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Numerical computing of elastic homogenized coefficients for periodic fibrous tissue

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Abstract

The homogenization theory in linear elasticity is applied to a periodic array of cylindrical inclusions in rectangular pattern extending to infinity in the inclusions axial direction, such that the deformation of tissue along this last direction is negligible. In the plane of deformation, the homogenization scheme is based on the average strain energy whereas in the third direction it is based on the average normal stress along this direction. Namely, these average quantities have to be the same on a Repeating Unit Cell (RUC) of heterogeneous and homogenized media when using a special form of boundary conditions forming by a periodic part and an affine part of displacement. It exists an infinity of RUCs generating the considered array. The computing procedure is tested with different choices of RUC to control that the results of the homogenization process are independent of the kind of RUC we employ. Then, the dependence of the homogenized coefficients on the microstructure can be studied. For instance, a special anisotropy and the role of the inclusion volume are investigated. In the second part of this work, mechanical traction tests are simulated. We consider two kinds of loading, applying a density of force or imposing a displacement. We test five samples of periodic array containing one, four, sixteen, sixty-four and one hundred of RUCs. The evolution of mean stresses, strains and energy with the numbers of inclusions is studied. Evolutions depend on the kind of loading, but not their limits, which could be predicted by simulating traction test of the homogenized medium.

Keywords: homogenization, elasticity, fibres, plane strain, Comsol Multiphysics modelling

1. Introduction

How a homogenized medium can be substituted to a nonhomogeneous material is of interest of a lot of research and industrial branches. The prediction of effective moduli of heterogeneous materials has been the topic of many works. The pioneers' contributions are surely the works of Hashin and Shtrikman [4] and Hill [5]. In the 1970s, significant progress was made with the mathematical theory of asymptotic homogenization of heterogeneous bodies formed by elastic, linear materials [1, 13]. That theory was later extended to nonlinear elasticity, with the work of Suquet [15] for instance, and even applied to nonelastic phenomena, e.g. to plasticity [16], or to physical couplings, e.g. to piezoelectric materials [11].

Concept of homogenization is attached with the notion of Representative Volume Element (RVE) and Repeating Unit Cell (RUC). RVEs are used for randomly heterogeneous materials with statically homogeneous microstructure (as concrete for instance). RUCs are used for the periodic structures (as textiles for instance). The work of Drago and Pindera [3] compares the theories of homogenization done on RVE and RUC. They determined how many inclusions the

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RVE must contain to be equivalent to a RUC (when the studied material is a regular array). Homogenization states that the average strain energy is the same on the representative piece of homogenized or heterogeneous material when both the homogenized and the heterogeneous samples are submitted to the same special boundary conditions. Differences between using RVE or RUC are due to the kind of boundary conditions that are applied. RVEs are studied under homogeneous boundary conditions whereas RUCs are studied using boundary conditions having a periodic part and a homogeneous part. Among other more complicated materials, Sanchez-Palencia [14] applied the notion of RUC to linearized elastic material. He obtained a determinist form of the homogenized coefficients from the energy calculated when mixing singles modes. Single modes are defined by choosing simple forms (e.g. pure stretch or shear) of the homogeneous part of the boundary conditions.

The aim of this contribution is the application of the homogenization method to a periodic media which microstructure is constituted of infinitely long cylindrical inclusions. This work is motivated by the mechanical modelling of fibrous tissues, in view of a further application to the human smooth muscle tissue whose modelling is of interest of the authors and their colleagues from several years [6, 7]. For sure, the structure of muscular tissues is more complex than the one of perfectly periodic tissues. However, as a first approximation of muscles, the assumption of periodicity is used in the present contribution. Employing numerical tools including the Finite Element method, we propose a computational procedure for calculating the homogenized coefficients and validating the homogenization theory for periodic fibrous tissues.

This paper is organized as follows. The sections two to four deal with the theoretical background. In the second section, the theory of elasticity regarding special material symmetries (orthotropy and isotropy) and deformation particularities (plane strain) is recalled. In the third section, we introduce the notion of RUC and we specify the boundary conditions. We give the scheme of homogenization in the fourth section. The next sections deal with our modelling. The problem setting is detailed in the fifth section. The sixth section deals with the numerical way of computing the homogenized coefficients. As a result, the dependence of the homogenized coefficients on the microstructure is studied. A special anisotropy and the role of the inclusion volume are investigated. In the seventh section, mechanical tests on periodic materials are simulated. Traction tests applying a density of force or imposing a displacement are done on samples containing various numbers of inclusions. We study the limits when the number of inclusions becomes large and we compare them with the results obtained from a sample of homogenized material. The paper finishes with conclusions in the eighth section.

2. Elastic materials, orthotropy and plane strain

Elastic material behaves accordingly with the Hooke's law [8],

$$\boldsymbol{\sigma} = \mathbf{D}\,\boldsymbol{\epsilon}, \tag{1}$$

where $\mathbf{D}$ is a fourth rank tensor of elastic stiffness obeying the symmetry relations $D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij}$ due to the symmetry of the stress and strain tensors $\sigma_{ij} = \sigma_{ji}$ and $\epsilon_{kl} = \epsilon_{lk}$ $(i, j, k, l = 1, 2, 3)$. In general three dimensional modelling, there are twenty-one independent components.

