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A thin rectangular viscoelastic orthotropic plate under transverse impuls loading

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Abstract

An analytical solution of stress wave propagation in a thin rectangular viscoelastic plate with special orthotropy under transverse impulse loading is presented. The solution is based on the approximate theory of thin plates using the Kirchhoff and the Rayleigh corrections. Constitutive equations for two-dimensional linear viscoelastic Maxwell model of solid are derived using the superposition principle. Transverse impulse loading has been allowed to effect on an arbitrary point of the plate surface and it has general behaviour in time. Results in the form of displacement, velocities and stress components are obtained.

Keywords: viscoelastic, orthotropic, plate, transient vibration

1. Introduction

Most of works concerning transient vibrations of thin plates focus above all on elastic isotropic problems while anisotropic (e.g. orthotropic) problems are solved sporadically, see [3]. A considerably less number of studies deals with the solution of transient vibration or, more precisely, with the solution of stress waves in viscoelastic isotropic plate. These few studies were carried out in the first half of the last century see [4] and [5]. The solution of transient vibration in viscoelastic anisotropic plate is at its beginning. The analysis of transient vibration or, more precisely, wave propagation in an orthotropic viscoelastic thin plate has been presented sporadically and therefore it was an object of authors' investigation, e.g. [9], [10], [6], [7], etc.

This contribution represents a part of investigation of rectangular plate vibrations caused by a transverse impulse loading. Previous solutions of isotropic or orthotropic elastic and isotropic viscoelastic plate vibrations can be found in [1], [2], [11], [12]. The new solution of vibrations (or more precisely wave propagation) in an orthotropic thin plate with viscoelastic behaviour described by several rheological models (Voigt-Kelvin [8], Maxwell [7], Zener-standard model, etc.) will be compared.

This study gives fundamentals and some improvements of the classical plate theory proposed by Kirchhoff. It is known that any plate theory is an approximation of the three-dimensional theory and this approximation always results in some loss of accuracy. Therefore, results obtained via approximate theories are compared with those obtained by three-dimensional finite element solution. In this research, four analytical models were used for the description of thin plate behaviour. In the first of them (known as the Kirchhoff model), the plate is assumed to be in the state of pure bending in which plane cross-sections of the plate

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remain plane and perpendicular to the midplane of the plate. Thus, shear deformation is not included in this model. A state of plane stress is also supposed and the effects of rotary inertia are neglected. In the second model (known as the Rayleigh model), the effects of rotary inertia are also included without any shear deformation. The third model (known as the Flügge model) incorporates shear deformation, but not rotary inertia effects. The fourth model (known as the Timoshenko-Mindlin model) takes into account both rotary inertia effects and shear deformations [2].

The aim of the research is to analyse deformations of thin plate under transient, impulse, impact loading and to obtain the dependence of fundamental mechanical quantities (i.e. displacement, deflection angles, velocities, accelerations, forces, moments, etc.) on time. These quantities are obtained by analytical methods for two-dimensional problem.

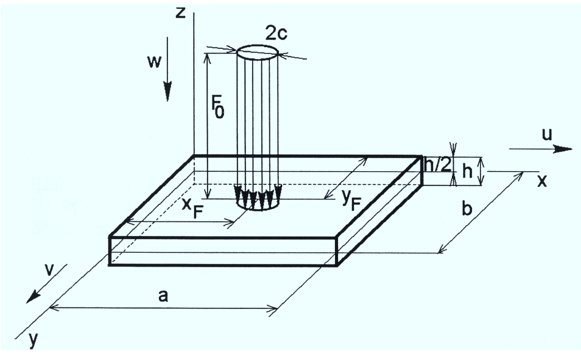

Fig. 1. Schema of solved problem.

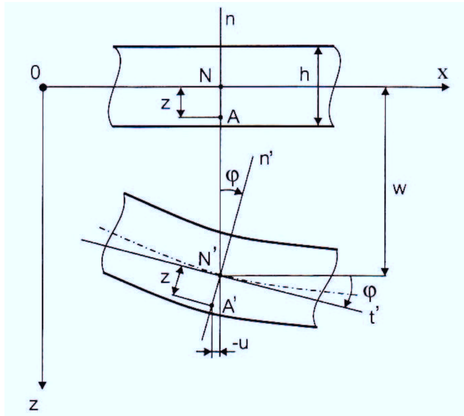

Fig. 2. Geometry of deformed plate.

