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Modal analysis of beam with piezoelectric sensors a actuators

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Abstract

One dimensional finite element is developed for the analysis of structures with applied piezoelectric sensors and actuators, i.e. smart structures, mechanical behavior of which can be controlled in real-time. The element is based on Euler-Bernoulli theory and it assumes bilinear distribution of electric field potential. Mathematical model was implemented in MATLAB environment. Sensitivity analysis is carried out for the case of modal analysis with and without piezo patches.

Keywords: piezo, finite element, modal

1. Introduction

Light-weight structures are nowadays necessary components in modern state-of-the-art products in all sorts of industries. The increasing requirements on structural performance call for the usage of embedded sensors and actuators, resulting in the construction of so-called smart or adaptive structures that can thus respond to loading conditions in real time. This enables for instance to suppress vibrations or to adapt the desired shape, provided that proper electronic control circuits are applied.

Since the first finite element implementation of the piezoelectric phenomenon by Allik and Hughes [1] in 1970, many researches have ‘equipped’ the standard structural finite elements with the piezoelectric capability to simulate the piezoelectric effect – sometimes using the similarity to the theory of thermo-elasticity. These early models concerned mainly 3D-solid elements, which are not suited for efficient analysis of laminated shell structures. In the recent years, the piezoelectric, beam, plate and shell elements are used more frequently. Cen et al [3], for example, developed a four-node plate element for laminated structures based on first-order shear deformation theory while Lee et al. [7] introduced a nine-node assumed strain element allowing, unlike other elements, for variable thickness. Hybrid laminated piezo plates are studied by Mitchell and Reddy [8] using higher-order shear deformation theory and layerwise approach for electric potential. Saravanos et al. [9] study dynamic behavior of smart laminated plates using the layerwise approach. Tzou et al. [11] investigate the control of smart conical shells using triangular finite elements. Kögl and Bucalem [6] introduced a MITC based element suitable for modeling of moderately thick sandwich smart structures. They stress the importance of quadratic variation of electric potential across the layer thickness to accurately model the electric field. Other various finite element approaches are summarized for example in the survey by Benjeddou [2]. Zhou et al. [12] study free vibrations of piezoelectric bimorphs by means of

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analytical solution. Heyliger [4] and later Heyliger and Wu [5] present exact analytical solution for laminated piezoelectric cylinder and sphere, respectively. Zemčík et al. [13] developed four-noded piezoelectric shell element and implemented into commercial code ANSYS.

The element proposed in this paper is two-noded and has two structural degrees of freedom (DOFs) at each node plus two DOFs for electric potential. The piezoelectric coupling is full and direct (i.e. non-iterative), and it is intended for the simulation of applied piezoelectric layers – patches.

2. Mathematical model

2.1. Constitutive equations

The theory of piezoelectric materials used here assumes symmetrical hexagonal piezoelectric structure – class 6*mm* (C_{6v}). Only the laminar piezoelectric effect (so-called d_{31} effect [10]) is considered, i.e., the material is polarized in the thickness direction.

The stress-strain law for each piezoelectric element can be written as

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbf{C}\,\boldsymbol{\varepsilon} - \mathbf{e}^{\mathrm{T}}\,\mathbf{E}\,, \\ \mathbf{D} &= \mathbf{e}\boldsymbol{\varepsilon} + \boldsymbol{\epsilon}\,\mathbf{E}\,, \end{aligned} \tag{1}$$

where $\mathbf{D}$ is the vector of electric flux density, $\boldsymbol{\epsilon}$ is the dielectric permittivity matrix, $\mathbf{e}$ is the piezoelectric coefficient matrix and $\mathbf{E}$ is the electric field vector.

The permittivity matrix $\boldsymbol{\epsilon}$ is defined as

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \tag{2}$$

and the piezoelectric matrix $\mathbf{e}$ as

$$\mathbf{e} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 \end{bmatrix} . \tag{3}$$

2.2. Analytical formulation

The beam element is based on Euler-Bernoulli theory. It has two nodes with one deflection w_i and one rotational φ_i DOF at each node (see fig. 1). Let the deflection $w(x)$ across the length be approximated by the polynomial:

$$w(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \mathbf{N}\mathbf{w}\,, \tag{4}$$

where $\mathbf{N}$ is matrix of approximation functions and the structural DOF vector is ordered as

$$\mathbf{w} = [w_n, \varphi_n, w_{n+1}, \varphi_{n+1}]^{\mathrm{T}}\,. \tag{5}$$