In case of *orthotropy* (i.e. if there are three perpendicular planes of elastic symmetry), then $D_{iikl} = 0$ if $k \neq l$, $D_{ijkk} = 0$ if $i \neq j$, and $D_{ijkl} = 0$ if $i \neq k$ and $j \neq l$. The number of remaining independent components is nine. Restricting the study to the case of plane strain where the plane of deformation corresponds to a plane of elastic symmetry (for instance $\epsilon_{3j} = 0$,

$\forall j$), the determination of the components D_{1313}, D_{2323} and D_{3333} is not necessary because they are multiplied by zero at every time. Six independent components remain. The first four ones enter the plane equations, which ones we can rewrite

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} D_{1111} & D_{1122} & 0 \\ D_{1122} & D_{2222} & 0 \\ 0 & 0 & D_{1212} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{pmatrix}, \tag{2}$$

stocking the plane strain and the plane stress components in three components vectors. The last two independent components of the tensor D enter in the equation giving the expression of the stress along the third direction,

$$\sigma_{33} = D_{1133}\,\epsilon_{11} + D_{2233}\,\epsilon_{22}. \tag{3}$$

The shear stresses σ_{13} and σ_{23} vanish under such conditions (orthotropic material, plane strain).

If the two planes of elastic symmetry perpendicular to the plane of deformation are equivalent, then the indices 1 and 2 are interchangeable. Thus $D_{1111} = D_{2222}$. The number of independent coefficients in the plane of deformation is three. In the third direction, the coefficients obey the relation $D_{1133} = D_{2233}$. A fourth independent coefficient is necessary to determine the perpendicular stress σ_{33}.

The above case has to be distinguished from the three dimensional *transverse isotropy* for which the additional cubic symmetry

$$D_{1212} = \frac{1}{2}\left(D_{1111} - D_{1122}\right), \tag{4}$$

holds, linking the shear coefficient D_{1212} to the traction ones D_{1111} and D_{1122}. The number of independent coefficients in the plane of deformation reduces to two. These coefficients are called the Lamé's coefficients. By convention, we use the symbols λ and μ, i.e. $\lambda \equiv D_{1122}$ and $\mu \equiv D_{1212}$. The behaviour law (2) becomes

$$\sigma_{ij} = \lambda \mathrm{tr}\boldsymbol{\epsilon}\delta_{ij} + 2\mu\epsilon_{ij}, \quad i, j = 1, 2. \tag{5}$$

Note that the additional independent coefficient D_{1133} is still necessary to determine the perpendicular stress σ_{33}.

If the isotropy of material is also respected along the third direction, then this last coefficient is nothing else but λ (i.e. $D_{iijj} = \lambda$ if $i \neq j$, $i, j = 1, 2, 3$). In this case, the Lamé's coefficients can be related to the engineering constants, i.e. the Young's modulus, E, and the Poisson's ratio, ν, thanks to the relationships,

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}. \tag{6}$$

3. Periodic media

A periodic structure is a heterogeneous body for which a *repeating unit cell* (RUC) can be defined. A periodic structure is in fact an array formed by RUCs assembled together side by side. Each RUC is indistinguishable from the next. It is not necessary to study the behaviour of the whole array to determine the mechanical characteristics of the medium. Indeed, they can be determined by the study on a single arbitrarily chosen RUC (the 'reference' RUC).

Consider a sample of periodic material. We study the deformation process resulting when imposing an affine displacement of the boundary of this sample. The local deformation, $\boldsymbol{\epsilon}$, at a point $\mathbf{x}$ within the reference RUC can be split into the overall deformation, $\boldsymbol{\Lambda}$, which would be the current one if the RUC were homogeneous, and a correction field, $\hat{\epsilon}$, which accounts for the presence of heterogeneities, i.e.

$$\boldsymbol{\epsilon} = \hat{\boldsymbol{\epsilon}} + \boldsymbol{\Lambda}. \tag{7}$$

Because of the affine boundary conditions, the overall deformation $\boldsymbol{\Lambda}$ is constant and is equal to the mean deformation [9], i.e.

$$\boldsymbol{\Lambda} = \frac{1}{V} \int_V \boldsymbol{\epsilon}(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \tag{8}$$

$(V \equiv \int_V d\mathbf{x}$ being the volume of the RUC) while the average of the correction $\hat{\boldsymbol{\epsilon}}$ vanishes,

$$\frac{1}{V} \int_V \hat{\boldsymbol{\epsilon}}(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 0. \tag{9}$$

That means that $\hat{\boldsymbol{\epsilon}}$ derives from a periodic displacement field, i.e. $\hat{\boldsymbol{\epsilon}} \equiv \boldsymbol{\epsilon}(\hat{\mathbf{u}}(\mathbf{x}))$ $\left(= \frac{1}{2} \left(\nabla \hat{\mathbf{u}} + \nabla^t \hat{\mathbf{u}}\right)\right)$ where $\hat{\mathbf{u}}$ is periodic. That is

$$\boldsymbol{\epsilon}(\mathbf{u}(\mathbf{x})) = \boldsymbol{\epsilon}(\hat{\mathbf{u}}(\mathbf{x})) + \boldsymbol{\Lambda}. \tag{10}$$

The form of the displacement field is obtained by integrating (10). A decomposition between a periodic part and an affine part is obtained,

$$\mathbf{u}(\mathbf{x}) = \hat{\mathbf{u}}(\mathbf{x}) + \boldsymbol{\Lambda}.\mathbf{x}. \tag{11}$$

