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A stochastic response of vibrating systems containing random parameters

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Abstract

The paper deals with an approach to the response limit estimation of the vibrating systems in the transient state. A gradient and perturbation methods for derivatives of the response with respect to the random parameters are shown. This sensitivity analysis can be performed for linear systems in the analytical way or in numerical way for both linear and nonlinear systems. In case of nonlinear systems it is supposed small deviations of random parameters from hyperbolic point of the system and that no bifurcations and changes of phase portrait will occur.
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1. Introduction

Let us assume the equation of motion of the vibrating system in form

$$\mathbf{M}(\mathbf{p})\ddot{\mathbf{q}}(\mathbf{p},t)+\mathbf{B}(\mathbf{p})\dot{\mathbf{q}}(\mathbf{p},t)+\mathbf{K}(\mathbf{p})\mathbf{q}(\mathbf{p},t)+\mathbf{f}_N(\mathbf{p},\dot{\mathbf{q}},\mathbf{q})=\mathbf{f}(\mathbf{p},t), \tag{1}$$

where $\mathbf{M}(\mathbf{p}),\mathbf{B}(\mathbf{p}),\mathbf{K}(\mathbf{p})\in\mathbf{R}^{n,n}$ are matrices of mass, damping and stiffness, respectively, and $\mathbf{q}(\mathbf{p},t),\ \mathbf{f}_N(\mathbf{p},\dot{\mathbf{q}},\mathbf{q}),\ \mathbf{f}(\mathbf{p},t)\in\mathbf{R}^n$ are vector of generalized displacements, vector of nonlinear forces and vector of external excitation, respectively. Vector of random parameters is marked by $\mathbf{p}\in\mathbf{R}^s$. The dot corresponds to the differentiation with respect to time. The random parameters are defined either by probability density functions, or by mean value (expectation) vector and covariance matrix, respectively

$$\overline{\mathbf{p}}=E\{\mathbf{p}\}=\begin{bmatrix}E\{p_1\}\\E\{p_2\}\\\vdots\\E\{p_s\}\end{bmatrix}\in\mathbf{R}^s,\qquad \boldsymbol{\Sigma}_\mathbf{p}=\begin{bmatrix}\sigma^2_{p_1}, & \sigma^2_{p_1p_2}, & \cdots, & \sigma^2_{p_1p_s}\\ \sigma^2_{p_2p_1}, & \sigma^2_{p_2}, & \cdots, & \sigma^2_{p_2p_s}\\ \vdots & \vdots & \ddots & \vdots\\ \sigma^2_{p_sp_1}, & \sigma^2_{p_sp_2}, & \cdots, & \sigma^2_{p_s}\end{bmatrix}\in\mathbf{R}^{s,s}. \tag{2}$$

The *(i,i)*-th element of the covariance matrix (along diagonal element) corresponds to variance of the *i*-th random parameter and the *(i,j)*-th element for $i\neq j$ (off-diagonal element) corresponds to covariance of *i*-th and *j*-th elements. Let us mark $\Delta\mathbf{p}=\mathbf{p}-\overline{\mathbf{p}}$. As can be seen

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the random vector variable $\Delta\mathbf{p}$ is centered. It means that mean value is equal to

$$E\{\Delta\mathbf{p}\} = E\{\mathbf{p} - \overline{\mathbf{p}}\} = E\{\mathbf{p}\} - E\{\overline{\mathbf{p}}\} = \overline{\mathbf{p}} - \overline{\mathbf{p}} = \mathbf{0} \in \mathbf{R}^s . \tag{3}$$

As a starting point we can write down the matrices and vectors in (1) into Taylor's power series according to random parameters [3]

$$\mathbf{K}(\mathbf{p}) \doteq \overline{\mathbf{K}} + \varepsilon \sum_{j=1}^{s} \frac{\partial \mathbf{K}}{\partial p_j} \Delta p_j + \frac{1}{2}\varepsilon^2 \sum_{j=1}^{s}\sum_{k=1}^{s} \frac{\partial^2 \mathbf{K}}{\partial p_j \partial p_k} \Delta p_j \Delta p_k \ldots,$$

$$\mathbf{M}(\mathbf{p}) \doteq \overline{\mathbf{M}} + \varepsilon \sum_{j=1}^{s} \frac{\partial \mathbf{M}}{\partial p_j} \Delta p_j + \frac{1}{2}\varepsilon^2 \sum_{j=1}^{s}\sum_{k=1}^{s} \frac{\partial^2 \mathbf{M}}{\partial p_j \partial p_k} \Delta p_j \Delta p_k \ldots,$$