The scheme of the problem solved is shown in fig. 1. The transverse pressure loading $p(x; y; t)$ with the resulting force F_0 is uniformly distributed over a small circle having diameter $2c$ and centre coordinates (x_F, y_F). This loading is applied to the upper face of a thin rectangular simply supported plate of dimensions $a \times b \times h$. The time and space dependence of applied loading is described by an arbitrary function. The plate material is supposed to be linear, viscoelastic, homogenous and orthotropic and principal material and body axes are considered identical (the plate with special orthotropy). Initial conditions of the problem are assumed zero, the plate is unloaded. Let u, v, w denote displacements in the directions of axes x, y, z, respectively and ρ is the material density.

2. Solution

When we use linear theory, in which the displacement components are small compared with the plate thickness h, the displacement components

$$u = u(x, y, z, t), \qquad v = v(x, y, z, t), \qquad w = w(x, y, t) \tag{1}$$

can be expressed by relations (see fig. 2)

$$u = -z\varphi_x , \qquad v = -z\varphi_y .$$

Strain components can be then written in the form

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial \varphi_x}{\partial x}; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \Rightarrow \quad \gamma_{xy} = -z\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right),$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial \varphi_y}{\partial y}; \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \Rightarrow \quad \gamma_{xz} = \frac{\partial w}{\partial x} - \varphi_x, \tag{2}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \to 0; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} \quad \Rightarrow \quad \gamma_{yz} = \frac{\partial w}{\partial y} - \varphi_y,$$

where $\varphi_x(x, y, t)$, $\varphi_y(x, y, t)$ are slopes of the plate normal corresponding to x and y directions, respectively.

For the Kirchhoff and the Rayleigh models, when shear strain $\gamma_{xz} = 0$ and $\gamma_{yz} = 0$, we can write

$$\gamma_{xz} = 0 \Rightarrow \varphi_x = \frac{\partial w}{\partial x} \Rightarrow \varepsilon_x = -z\frac{\partial^2 w}{\partial x^2} \qquad \gamma_{yz} = 0 \Rightarrow \varphi_y = \frac{\partial w}{\partial y} \Rightarrow \varepsilon_y = -z\frac{\partial^2 w}{\partial y^2},$$

$$\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}. \tag{3}$$

The constitutive equations for orthotropic linear viscoelastic Maxwell model are derived using the principle of superposition. The rheological equation of the orthotropic Maxwell solid can be expressed in tensor notation as

$$a_{ijkl}\sigma_{kl} + c_{ijkl}\dot{\sigma}_{kl} = d_{ijmn}\dot{\varepsilon}_{mn}, \tag{4}$$

where the vector σ_{kl} components of stress tensor is expressed in the form

$$\sigma_{kl} = \{\sigma_x = \sigma_{11}, \sigma_y = \sigma_{22}, \sigma_z = \sigma_{33}; \quad \tau_{xy} = \sigma_{12}, \tau_{xz} = \sigma_{13}, \tau_{yx} = \sigma_{23}\}^T,$$

the vector $\dot{\sigma}_{kl}$ components of stress rate tensor is expressed in the form

$$\dot{\sigma}_{kl} = \{\dot{\sigma}_x = \dot{\sigma}_{11}, \dot{\sigma}_y = \dot{\sigma}_{22}, \dot{\sigma}_z = \dot{\sigma}_{33}; \quad \dot{\tau}_{xy} = \dot{\sigma}_{12}, \dot{\tau}_{xz} = \dot{\sigma}_{13}, \dot{\tau}_{yz} = \dot{\sigma}_{23}\}^T,$$

and the vector $\dot{\varepsilon}_{mn}$ components of strain rate tensor is expressed in the form

$$\dot{\varepsilon}_{mn} = \{\dot{\varepsilon}_x = \dot{\varepsilon}_{11}, \dot{\varepsilon}_y = \dot{\varepsilon}_{22}, \dot{\varepsilon}_z = \dot{\varepsilon}_{33}, \dot{\gamma}_{xy} = \dot{\varepsilon}_{12}, \dot{\gamma}_{xz} = \varepsilon_{13}; \quad \dot{\gamma}_{yx} = \dot{\varepsilon}_{23}\}^T.$$

Then for the case of orthotropic continuum, the general rheological equation (4) reduces to