Consider a particular point $\mathbf{x}_0$ in the arbitrarily chosen reference RUC and its corresponding point in a neighbouring RUC (i.e. the point with the same local position within the RUC). The points are separed by a vector, $\mathbf{d}$, of translation invariance. Since $\hat{\mathbf{u}}$ is periodic, then $\hat{\mathbf{u}}(\mathbf{x}_0 + \mathbf{d}) = \hat{\mathbf{u}}(\mathbf{x}_0)$, and the decomposition (11) is equivalent to

$$\mathbf{u}(\mathbf{x}_0 + \mathbf{d}) = \mathbf{u}(\mathbf{x}_0) + \boldsymbol{\Lambda} \cdot \mathbf{d}. \tag{12}$$

Because RUCs are assembled together side by side, borders are shared by neighbouring RUCs. The image by translation $\mathbf{d}$ of one side of the reference RUC in the neighbouring RUC coincides with the opposite side of the reference RUC. In other words, if $\mathbf{x}_0$ is on one side of the RUC, then $\mathbf{x}_0 + \mathbf{d}$ is on the opposite side of the RUC. Equation (12) with $\mathbf{x}_0$ on the border of the RUC will be used in following, when considering the problem of mechanical equilibrium at the micro-level (i.e. within the RUC), to define the special boundary conditions of the RUC.

4. Homogenization of periodic elastic structures

The formula we use to calculate the effective elastic coefficients (i.e. the values of the components of the stiffness tensor $\mathbf{D}^H$ of equivalent homogeneous material) was obtained by Sanchez-Palencia [14] as follows

$$D^H_{rspq} = \frac{1}{V} \int_V D_{ijkl} \, \epsilon^{(rs)}_{ij} \, \epsilon^{(pq)}_{kl} \, \mathrm{d}\mathbf{x}. \tag{13}$$

In Eq. (13) the subscripts correspond to the tensor coordinates and the rule of summation on repeated indices holds. The superscripts in parentheses are used to define the mode on which the test deformation is calculated. For instance, $\boldsymbol{\epsilon}^{(rs)}$ is the deformation field within a RUC which is solicited under the mode "rs" of loading. More precisely, the field $\boldsymbol{\epsilon}^{(rs)}$ is the local field of deformation within the RUC imposing boundary conditions of type (12) where $\boldsymbol{\Lambda}$ is replaced by a given tensor $\boldsymbol{\Lambda}^{(rs)}$. That is $\boldsymbol{\epsilon}^{(rs)} \equiv \boldsymbol{\epsilon}(\mathbf{u}^{(rs)})$ where $\mathbf{u}^{(rs)}$ is the solution of the problem of mechanical equilibrium

$$\begin{aligned} &\nabla.(\mathbf{D}(\mathbf{x}) : \boldsymbol{\epsilon}(\mathbf{u}^{(rs)}(\mathbf{x}))) = 0, \quad \mathbf{x} \in V, \\ &\mathbf{u}(\mathbf{x}_0 + \mathbf{d}) = \mathbf{u}(\mathbf{x}_0) + \boldsymbol{\Lambda}^{(rs)}.\mathbf{d}, \quad \mathbf{x}_0 \in \partial V. \end{aligned} \tag{14}$$

For the two dimensional modelling, there are typically three single modes of loading. They are defined through the overall deformation $\boldsymbol{\Lambda}^{(rs)}$, with

$$\begin{aligned} &\text{if } \ r, s = 1 \ \text{ then } \ \boldsymbol{\Lambda}^{(11)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ &\text{if } \ r, s = 1, 2 \ \text{ then } \ \boldsymbol{\Lambda}^{(12)} = \boldsymbol{\Lambda}^{(21)} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \\ &\text{if } \ r, s = 2 \ \text{ then } \ \boldsymbol{\Lambda}^{(22)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{15}$$

For illustration, consider a homogeneous RUC solicited under the mode "rs". Since the medium is homogeneous, the correction in deformation $\hat{\boldsymbol{\epsilon}}^{(rs)}$ vanishes. Thus, the local field of deformation is equal to the overall one, i.e. $\boldsymbol{\epsilon}^{(rs)} = \boldsymbol{\Lambda}^{(rs)}$. Noticing from (15) that $\Lambda_{ij}^{(rs)} = \frac{1}{2}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})$, the relation (13) leads without surprise to $\mathbf{D}^H = \mathbf{D}$, i.e. the homogenized stiffness of a homogeneous material is its true stiffness.

5. Problem setting

We consider a periodic array of cylindrical inclusions with elliptic cross-section in a rectangular pattern extending to infinity in the plane transverse to the inclusion axis (say the third spatial direction), as shown in fig. 1. The assumption of *plane strain* is suitable for such material. The deformation components ϵ_{13}, ϵ_{23} and ϵ_{33} remain equal to zero at any time. Moreover, this material is *orthotropic*: it admits three perpendicular planes of elastic symmetry. This considerably reduces the number of independent stiffness coefficients as explained in section 2. In the cross-section of the array, there exits an infinity of RUCs generating the two dimensional lattice. Some of them are illustrated in fig. 2. The smallest one is a rectangle with a single inclusion in the center, or a rectangle with four quarters of inclusions at corners, or even a rectangle with two halfs of inclusions at opposite sides. Larger RUCs containing more full inclusions or more portions of inclusions can be also defined. To be relevant, the homogenization theory has to lead to the same homogenized coefficients independently of the kind of RUC we employ. One of our aims is to test the homogenization procedure comparing the value of homogenized coefficients for various kinds of RUC.