$$\mathbf{B}(\mathbf{p}) \doteq \overline{\mathbf{B}} + \varepsilon \sum_{j=1}^{s} \frac{\partial \mathbf{B}}{\partial p_j} \Delta p_j + \frac{1}{2}\varepsilon^2 \sum_{j=1}^{s}\sum_{k=1}^{s} \frac{\partial^2 \mathbf{B}}{\partial p_j \partial p_k} \Delta p_j \Delta p_k \ldots,$$

$$\mathbf{q}(\mathbf{p},t) \doteq \overline{\mathbf{q}}(t) + \varepsilon \sum_{j=1}^{s} \frac{\partial \mathbf{q}(t)}{\partial p_j} \Delta p_j + \frac{1}{2}\varepsilon^2 \sum_{j=1}^{s}\sum_{k=1}^{s} \frac{\partial^2 \mathbf{q}(t)}{\partial p_j \partial p_k} \Delta p_j \Delta p_k \ldots, \tag{4}$$

$$\mathbf{f}(\mathbf{p},t) \doteq \overline{\mathbf{f}}(t) + \varepsilon \sum_{j=1}^{s} \frac{\partial \mathbf{f}(t)}{\partial p_j} \Delta p_j + \frac{1}{2}\varepsilon^2 \sum_{j=1}^{s}\sum_{k=1}^{s} \frac{\partial^2 \mathbf{f}(t)}{\partial p_j \partial p_k} \Delta p_j \Delta p_k \ldots..,$$

$$\mathbf{f}_N(\mathbf{p},\mathbf{q},\dot{\mathbf{q}}) \doteq \mathbf{f}_N(\overline{\mathbf{p}},\overline{\mathbf{q}},\dot{\overline{\mathbf{q}}}) + \varepsilon \sum_{i=1}^{s} \left(\frac{\partial \mathbf{f}_N(\overline{\mathbf{p}},\overline{\mathbf{q}},\dot{\overline{\mathbf{q}}})}{\partial p_i} + \frac{\partial \mathbf{f}_N(\overline{\mathbf{p}},\overline{\mathbf{q}},\dot{\overline{\mathbf{q}}})}{\partial \dot{\mathbf{q}}} \frac{\partial \dot{\mathbf{q}}}{\partial p_i} + \frac{\partial \mathbf{f}_N(\overline{\mathbf{p}},\overline{\mathbf{q}},\dot{\overline{\mathbf{q}}})}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial p_i} \right) + \varepsilon^2 \frac{1}{2} \ldots\ldots,$$

where $\overline{\mathbf{q}} = \mathbf{q}(\overline{\mathbf{p}}) = \mathbf{q}(E\{\mathbf{p}\})$, $\overline{\mathbf{K}} = \mathbf{K}(E\{\mathbf{p}\})$, $\overline{\mathbf{f}} = \mathbf{f}(E\{\mathbf{p}\})$ etc. and ε is small parameter. To obtain the derivatives of state variables we can use two methods. The first is perturbation method whose application means substitute the relations (4) into the equation of motion (1) and compare terms with the same power of parameter ε. The second is called a gradient method starting by the direct derivation of equation of motion with respect to the random parameter. The obtained derivatives of the state variables (vectors of generalized displacements, velocities and accelerations) are substituted into the fourth relation of (4). Having performed this substitution we can apply the mean operator to the result equation for $\varepsilon = 1$ and than obtain the time dependent mean vector of generalized displacements and corresponding time dependent covariance matrix. Using the six sigma criterion approach we can determine the time dependent upper and lower limits of the individual displacements in the probability sense.

2. The sensitivity analysis – gradient method

Because the results of perturbation and gradient method are identical which can be simply proved we can show the use of the second one only. There is presented in [1] and [2] the analytical solution of sensitivity analysis. This is possible only for linear case of systems when analytical solution of the equation of motion is available. We now pay an attention to the cases when the analytical solution is generally not available e.g. nonlinear systems. Let us mark $\mathbf{q}^{(i)}(\mathbf{p}) = \frac{\partial \mathbf{q}(\overline{\mathbf{p}})}{\partial p_i}$, $\mathbf{q}^{(ij)}(\mathbf{p}) = \frac{\partial^2 \mathbf{q}(\overline{\mathbf{p}})}{\partial p_i \partial p_j}$, etc. and let us differentiate the equation of motion (1) with respect to the i-th random parameter and come to (briefly written)