Then, the axial displacement can be written as

$$u(x, z) = z\varphi(x) = z\frac{\partial w}{\partial x} \tag{6}$$

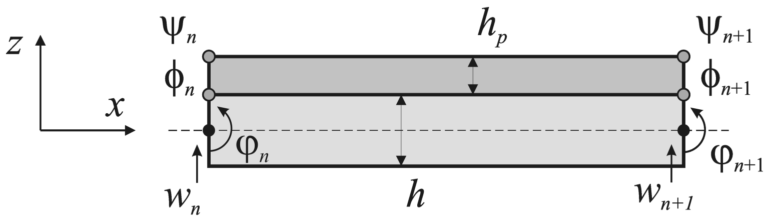

Fig. 1. Element geometry example. Piezo material (dark grey) and its supporting structure (light grey) sharing common nodes (degrees of freedom).

and, consequently, the axial strain is

$$\varepsilon(x,z) = \frac{\partial u}{\partial x} = z\frac{\partial^2 w}{\partial x^2} = \mathbf{Bw} \; , \tag{7}$$

where $\mathbf{B}$ is strain-displacement matrix consisting of shape functions derivatives.

Similarly, let the electric field potential $\phi(x,z)$ be approximated by bi-linear function

$$\phi(x,z) = a_4 + a_5 x + a_6 z + a_7 xz \; . \tag{8}$$

Hence, the electric field intensity vector $\mathbf{E}$ is

$$\mathbf{E} = -\nabla\phi = \mathbf{\Phi}\boldsymbol{\phi} \; , \tag{9}$$

where $\mathbf{\Phi}$ is the electric field intensity-potential matrix and the electrical DOF vector is ordered as

$$\boldsymbol{\phi} = [\phi_n, \psi_n, \phi_{n+1}, \psi_{n+1}]^{\mathrm{T}} \tag{10}$$

with ϕ_i and ψ_i being the potential values on the lower and upper surfaces, respectively (see fig. 1).

2.3. Variational principle

The equations of motion of a piezoelectric structure can be derived from the Lagrangian and the virtual work which must include both the mechanical and the electrical contributions. The potential energy density P of a piezoelectric material includes contributions from the strain energy and from the electrostatic energy, hence

$$P = \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon} = \tfrac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{C}\boldsymbol{\varepsilon} - \mathbf{E}^{\mathrm{T}}\mathbf{e}\boldsymbol{\varepsilon} - \tfrac{1}{2}\mathbf{E}^{\mathrm{T}}\boldsymbol{\epsilon}\mathbf{E} \tag{11}$$

while the kinetic energy density is simply

$$K = \tfrac{1}{2}\rho(\dot{w})^2 \; . \tag{12}$$

Let us confine to case without external mechanical forces and electric charge. The Lagrangian can then be written in the form

$$L = \int_V (K - P)\,\mathrm{d}V \; . \tag{13}$$

Using the variation principle, the condition

$$\delta L = 0 \tag{14}$$

must be satisfied for any arbitrary variation of the displacements and electrical potentials, thus the resulting equations of motion with the assumptions made above are assembled as

$$\begin{bmatrix} \mathbf{M}_{uu} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{w}} \\ \ddot{\boldsymbol{\phi}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\phi} \\ \mathbf{K}_{u\phi}^{\mathrm{T}} & \mathbf{K}_{\phi\phi} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \boldsymbol{\phi} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} , \tag{15}$$

where the submatrices (element stiffness, piezoelectric coupling, capacitance and mass matrix) are

$$\begin{aligned} \mathbf{K}_{uu} &= \int_V \mathbf{B}^{\mathrm{T}}\,\mathbf{C}\,\mathbf{B}\,\mathrm{d}V & [4\times 4]\,, \\ \mathbf{K}_{u\phi} &= -\int_V \mathbf{B}^{\mathrm{T}}\,\mathbf{e}\,\boldsymbol{\Phi}\,\mathrm{d}V & [4\times 4]\,, \\ \mathbf{K}_{\phi\phi} &= -\int_V \boldsymbol{\Phi}^{\mathrm{T}}\,\boldsymbol{\epsilon}\,\boldsymbol{\Phi}\,\mathrm{d}V & [4\times 4]\,, \\ \mathbf{M}_{uu} &= \rho\int_V \mathbf{N}^{\mathrm{T}}\,\mathbf{N}\,\mathrm{d}V & [4\times 4]\,. \end{aligned} \tag{16}$$

2.4. Modal analysis

In order to perform the modal analysis the system must be statically condensed, hence the set of equation is modified to

$$\mathbf{M}_{uu}\ddot{\mathbf{w}} + \left(\mathbf{K}_{uu} - \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{u\phi}^{\mathrm{T}}\right)\mathbf{w} = \mathbf{0} \tag{17}$$

and the problem reduces to the eigenvalue analysis of the matrix

$$\mathbf{A} = {\mathbf{M}_{uu}}^{-1}\left(\mathbf{K}_{uu} - \mathbf{K}_{u\phi}\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{u\phi}^{\mathrm{T}}\right) . \tag{18}$$

The corresponding values for electrical DOFs can be retrieved using

$$\boldsymbol{\phi} = -\mathbf{K}_{\phi\phi}^{-1}\mathbf{K}_{u\phi}^{\mathrm{T}}\mathbf{w} . \tag{19}$$

The set of equations in eq. (15) for single element is expanded accordingly in finite element analysis when joining element with common nodes (see fig. 1).