In the second part of this work, we compare the behaviour of a sample of homogenized material with the real behaviour of a sample of periodic array in dependence on the number of RUCs contained in the sample. We verify that the larger the number of RUCs is (i.e. the smaller the dimension of the RUC is, if we consider that the dimension of the sample is constant) the

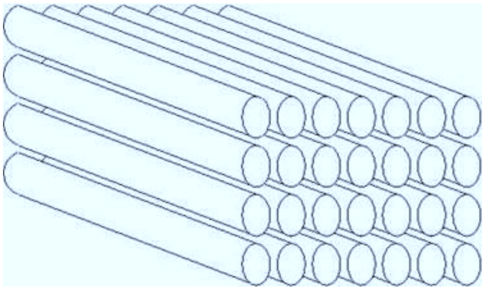

Fig. 1. Periodic fibrous tissues

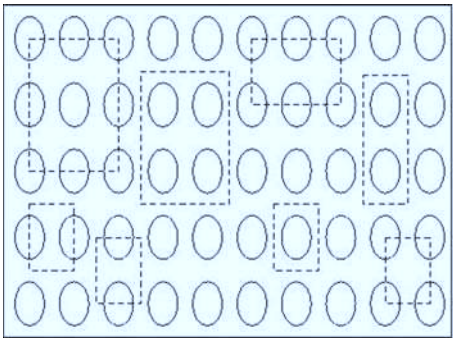

Fig. 2. Various choices of RUC in the cross-section

smaller the error done when using the homogenized medium instead of the heterogeneous one is.

The two phases of the considered heterogeneous material are elastic and isotropic. Their engineering constants are denoted E^i and ν^i for the inclusions and E^m and ν^m for the matrix. These moduli are related to the local stiffness coefficients $D_{ijkl}(\mathbf{x})$ ($\mathbf{x} \in V$ is the position within the RUC) by the intermediary of the Lamé's coefficients as developed in section 2.

6. Homogenization computing

6.1. Plane strain hypothesis

The homogenization process is done under the assumption of plane strain. The plane restriction of the homogenization scheme (13) yields to four independent coefficients D^H_{1111}, D^H_{1212} D^H_{2222} and D^H_{1122}. If the geometry of the array respects symmetry, i.e. if the RUC is square and the cross-section of the inclusion is round, then this number decreases to three (the indices 1 and 2 are interchangeable). However, in every case, the shear coefficient remains independent from the others (the cubic symmetry (4) does not hold for the considered fibres arrangement). Because of the plain strain assumption, the stress component σ_{33} does not enter the expression of stored strain energy. To determine the homogeneous coefficients D^H_{ii33} we use a new assumption. We claim that the mean stress $< \sigma_{33} > \left(\equiv \frac{1}{V} \int_V \sigma_{33}\, \mathrm{d}v \right)$ has to be the same for both the heterogeneous and the homogeneous RUC under every single mode of special boundary conditions. We obtain the second homogenization scheme

$$D^H_{ii33} = \frac{1}{V} \int_V D_{jj33} \epsilon^{(ii)}_{jj}\, \mathrm{d}v, \quad i = 1, 2. \tag{16}$$

where the subscripts correspond to the tensor coordinates and the superscripts in parentheses refer to the single mode of boundary conditions.

6.2. Numerical methods for the determination of the homogenized coefficients

The problem of mechanical equilibrium of heterogeneous structures is not analytically solvable in general case. This is the reason why we use the Finite Element Method. The commercial code Comsol Multiphysics [2] is employed because of its ability to model boundary conditions of the form (12). Three finite-element analysis are performed on an isolated RUC to calculate

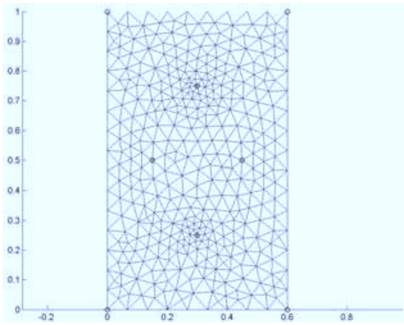

Fig. 3. Automatic Comsol's mesh

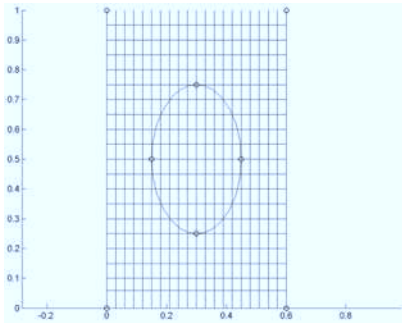

Fig. 4. Rectangular spatial indentation

the fields $\boldsymbol{\epsilon}^{(rs)}$ as solutions of the mechanical problem (14) with "rs" = "11","12" and "22", respectively. The mesh we use is the one automatically generated by Comsol Multiphysics. It is formed by triangles as shown in fig. 3. On the other hand, the integrals (13) and (16) are numerically approximated as the sum of areas,