$$\mathbf{M}^{(i)}\ddot{\mathbf{q}} + \overline{\mathbf{M}}\ddot{\mathbf{q}}^{(i)} + \mathbf{B}^{(i)}\dot{\mathbf{q}} + \overline{\mathbf{B}}\dot{\mathbf{q}}^{(i)} + \mathbf{K}^{(i)}\mathbf{q} + \overline{\mathbf{K}}\mathbf{q}^{(i)} + \frac{\partial \overline{\mathbf{f}}_N}{\partial p_i} + \frac{\partial \overline{\mathbf{f}}_N}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}}^{(i)} + \frac{\partial \overline{\mathbf{f}}_N}{\partial \mathbf{q}}\mathbf{q}^{(i)} = \mathbf{f}^{(i)}. \tag{5}$$

This equation should be written down and solved s times for $\ddot{\mathbf{q}}^{(i)}$, $\dot{\mathbf{q}}^{(i)}, \mathbf{q}^{(i)}$, $i = 1,2,....,s.$ From this reason we can rewrite (5) into form

$$\overline{\mathbf{M}}\ddot{\mathbf{q}}^{(i)} + \left(\overline{\mathbf{B}} + \frac{\partial \overline{\mathbf{f}}_N}{\partial \dot{\mathbf{q}}}\right)\dot{\mathbf{q}}^{(i)} + \left(\overline{\mathbf{K}} + \frac{\partial \overline{\mathbf{f}}_N}{\partial \mathbf{q}}\right)\mathbf{q}^{(i)} = \mathbf{f}^{(i)} - \mathbf{M}^{(i)}\ddot{\overline{\mathbf{q}}} - \mathbf{B}^{(i)}\dot{\overline{\mathbf{q}}} - \mathbf{K}^{(i)}\overline{\mathbf{q}} - \frac{\partial \overline{\mathbf{f}}_N}{\partial p_i},\ i = 1,2,....,s, \tag{6}$$

where
$$\frac{\partial \overline{\mathbf{f}}_N}{\partial \dot{\mathbf{q}}} = \frac{\partial \overline{\mathbf{f}}_N(\overline{\mathbf{p}}, \dot{\overline{\mathbf{q}}}, \overline{\mathbf{q}})}{\partial \dot{\mathbf{q}}},\quad \frac{\partial \overline{\mathbf{f}}_N}{\partial \mathbf{q}} = \frac{\partial \overline{\mathbf{f}}_N(\overline{\mathbf{p}}, \dot{\overline{\mathbf{q}}}, \overline{\mathbf{q}})}{\partial \mathbf{q}},\quad \frac{\partial \overline{\mathbf{f}}_N}{\partial p_i} = \frac{\partial \overline{\mathbf{f}}_N(\overline{\mathbf{p}}, \dot{\overline{\mathbf{q}}}, \overline{\mathbf{q}})}{\partial p_i}. \tag{7}$$

To express the fourth relation of (4) with $\varepsilon = 1$ we have to solve (1) in numerical way for $\mathbf{p} = \overline{\mathbf{p}}$. From the solution follows $\overline{\mathbf{q}}(t) = \mathbf{q}(\overline{\mathbf{p}}, t)$, $\dot{\overline{\mathbf{q}}}(t) = \dot{\mathbf{q}}(\overline{\mathbf{p}}, t)$, $\ddot{\overline{\mathbf{q}}}(t) = \ddot{\mathbf{q}}(\overline{\mathbf{p}}, t)$. Simultaneously we can solve s equations (6) because the state variables of the equation of motion are contained in the excitation (right hand side of (6)) of the sensitivity equations. Using the mean operator to the fourth relation of (4) we can write ($\varepsilon = 1$)

$$\boldsymbol{\mu}_{\mathbf{q}}(t) \doteq E\{\mathbf{q}(\mathbf{p}, t)\} \doteq \overline{\mathbf{q}}(t) + \sum_{j=1}^{s} \frac{\partial \mathbf{q}(t)}{\partial p_j} \underbrace{E\{\Delta p_j\}}_{0} + \frac{1}{2}\sum_{j=1}^{s}\sum_{k=1}^{s} \frac{\partial^2 \mathbf{q}(t)}{\partial p_j \partial p_k} \underbrace{E\{\Delta p_j \Delta p_k\}}_{\sigma_{ij}^2} ...., \tag{8}$$

and then
$$\boldsymbol{\mu}_{\mathbf{q}}(t) \doteq \overline{\mathbf{q}}(t) + \frac{1}{2}\sum_{j=1}^{s}\sum_{k=1}^{s} \frac{\partial^2 \mathbf{q}(t)}{\partial p_j \partial p_k} \sigma_{ij}^2 . \tag{9}$$