3. Numerical tests

Two types of sensitivity analyses were carried out using MATLAB code. One for pure structural behavior and one for fully coupled piezoelectric (hybrid) system. The first test concerned simple steel cantilever beam clamped at $x = 0$. Up to three lowest eigenfrequencies were obtained for total length $l = 1\,\mathrm{m}$ modeled with 1 , 2, 4, 8 and 16 elements, respectively. The thickness was $h = 0.01\,\mathrm{m}$, width $b = 0.1\,\mathrm{m}$ and material properties are shown in tab. 1.

PVDF (polyvynilidene fluoride) piezopatch of thickness $h_p = 1\,\mathrm{mm}$ was modeled across the whole length of the steel beam in the second test. The lower surface of the patch was grounded, i.e. $\phi_i = 0\,\mathrm{V}$, while the upper surface was left open. The piezoelectric properties are again in tab. 1.

The resulting values of obtained eigenfrequencies are summarized in tab. 2 and 3. Very good convergence is obvious from the results in both cases. The shapes for the first three eigenmodes are shown in fig. 2.

		Steeel	Piezo
E	[GPa]	210	2
ρ	[kg/m^3]	7850	1800
e_{31}	[N/m^2]	-	0.044
$\epsilon_{11} = \epsilon_{33}$	[nF/m]	-	1.062

Tab. 1. Material properties.

Elements	ω_1 [Hz]	ω_2 [Hz]	ω_3 [Hz]
1	8.39	82.71	-
2	8.36	52.81	178.60
4	8.36	52.42	147.74
8	8.36	52.37	146.70
16	8.36	52.36	146.61

Tab. 2. Summary of lowest eigenfrequencies for steel beam.

Elements	ω_1 [Hz]	ω_2 [Hz]	ω_3 [Hz]
1	8.35	82.29	-
2	8.32	52.53	177.69
4	8.31	52.16	147.00
8	8.31	52.10	146.96
16	8.31	52.10	145.88

Tab. 3. Summary of lowest eigenfrequencies for hybrid steel/piezo beam.

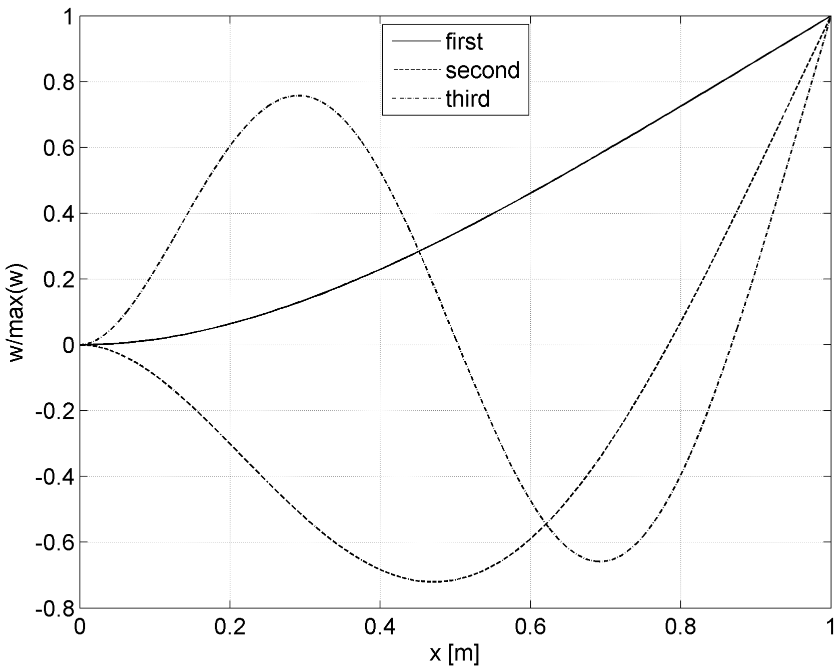

Fig. 2. First three normalized eigenmodes.

4. Conclusion

One dimensional beam element suitable for the analysis of structures with applied piezoelectric sensors and actuators, i.e. smart structures, is developed and implemented in MATLAB code. The element is based on Euler-Bernoulli theory and it assumes bilinear distribution of electric field potential. Sensitivity analysis is carried out for the case of modal analysis of steel beam with and without applied piezo patches. The results of the analysis in terms of the first three lowest eigenfrequencies show very good convergence of the element.

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