$$\int\int g(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sum_n \sum_m (g(x^{n+1},y^{m+1}) + g(x^{n+1},y^m) + g(x^n,y^{m+1}) + g(x^n,y^m))\frac{\delta x \delta y}{4}, \tag{17}$$

where $g(x,y) \equiv D_{ijkl}(x,y)\,\epsilon_{ij}^{(rs)}(x,y)\,\epsilon_{kl}^{(pq)}(x,y)$, or $g(x,y) \equiv D_{jj33}(x,y)\;\epsilon_{jj}^{(rr)}(x,y)$, with $r,s,p,q = 1,2$. In the numerical scheme (17) we use a regular spatial indentation with constant increments δx and δy, i.e. $x^{n+1} = x^n + \delta x$ and $y^{m+1} = x^m + \delta y$. This rectangular indentation does not coincide with the mesh employed by the Finite Element analysis, see fig. 4. An approximation of $\epsilon_{ij}^{(rs)}(x^n,y^m)$ is obtained as the weighted average of $\epsilon_{ij}^{(rs)}(x^{mesh},y^{mesh})$,

$$\epsilon_{ij}^{(rs)}(x^n,y^m) \approx \frac{\displaystyle\sum_{neighb.} w_{mesh}^{n,m}\;\;\epsilon_{ij}^{(rs)}(x^{mesh},y^{mesh})}{\displaystyle\sum_{neighb.} w_{mesh}^{n,m}}, \tag{18}$$

where (x^n,y^m) is a point of the regular indentation and (x^{mesh},y^{mesh}) are nodes from the Comsol mesh. The sums $\displaystyle\sum_{neighb.}$ act on all nodes (x^{mesh},y^{mesh}) which are sited in the near neighbouring of (x^n,y^m). Here, we consider the disk centered on (x^n,y^m) which radius is equal to $\Delta = (\delta x^2 + \delta y^2)^{1/2}$. If no node of the mesh is sited within the disk, the radius is rather chosen as $\Delta = 2(\delta x^2 + \delta y^2)^{1/2}$. If no node is even sited within this second disk, the radius is chosen as $\Delta = 3(\delta x^2 + \delta y^2)^{1/2}$, etc. The maximum distance separating a node (x^{mesh},y^{mesh}) sited in the disk from the center (x^n,y^m) is the radius Δ. The averaging (18) is weighted by the difference between this maximal distance and the current distance separating the considered mesh's node from (x^n,y^m), i.e. the weight $w_{mesh}^{n,m}$ is chosen as

$$w_{mesh}^{n,m} \equiv \Delta - \sqrt{(x^n - x^{mesh})^2 + (y^m - y^{mesh})^2}. \tag{19}$$

6.3. Results of homogenization

A Matlab function calculating the homogenized coefficients is generated by the way described in subsection 6.2. We obtain the evolution of homogenized stiffness coefficients in dependence

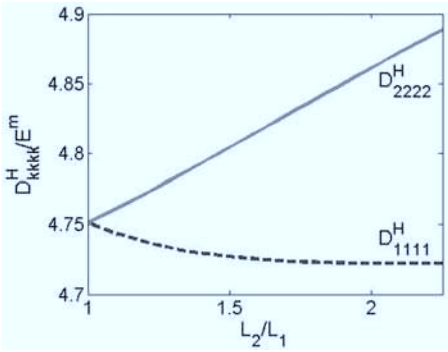

Fig. 5. Dependence of the homogenized traction coefficients D^H_{1111} and D^H_{2222} on the anisotropy parameter L_2/L_1 with $r_1/L_1 = r_2/L_2 = 0.25$, $E^i/E^m = 10$ and $\nu^i = \nu^m = 0.45$

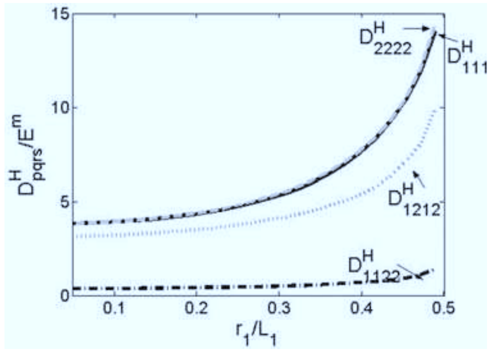

Fig. 6. Dependence of the homogenized traction coefficients $D^H_{ijkl}, i, \ldots, l = 1, 2$, on the inclusion size. With $L_1/L_2 = r_1/r_2 = 1/2$, $E^i/E^m = 10$, and $\nu^i = \nu^m = 0.45$

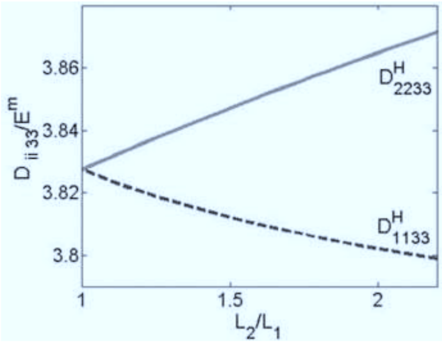

Fig. 7. Dependence of the homogenized traction coefficients D^H_{1133} and D^H_{2233} on the anisotropy parameter L_2/L_1 with $r_1/L_1 = r_2/L_2 = 0.25$, $E^i/E^m = 10$ and $\nu^i = \nu^m = 0.45$