Respecting only terms up to the first derivatives in the fourth relation of (4) we can rewrite (9) into form

$$\boldsymbol{\mu}_{\mathbf{q}}(t) \doteq \overline{\mathbf{q}}(t). \tag{10}$$

Inspite of (10) it can be told that mean values of the state variables generally does not correspond to the state variables for mean values of random parameters. The relation (10) holds in case of respecting the fourth relation of (4) up to the first derivatives, only. Let us focus on the covariance calculation. Covariance of the real random variables x and y is defined as

$$\sigma_{xy}^2 = E\{(x - \mu_x)(y - \mu_y)\}. \tag{11}$$

It can be written for *i*-th and *j*-th random parameter in form

$$\sigma^2_{p_i p_j} = E\{(p_i - \mu_{p_i})(p_j - \mu_{p_j})\} = E\{(p_i - \bar{p}_i)(p_j - \bar{p}_j)\}. \tag{12}$$

This covariance can be understood as element of the so called covariance matrix which can be written in compact form

$$\boldsymbol{\Sigma}_{\mathbf{p}} = E\{(\mathbf{p} - \bar{\mathbf{p}})(\mathbf{p} - \bar{\mathbf{p}})^T\} = \begin{bmatrix} \sigma^2_{p_1}, & \sigma^2_{p_1 p_2}, & \cdots, & \sigma^2_{p_1 p_s} \\ \sigma^2_{p_2 p_1}, & \sigma^2_{p_2}, & \cdots, & \sigma^2_{p_2 p_s} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2_{p_s p_1}, & \sigma^2_{p_s p_2}, & \cdots, & \sigma^2_{p_s} \end{bmatrix} \in \mathbf{R}^{s,s}. \tag{13}$$

The covariance matrix of displacement vector will be time dependent, because the mean value is time dependent too. Taking into account (10) and the fourth relation of (4) we can write

$$\boldsymbol{\Sigma}_{\mathbf{q}}(t) = E\{[\mathbf{q}(t) - \bar{\mathbf{q}}(t)][\mathbf{q}(t) - \bar{\mathbf{q}}(t)]^T\} = E\left\{\sum_{i=1}^{s}\sum_{j=1}^{s} \frac{\partial \mathbf{q}(t)}{\partial p_i}\frac{\partial \mathbf{q}(t)}{\partial p_j}\Delta p_i \Delta p_j\right\} =$$

$$= \sum_{i=1}^{s}\sum_{j=1}^{s} \frac{\partial \mathbf{q}(t)}{\partial p_i}\frac{\partial \mathbf{q}(t)}{\partial p_j} E\{\Delta p_i \Delta p_j\} = \sum_{i=1}^{s}\sum_{j=1}^{s} \frac{\partial \mathbf{q}(t)}{\partial p_i}\frac{\partial \mathbf{q}(t)}{\partial p_j}\sigma^2_{p_i p_j}. \tag{14}$$

Because the covariance matrix of the random parameters is time independent the relation (14) can be written in compact form

$$\boldsymbol{\Sigma}_{\mathbf{q}}(t) = \frac{\partial \mathbf{q}(t)}{\partial \mathbf{p}} \boldsymbol{\Sigma}_{\mathbf{p}} \frac{\partial \mathbf{q}^T(t)}{\partial \mathbf{p}}, \tag{15}$$

where

$$\boldsymbol{\Sigma}_{\mathbf{q}}(t) = \begin{bmatrix} \sigma^2_{q_1}(t), & \sigma^2_{q_1 q_2}(t), & \cdots, & \sigma^2_{q_1 q_n}(t) \\ \sigma^2_{q_2 q_1}(t), & \sigma^2_{q_2}(t), & \cdots, & \sigma^2_{q_2 q_n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^2_{q_n q_1}(t), & \sigma^2_{q_n q_2}(t), & \cdots, & \sigma^2_{q_n}(t) \end{bmatrix} \in \mathbf{R}^{n,n} \tag{16}$$

and

$$\frac{\partial \mathbf{q}(t)}{\partial \mathbf{p}} = \begin{bmatrix} \frac{\partial q_1(t)}{\partial p_1}, & \frac{\partial q_1(t)}{\partial p_2}, & \ldots, & \frac{\partial q_1(t)}{\partial p_s} \\ \frac{\partial q_2(t)}{\partial p_1}, & \frac{\partial q_2(t)}{\partial p_2}, & \ldots, & \frac{\partial q_2(t)}{\partial p_s} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q_n(t)}{\partial p_1}, & \frac{\partial q_n(t)}{\partial p_2}, & \ldots, & \frac{\partial q_n(t)}{\partial p_s} \end{bmatrix} \in \mathbf{R}^{n,s}. \tag{17}$$

