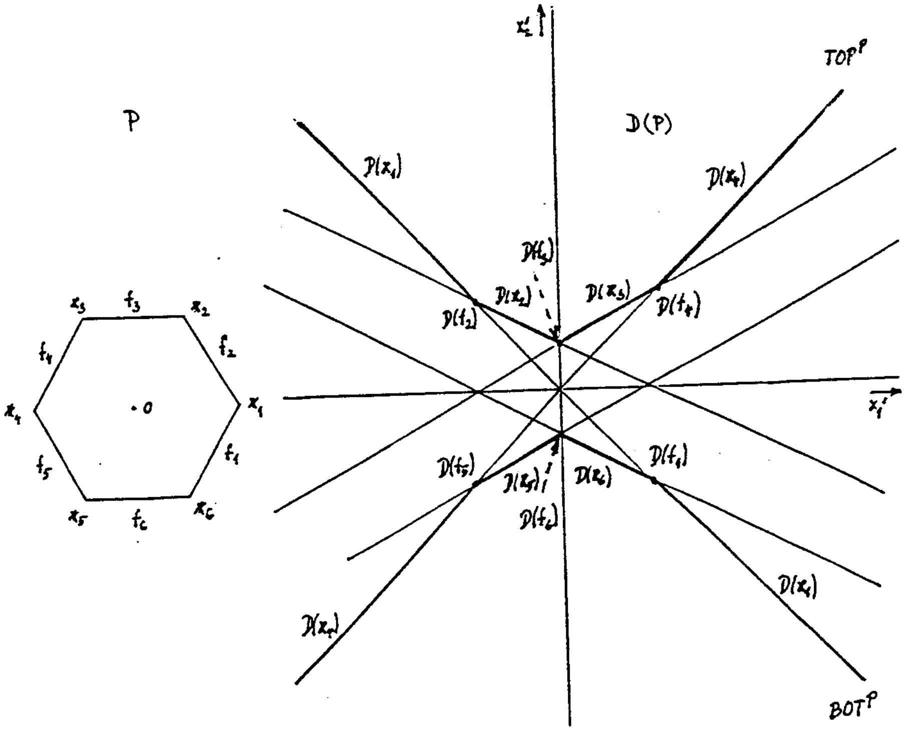
f_1 : $a = [1.732, -1]^T$, $c = 1.732$
f_2 : $a = [1.732, 1]^T$, $c = 1.732$
f_3 : $a = [0,1]^T$, $c = 0.866$
f_4 : $a = [-1.732, 1]^T$, $c = 1.732$
f_5 : $a = [-1.732, -1]^T$, $c = 1.732$
f_6 : $a = [0,-1]^T$, $c = 0.866$

Dual images of the vertices :

$D(x_1)$: $x_2' = -x_1'$
$D(x_2)$: $x_2' = -0.5\ x_1' + 0.866$
$D(x_3)$: $x_2' = 0.5\ x_1' + 0.866$
$D(x_4)$: $x_2' = x_1'$
$D(x_5)$: $x_2' = 0.5\ x_1' - 0.866$
$D(x_6)$: $x_2' = -0.5\ x_1' - 0.866$

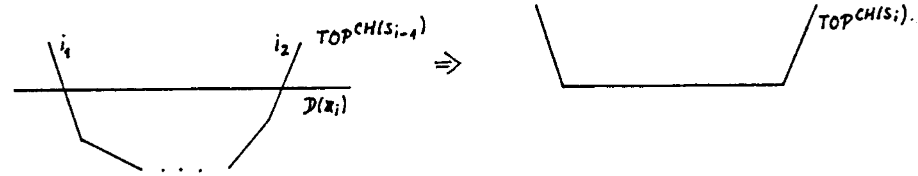
Dual images of the edges :

$D(f_1) = [1.732, -1.732]$
$D(f_2) = [-1.732, 1.732]$
$D(f_3) = [0, 0.866]$
$D(f_4) = [1.732, 1.732]$
$D(f_5) = [-1.732, -1.732]$
$D(f_6) = [0, -0.866]$



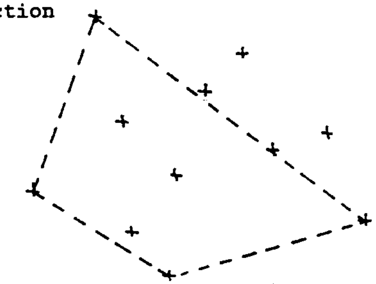
The convex polygon P and its dual representation

Fig. 3



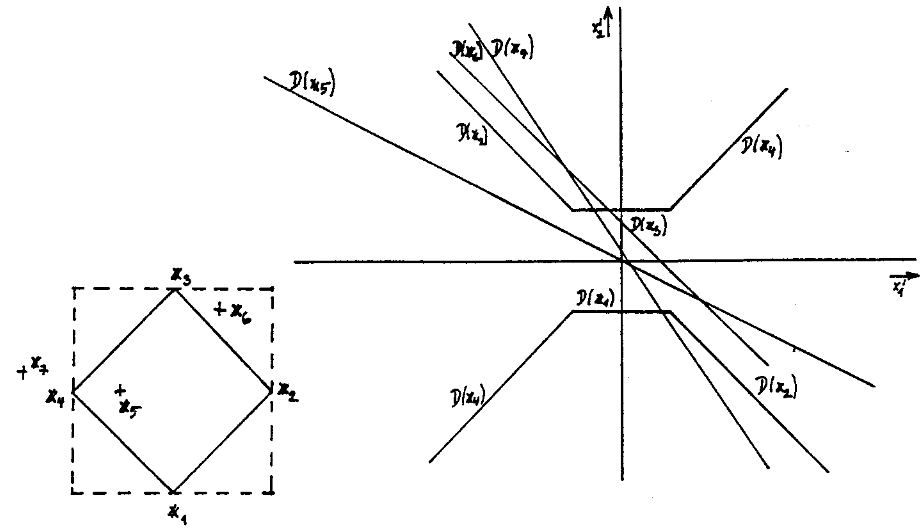
TOP reconstruction

Fig. 4



The initial convex hull for a set of points

Fig. 5



Examples of mutual position of $TOP/BOT^{CH(S_4)}$ to $D(x_5)$, $D(x_6)$ and $D(x_7)$

Fig. 6