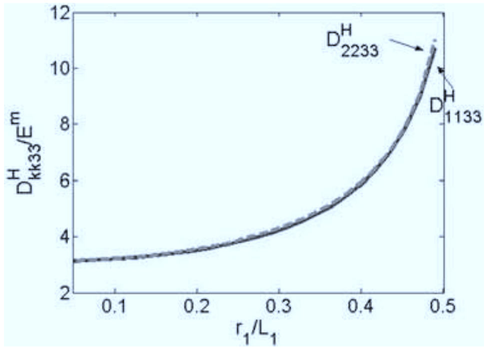

Fig. 8. Dependence of the homogenized traction coefficients D^H_{1133} and D^H_{2233} on the inclusion size. With $L_1/L_2 = r_1/r_2 = 1/2$, $E^i/E^m = 10$ and $\nu^i = \nu^m = 0.45$

on various material parameters as plotted in figs. 5 to 8 for instance. We consider a regular periodic structure having the following property. At the rest configuration, the geometry of the inclusion is proportional to its space distribution. The ratio $h \equiv r_i/L_i$ is independent of the space orientation i, i.e. $r_2/L_2 = r_1/L_1$, where r_1 and r_2 are the radii of the elliptic inclusion and L_1 and L_2 are the distances separating two neighbouring inclusions in the first and the second direction, respectively. The anisotropy of the array is uniquely characterized by the ratio $a \equiv L_2/L_1 = r_2/r_1$. Its influence on the homogenized coefficients is studied. We observe that the cross traction coefficient D^H_{1122} and the shear one D^H_{1212} are unchanged by moving the anisotropy parameter. On the contrary, the traction coefficients D^H_{1111} and D^H_{2222} and the cross traction coefficients D^H_{1133} and D^H_{2233} move. When L_2/L_1 increases, D^H_{1111} and D^H_{1133} decrease whereas D^H_{2222} and D^H_{2233} increase. Moreover, D^H_{2222} and D^H_{2233} linearly increase with L_2/L_1 as it is highlighted in figs. 5 and 7. In figs. 6 and 8, the influence of the size of the inclusion on the homogenized coefficient is studied. The inclusion was chosen stiffer than the matrix. As a consequence, when the volume of the inclusion increases, the values of the homogenized coefficients increase too. Note that the differences between the traction coefficients $D^H_{2222} - D^H_{1111}$ and $D^H_{2233} - D^H_{1133}$ are small in comparison to the difference between the other coefficients. However, these differences are not zero since the considered RUC is not symmetric ($L_1 \neq L_2$). Curves in figs. 6 and 8 are close each other but they do not coincide. The results shown in figs. 5 to 8 were obtained equivalently using any kind of RUC. It has been verified that these results do no depend whether the RUC is centered on the inclusion or not. They neither depend whether it contains one inclusion or more.

7. Traction tests

7.1. Test settings

Five samples of periodic array containing one, four, sixteen, sixty-four and one hundred RUCs and a sample of homogenized material are considered. RUCs are square. They are centered on the round inclusion. The ratio between the length of the RUC side, L, and the radius of the inclusion, r, is chosen as $L/r = 0.25$. The dimensions of all samples are the same. Thus, the RUC dimensions decrease when the number of inclusions increases. Engineering constants are $E^i = 4 \cdot 10^4$ Pa, $E^m = 4 \cdot 10^3$ Pa, $\nu^i = \nu^m = 0.45$ for heterogeneous samples. The homogenized material is characterized by $\boldsymbol{\sigma} = \mathbf{D^H}\, \boldsymbol{\epsilon}$, where $\mathbf{D^H}$ has been calculated with the homogenization process developed in section 6. Traction tests are simulated by applying a density of force or imposing a displacement on horizontal borders. Because of the symmetry of the sample and the loading, only one quarter of the sample is studied. Consider the right up quarter. The boundary conditions are the following. A density of force, fi, or a displacement, vi, is applied on the top border. The right border is free of stress. Because of the symmetries, mixed boundary conditions are applied to remaining borders. We apply $v = 0$, $\sigma_{11} = 0$ on the bottom border, whereas $u = 0$, $\sigma_{22} = 0$ on the left border. Loadings vi and fi are determined in such a way that the strain energy for homogenized material is the same in both kinds of traction tests. To carry out the computing, the commercial finite element code Comsol Multiphysics is employed. Differential equations are input according to inner code modes. *Structural mechanics plane strain application* mode is used for heterogeneous samples, which can study the displacements, stresses, and strains in an in-plane loaded body assuming plane strain when engineering constants are given. *Coefficient form* mode is used for homogenized materials, which solves the system of linear differential equations when coefficients entering equations are given. More details about modes can be found in the Comsol manual [2]. Applications of the *coefficient form* mode to the mechanical equilibrium can be found e.g. in [10] (isotropic viscoelastic material) and in [12] (orthotropic elastic material).

7.2. Results of traction tests

Figs. 9 and 10 show strain energy density distributions for every sample. With the chosen material constants, the minimum of energy is localized in inclusions. The deformed shape of samples can be also observed in these figures. The deformation on the border of samples is not homogeneous but waved by the presence of inclusions (however, when a displacement is applied on the top border, deformation is homogeneous on this border from definition).