Applying the six sigma criterion we can roughly determine upper and lower limits of the state variables. It means that we have to determine firstly standard deviations of displacements as follows

$$\sigma_{q_i} = \sqrt{\sigma_{q_i}^2}, \quad i = 1,2,.......,n \tag{18}$$

and than we can write for approximate displacement limits relations

$$\mathbf{q}_{\min}(t) = \overline{\mathbf{q}}(t) - 3diag\left\{\sqrt{\mathbf{\Sigma}_{\mathbf{q}}(t)}\right\}, \qquad \mathbf{q}_{\max}(t) = \overline{\mathbf{q}}(t) + 3diag\left\{\sqrt{\mathbf{\Sigma}_{\mathbf{q}}(t)}\right\}, \tag{19}$$

$$diag\left\{\sqrt{\mathbf{\Sigma}_{\mathbf{q}}(t)}\right\} = \begin{bmatrix} \sigma_{q_1}(t) \\ \sigma_{q_2}(t) \\ \vdots \\ \sigma_{q_n}(t) \end{bmatrix}. \tag{20}$$

3. Linear system with random parameters

Equation of motion of such system has form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{B}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t). \tag{21}$$

Differentiating equation of motion with respect to i-th random parameter we can come to

$$\overline{\mathbf{M}}\ddot{\mathbf{q}}^{(i)}(t) + \overline{\mathbf{B}}\dot{\mathbf{q}}^{(i)}(t) + \overline{\mathbf{K}}\mathbf{q}^{(i)}(t) = \underbrace{\overline{\mathbf{f}}^{(i)}(t) - \mathbf{M}^{(i)}\ddot{\overline{\mathbf{q}}}(t) - \mathbf{B}^{(i)}\dot{\overline{\mathbf{q}}}(t) - \mathbf{K}^{(i)}\overline{\mathbf{q}}(t)}_{\mathbf{f}^*(t)}. \tag{22}$$

The one step method of integration (method of average acceleration) uses the following scheme

$$\dot{\mathbf{q}}_{t+\Delta t} = \dot{\mathbf{q}}_t + \frac{1}{2}\left(\ddot{\mathbf{q}}_t + \ddot{\mathbf{q}}_{t+\Delta t}\right), \quad \mathbf{q}_{t+\Delta t} = \mathbf{q}_t + \Delta t\dot{\mathbf{q}}_t + \frac{\Delta t^2}{4}\left(\ddot{\mathbf{q}}_t + \ddot{\mathbf{q}}_{t+\Delta t}\right). \tag{23}$$

Relations (23) express vectors of generalized velocities and displacements in following step by means of state variable vectors from foregoing time step and generalized acceleration vector in the following step. Substituting (23) into the equation of motion (21) for time $t + \Delta t$ we can write

$$\mathbf{A}\ddot{\overline{\mathbf{q}}}_{t+\Delta t} = \tilde{\overline{\mathbf{f}}}_{t+\Delta t}, \tag{24}$$

where

$$\mathbf{A} = \mathbf{M} + \frac{\Delta t}{2}\mathbf{B} + \frac{\Delta t^2}{4}\mathbf{K}, \quad \tilde{\overline{\mathbf{f}}}_{t+\Delta t} = \overline{\mathbf{f}}_{t+\Delta t} - \left(\mathbf{B} + \Delta t\mathbf{K}\right)\dot{\overline{\mathbf{q}}}_t - \mathbf{K}\overline{\mathbf{q}}_t - \left(\frac{\Delta t}{2}\mathbf{B} + \frac{\Delta t^2}{4}\mathbf{K}\right)\ddot{\overline{\mathbf{q}}}_t. \tag{25}$$