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \dot{\sigma}_x \\ \dot{\sigma}_y \\ \dot{\sigma}_z \\ \dot{\tau}_{xy} \\ \dot{\tau}_{yz} \\ \dot{\tau}_{xz} \end{Bmatrix} =$$

$$= \begin{bmatrix} d_{11} & d_{12} & d_{13} & 0 & 0 & 0 \\ d_{21} & d_{22} & d_{23} & 0 & 0 & 0 \\ d_{31} & d_{32} & d_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{66} \end{bmatrix} \begin{Bmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \\ \dot{\varepsilon}_z \\ \dot{\gamma}_{xy} \\ \dot{\gamma}_{xz} \\ \dot{\gamma}_{yz} \end{Bmatrix}.$$

For the derivation of equation, describing the vertical vibration of thin viscoelastic orthotropic rectangular plate for Maxwell model, it is useful to write constitutive equations in integral form [6]

$$\sigma_{ij} = b_{ijkl}\varepsilon_{kl} - \int_0^\tau L_{ijkl}(t-\tau)\varepsilon_{kl}(\tau)d\tau \,.$$

Introducing constants

$$\delta_x = \frac{E_x}{\lambda_x}, \qquad \delta_y = \frac{E_y}{\lambda_y}, \qquad \delta_{xy} = \frac{G}{\eta},$$

under the assumptions

$$\delta_x = \frac{\nu_y \delta_y}{\mu_y}, \qquad \delta_y = \frac{\nu_x \delta_x}{\mu_x} \quad \text{and} \quad \frac{\nu_x \nu_y}{\mu_x \mu_y} = 1, \tag{5}$$

and supposing that the potential exists and using the Betti's theorem

$$\frac{\mu_x}{E_x} = \frac{\mu_y}{E_y} \quad \text{or} \quad \frac{\nu_x}{\lambda_x} = \frac{\nu_y}{\lambda_y}, \text{ respectively,}$$

stress-strain relations can be written as [7]

$$\sigma_x = \frac{E_x}{1-\mu_x.\mu_y}.\left[\varepsilon_x + \mu_y\varepsilon_y - \delta_x\int_0^t (\varepsilon_x + \mu_y\varepsilon_y)\, e^{-\delta_x(t-\tau)}d\tau\right],$$

$$\sigma_y = \frac{E_y}{1-\mu_x.\mu_y}.\left[\varepsilon_y + \mu_x\varepsilon_x - \delta_y\int_0^t (\varepsilon_y + \mu_x\varepsilon_x)\, e^{-\delta_y(t-\tau)}d\tau\right], \tag{6}$$

$$\tau_{xy} = G\left[\gamma_{xy} - \delta_{xy}\int_0^t \gamma_{xy}\, e^{-\delta_{xy}(t-\tau)}d\tau\right],$$

where E and G represent the Young modulus of elasticity and the shear modulus, λ and η are coefficients of normal and tangential (shear) viscosity and μ, v denote Poisson ratios corresponding to elastic and viscous part of the Maxwell model, respectively.

Substituting relations (1) and (3) into equations (6) we obtain constitutive relations for the Kirchhoff and the Rayleigh model of thin plate

$$\sigma_x = \frac{-z\,E_x}{1-\mu_x\mu_y}\left[\frac{\partial^2 w}{\partial x^2} + \mu_y\frac{\partial^2 w}{\partial y^2} - \delta_x\int_0^t\left(\frac{\partial^2 w}{\partial x^2} + \mu_y\frac{\partial^2 w}{\partial y^2}\right)e^{-\delta_x(t-\tau)}d\tau\right],$$

$$\sigma_y = \frac{-zE_y}{1-\mu_x\mu_y}\left[\frac{\partial^2 w}{\partial y^2} + \mu_x\frac{\partial^2 w}{\partial x^2} - \delta_y\int_0^t\left(\frac{\partial^2 w}{\partial y^2} + \mu_x\frac{\partial^2 w}{\partial x^2}\right)e^{-\delta_y(t-\tau)}d\tau\right], \tag{7}$$

$$\tau_{xy} = -2zG\left[\frac{\partial^2 w}{\partial x\partial y} - \delta_{xy}\int_0^t \frac{\partial^2 w}{\partial x\partial y}e^{-\delta_{xy}(t-\tau)}d\tau\right].$$

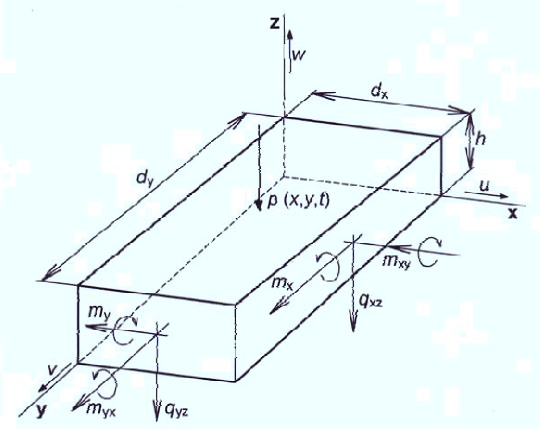

Fig. 3. Differential element of plate subjected to external load with all internal effects (forces and moments per unit length).