Mean stresses, strain and total strain energy are calculated for every sample. We use the superscript H for quantities measured in homogenized sample, e.g. ϵ^H_{11}, σ^H_{22}. For all samples, the mean stress $\langle\sigma_{11}\rangle$ and the mean deformation in shear $\langle\epsilon_{12}\rangle$ are zero. Following observations are done in the direction of loading. When a density force is applied, the mean stress $\langle\sigma_{22}\rangle$

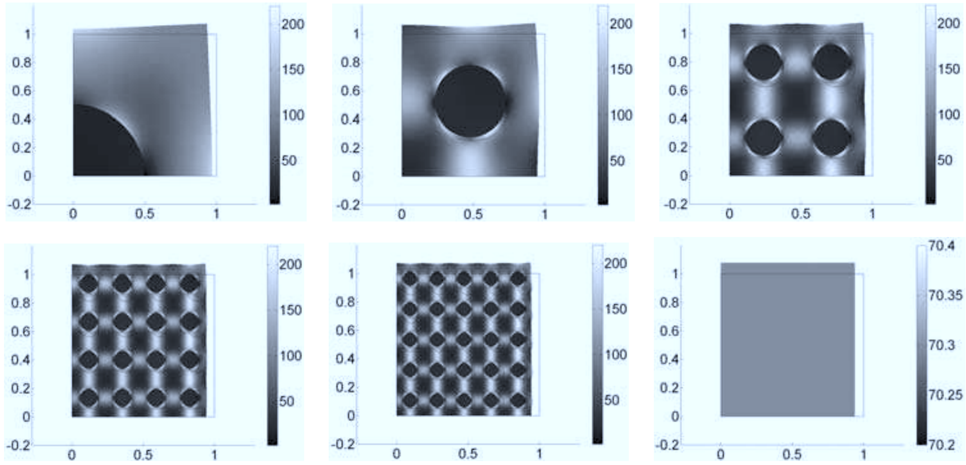

Fig. 9. Strain energy density distribution after a traction test applying the density force $fi = 10^3$ Pa in samples of one, four, sixteen, sixty-four and one hundred inclusions and the sample of homogenized material

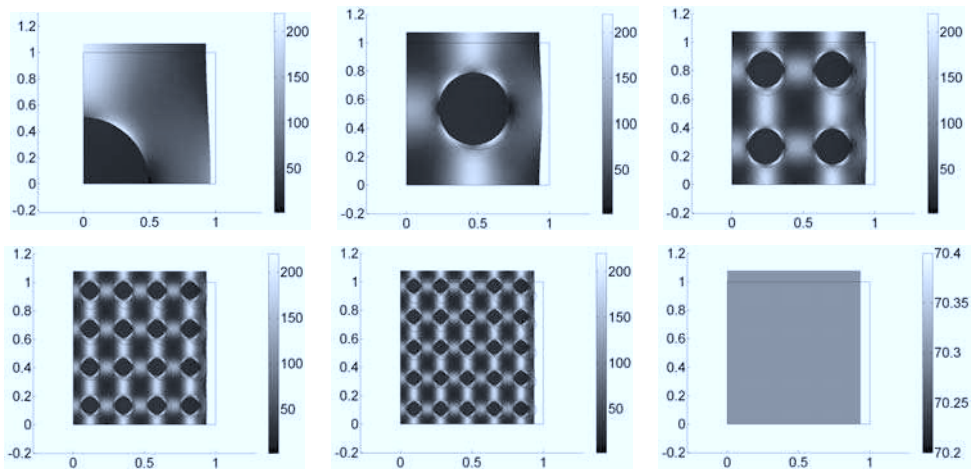

Fig. 10. Strain energy density distribution after a traction test imposing the displacement $vi = 0.140\,6$ m in samples of one, four, sixteen, sixty-four and one hundred inclusions and the sample of homogenized material

is independent of the number of inclusions and $\langle\sigma_{22}\rangle = \sigma_{22}^H$. On the other hand, the mean strain $\langle\epsilon_{22}\rangle$ decreases when the number of inclusions rises. Nevertheless, it tends to ϵ_{22}^H. On the contrary, when a displacement is imposed, the mean strain is unchanged and $\langle\epsilon_{22}\rangle = \epsilon_{22}^H$, whereas the mean stress $\langle\sigma_{22}\rangle$ increases with the number of inclusions and tends to σ_{22}^H. As a result, since $\langle\boldsymbol{\epsilon} : \boldsymbol{\sigma}\rangle = \langle\boldsymbol{\epsilon}\rangle : \langle\boldsymbol{\sigma}\rangle$, the strain energy decreases with the number of inclusions in one case whereas it increases in the latter, but in both cases it tends to the homogenized strain energy. These results are shown in fig. 11. In addition, the dependence of the mean transverse stress $\langle\sigma_{33}\rangle$ on the number of inclusions is shown in fig. 12. When a density of force is applied, $\langle\sigma_{33}\rangle$ is unchanged and $\langle\sigma_{33}\rangle = \sigma_{33}^H$. When a displacement is imposed, it decreases