Then we solve the equation (24) for $\ddot{\mathbf{q}}_{t+\Delta t}$ in each time step and simultaneously calculate the generalized velocity and displacement vector in the same time according to (23). Initial vectors of the generalized displacements $\mathbf{q}(0)=\mathbf{q}_0$ and velocities $\dot{\mathbf{q}}(0)=\dot{\mathbf{q}}_0$ are known. It is still necessary to determine the initial vector of accelerations. It can be achieved by substitution $t = 0$ into the equation of motion (21) as follows

$$\ddot{\mathbf{q}}_0 = \overline{\mathbf{M}}^{-1}\left(\bar{\mathbf{f}}_0 - \overline{\mathbf{B}}\dot{\mathbf{q}}_0 - \overline{\mathbf{K}}\mathbf{q}_0\right). \tag{26}$$

In the same time we solve the sensitivity problem for derivative of the state vectors with respect to the i-th random parameter. The procedure is the same except $\mathbf{f}(t)$ which must be replaced by $\mathbf{f}^*(t)$ defined in (22). Then we can solve the equations

$$\mathbf{A}\ddot{\mathbf{q}}_{t+\Delta t}^{(i)} = \mathbf{b}_{t+\Delta t}^{(i)},\ i = 1,2,....,s, \tag{27}$$

where

$$\mathbf{b}_{t+\Delta t}^{(i)} = \underbrace{\bar{\mathbf{f}}_{t+\Delta t}^{(i)}(t) - \mathbf{M}^{(i)}\ddot{\bar{\mathbf{q}}}_{t+\Delta t}(t) - \mathbf{B}^{(i)}\dot{\bar{\mathbf{q}}}_{t+\Delta t}(t) - \mathbf{K}^{(i)}\bar{\mathbf{q}}_{t+\Delta t}(t)}_{\mathbf{f}_{t+\Delta t}^{*}(t)} - \left(\overline{\mathbf{B}} + \Delta t\overline{\mathbf{K}}\right)\dot{\mathbf{q}}_t^{(i)} - \mathbf{K}\mathbf{q}_t^{(i)} - \left(\frac{\Delta t}{2}\overline{\mathbf{B}} + \frac{\Delta t^2}{4}\overline{\mathbf{K}}\right)\ddot{\mathbf{q}}_t^{(i)} \tag{28}$$

In each time step we then calculate the sensitivity state variables

$$\dot{\mathbf{q}}_{t+\Delta t}^{(i)} = \dot{\mathbf{q}}_t^{(i)} + \frac{1}{2}\left(\ddot{\mathbf{q}}_t^{(i)} + \ddot{\mathbf{q}}_{t+\Delta t}^{(i)}\right), \tag{29}$$

$$\mathbf{q}_{t+\Delta t}^{(i)} = \mathbf{q}_t^{(i)} + \Delta t\dot{\mathbf{q}}_t^{(i)} + \frac{\Delta t^2}{4}\left(\ddot{\mathbf{q}}_t^{(i)} + \ddot{\mathbf{q}}_{t+\Delta t}^{(i)}\right). \tag{30}$$

The initial conditions of the sensitivity problem are independent of random parameters and then

$$\mathbf{q}^{(i)}(0) = \mathbf{0},\ \dot{\mathbf{q}}^{(i)}(0) = \mathbf{0}. \tag{31}$$

From this reason the derivative of the initial acceleration vector with respect of the i-th random parameter can be expressed as

$$\ddot{\mathbf{q}}_0^{(i)} = \overline{\mathbf{M}}^{-1}\left[\bar{\mathbf{f}}^{(i)}(0) - \mathbf{M}^{(i)}\ddot{\bar{\mathbf{q}}}(0) - \mathbf{B}^{(i)}\dot{\bar{\mathbf{q}}}(0) - \mathbf{K}^{(i)}\bar{\mathbf{q}}(0)\right], \tag{32}$$

which can be rewritten respecting (26) into form

$$\ddot{\mathbf{q}}_0^{(i)} = \overline{\mathbf{M}}^{-1}\left[\bar{\mathbf{f}}_0^{(i)} - \mathbf{M}^{(i)}\overline{\mathbf{M}}^{-1}\left(\bar{\mathbf{f}}_0 - \overline{\mathbf{B}}\dot{\bar{\mathbf{q}}}_0 - \overline{\mathbf{K}}\bar{\mathbf{q}}_0\right) - \mathbf{B}^{(i)}\dot{\bar{\mathbf{q}}}_0 - \mathbf{K}^{(i)}\bar{\mathbf{q}}_0\right]. \tag{33}$$

The number of equations (27) to be solved is equal to s.