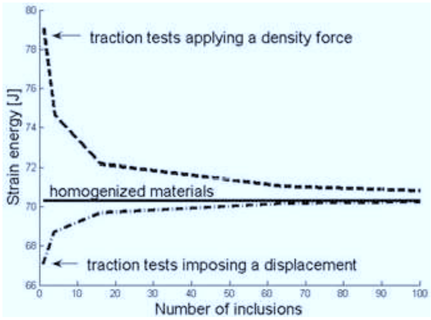

Fig. 11. Dependence of the total strain energy on the number of inclusions for traction tests with $fi = 10^3$ Pa and $vi = 0.14$ m. Curves tend to the homogenized energy $\boldsymbol{\Omega}^H = 70.30$ J

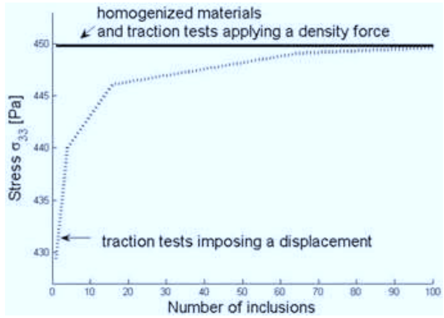

Fig. 12. Dependence of the transverse stress on the number of inclusions for traction tests with $fi = 10^3$ Pa and $vi = 0.14$ m. Curves tend to the homogenized stress $\sigma_{33}^H = 450$ Pa

with the number of inclusions and tends to σ_{33}^H. We conclude that in both plane of deformation and transverse direction, as well as for both kinds of loading, the mechanical behaviour of the heterogenous sample becomes closer to the homogenized one when the number of inclusions increases.

8. Conclusions

We have developed a numerical procedure for computing the homogenized coefficients of elastic fibrous tissue with the help of the Finite Element code Comsol Multiphysics and the computing environment Matlab. Tissue is considered as periodic, and the assumption of plane strain is used. The number of independent effective stiffness coefficients to be determined is six in general case. It reduces to four if the fibres cross-section are round and the arrangement of fibres is square. In fact, four coefficients (resp. three if symmetry) enter the plane behaviour equations and two ones (resp. one) are involved in the definition of stress along the third direction. This is the reason why two different homogenization schemes, (13) and (16), are used. These schemes are tested with different choices of Repeating Unit Cell. It has been controlled that the results of the homogenization process are independent whether the RUC contains only one inclusion or more inclusions, or whether the inclusions contained in the RUC are full or parceled out. On the contrary, the number of inclusions contained in the sample influences the values of the total strain energy and the mean transverse stress developed during mechanical traction tests. Nevertheless, we showed the tendency of these quantities toward the homogenized values when the number of inclusions increases. This is in accordance with the ideas of asymptotic homogenization which claims the homogenized medium is the limit of the heterogeneous medium when the size of heterogeneities tends to zero.

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References

[1] A. Bensoussan, J. L. Lions and G. Papnicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, 1978.

[2] Comsol Multiphysics, user's book – version 3.2, 2005.

[3] A. Drago, M. J. Pindera, Micro-macromechanical analysis of heterogeneous materials: Macroscopically homogeneous vs periodic microstructures, Composites Science and Technology 67 (2007) 1 243–1 263.

[4] Z. Hashin and S. Shtrikman, On some variational principles in anisotropic and nonhomogeneous elasticity, J. Mech. Phys. Solids 10 (1962) 335–342.

[5] R. Hill, Elastic properties of reinforced solids: Some theoretical principle, J. Mech. Phys. Solids 11 (1963) 357–372.

[6] M. Holeček, 0. Cervená , F. Poirier, Scale continuum approach in biomechanics: a simple simulation of a microstructural control of tissues' stiffness, Mathematics and Computers in Simulation 61 (2003) 583–590.

[7] M. Holeček, F. Moravec, Hyperelastic model of a material whose microstructure is formed by "balls and springs" International Journal of Solids and Structures, 43 (2006) 7 393–7 406.

[8] M. Kachanov, B. Shafiro, I. Tsukrov, Handbook of Elasticity Solutions, Kluwer Academic Publishers, 2003.

[9] J. C. Michel, H. Moulinec, P. Suquet, Effective properties of composite materials with periodic microstructure: a computational approach. Comput. Methods Appl. Mech. Engrg. 172 (1999) 109–143.

[10] F. Moravec, N. Letzelter, On the modeling of the linear viscoelastic behaviour of biological materials using Comsol Multiphysics: In Applied and Computational Mechanics 1(1) (2007) 175–184.

[11] E. Rohan, B. Miara, Homogenization and shape sensitivity of microstructures for design of piezoelectric bio-materials. In Mechanics of Advanced Materials and Structures 13(6) (2006) 473–485.

[12] S. Roman, Stage de modélisation de tissus périodiques et calculs des coefficients élastiques homogénéisés, Master Report, ISBS Paris 12, 2008.

[13] E. Sanchez-Palencia, Comportement local et macroscopique d'un type de milieux plysiques hétérogènes, Int. J. Engrg. Sci. 12 (1974) 331–351.

[14] E. Sanchez-Palencia, Non-Homogeneous Media and Vibration Theory, Springer, 1980.

[15] P. Suquet, Elements of homogenization theory for inelastic solids mechanics. In: Sanchez-Palencia, E., Zaoui, A. (Eds.), Homogenization Techniques for Composites Media. Springer-Verlag, Berlin, 1997, pp. 194–275.

[16] J. L. Teply and G. Dvorak, Bounds on overall instanteneous properties of elastic-plastic composites, J. Mech. Phys. Solids 36 (1988) 29–58.