4. Example

Let us present very simple example. Vibrating system with 2 DOF is depicted in fig. 1. We take into account two cases of excitation. In both cases the random parameters are stiffness of the springs which have the uniform probability density function. The first case is the harmonic excitation and the second one is the random excitation - white noise independent of the random parameters. But this noise will be the same for all realizations. The responses in the place "one" are depicted in fig. 2 and 3, respectively. The displacement $\bar{q}_1(t)$

corresponding to the mean values of random parameters and upper and lower limits are depicted at the left hand sides of both figures. The families of responses modelled by the Monte Carlo method for randomly changing parameters of stiffness are depicted at the right hand side of both figures.

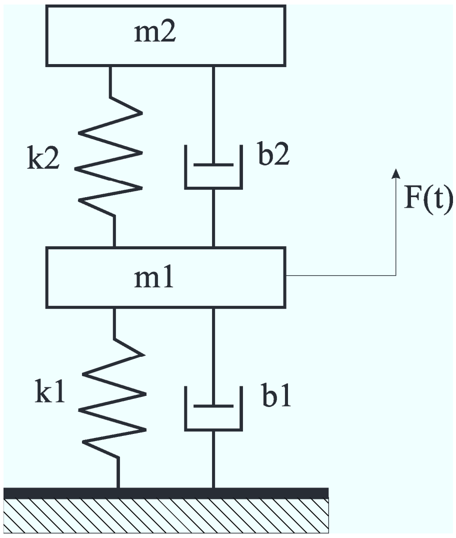

Fig. 1. Simple linear system with 2 DOF.

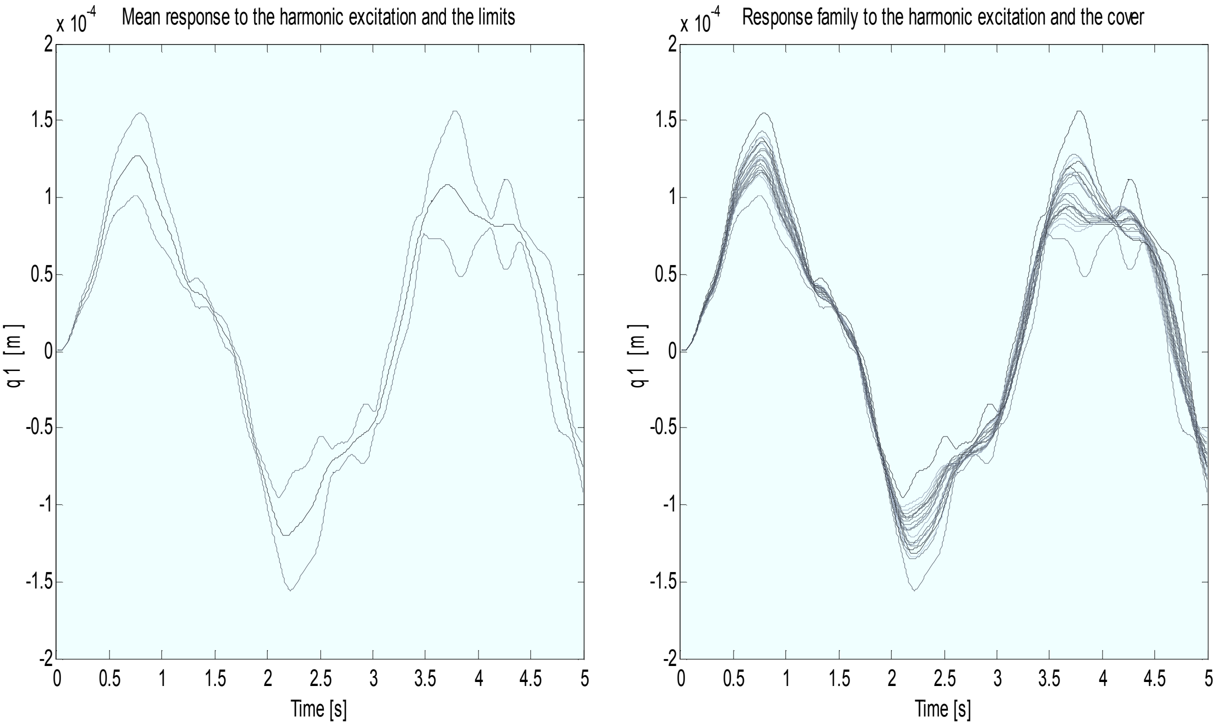

Fig. 2. Response to harmonic excitation.

In case of process $x(t)$ having normal probability density function, probability

$$P\{\mu_x - 3\sigma_x \le x(t) \le \mu_x + 3\sigma_x\} = 0.9973 \ . \tag{34}$$

Despite the dynamic response does not have generally normal probability density function we can see in figs. 2 and 3 that the estimation accuracy of upper and lower limits is sufficient.

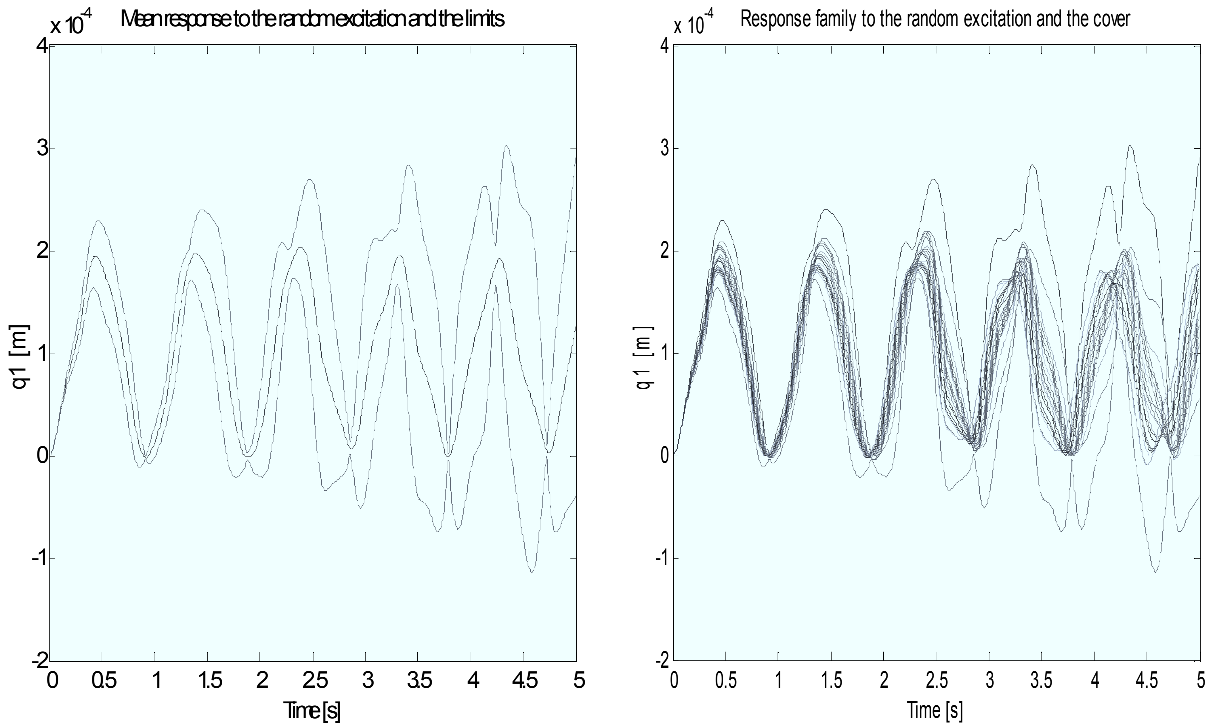

Fig. 3. Response to the harmonic excitation with noise.

5. Conclusion

The paper deals with approach to the response limit evaluating of vibrating systems. This approach is applied to the eigenvalue problem, harmonic amplitudes of the displacements (both for linear systems), transient response (linear and nonlinear systems) and linear static problems. The static problems can be solved exactly in the direct way by the transformation of probability density function of the random parameters to those one of the generalized displacements but the application of perturbation or gradient method is much more simpler.

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References

[1] J. Dupal, J. Vimmr, J. Králíček. Perturbation approach to the solution to eigenvalues and vibration amplitudes of the vertical model of the sitting man. Report to the project FT-TA/024 of the Ministry of Industry and Trade (In Czech).

[2] J. Dupal, J. Vimmr, J. Králíček. The use of a perturbation approach to the response statistical parameter determination of the linear dynamic systems containing random parameters in transient state. Report to the project FT-TA/024 of the Ministry of Industry and Trade (In Czech).

[3] M. Kleiber, T.D. Hien. The Stochastic Finite Element Method. John Wiley & Sons, 1992.