Internal effects arising in the plate due to external load can be expressed by following integral relations:

- bending moments $m_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz$, $m_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y z dz$,

- twisting moment $m_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} z dz$, (8)

- shearing forces $q_{xz} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} dz$, $q_{yz} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} dz$.

Introducing equations (7) into equations (8) we obtain internal moments as the functions of plate deflection *w*

$$m_x = -D_x\left[\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2} - \delta_x \int_0^t \left(\frac{\partial^2 w}{\partial x^2} + \mu_y \frac{\partial^2 w}{\partial y^2}\right) e^{-\delta_x (t-\tau)} d\tau\right],$$

$$m_y = -D_y\left[\frac{\partial^2 w}{\partial y^2} + \mu_x \frac{\partial^2 w}{\partial x^2} - \delta_y \int_0^t \left(\frac{\partial^2 w}{\partial y^2} + \mu_y \frac{\partial^2 w}{\partial x^2}\right) e^{-\delta_y (t-\tau)} d\tau\right], \tag{9}$$

$$m_{xy} = m_{yx} = -2D_{xy}\left[\frac{\partial^2 w}{\partial x \partial y} - \delta_{xy} \int_0^t \frac{\partial^2 w}{\partial x \partial y} e^{-\delta_{xy} (t-\tau)} d\tau\right],$$

where $D_x = \dfrac{E_x h^3}{12(1-\mu_{xy}\mu_{yx})}$, $D_y = \dfrac{E_y h^3}{12(1-\mu_{xy}\mu_{yx})}$, $D_{xy} = \dfrac{G\, h^3}{12}$.

With respect to fig. 3, the equation of motion in vertical direction can be written for all models as

$$\frac{\partial q_{xz}}{\partial x}+\frac{\partial q_{yz}}{\partial y}+p(x,y,t)=\rho\, h\frac{\partial^2 w}{\partial t^2} \tag{10}$$

and the moment condition of dynamical equilibrium

$$q_{xz}-\frac{\partial m_x}{\partial x}-\frac{\partial m_{xy}}{\partial y}=\Phi_x\,, \qquad q_{yz}-\frac{\partial m_y}{\partial y}-\frac{\partial m_{yx}}{\partial x}=\Phi_y\,, \tag{11}$$

where right-hand sides of $(8)_2$ depend on model used:
- for the Kirchhoff model $\Phi_{xK}=0$, $\Phi_{yK}=0$,

- for the Rayleigh model $\Phi_{xR}=J_\rho\dfrac{\partial^3 w}{\partial x\,\partial t^2}$, $\Phi_{yR}=J_\rho\dfrac{\partial^3 w}{\partial y\,\partial t^2}$, $J_\rho=\dfrac{\rho\, h^3}{12}$. (12)

Substituting q_{xz} and q_{yz} from equations $(8)_2$ into equation $(8)_1$ we obtain

$$\frac{\partial^2 m_x}{\partial x^2}+2\frac{\partial^2 m_{xy}}{\partial x\,\partial y}+\frac{\partial^2 m_y}{\partial y^2}+\frac{\partial \Phi_x}{\partial x}+\frac{\partial \Phi_y}{\partial y}+p(x,y,t)=\rho\, h\frac{\partial^2 w}{\partial t^2}\,. \tag{13}$$

Finally, introducing equations (9) into equation (13) we obtain the final motion equation for the Kirchhoff model ($\Phi_{xK}=0,\ \Phi_{yK}=0$) and for the Rayleigh model ($\Phi_{xR}=J_\rho\dfrac{\partial^3 w}{\partial x\partial t^2}$, $\Phi_{yR}=J_\rho\dfrac{\partial^3 w}{\partial y\partial t^2}$) in the form

$$\begin{aligned}
&D_x\frac{\partial^4 w}{\partial x^4}+D_y\frac{\partial^4 w}{\partial y^4}+2\left(D_\mu+2.D_{xy}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}-D_x\,\delta_x\int_0^t\frac{\partial^4 w}{\partial x^4}\,e^{-\delta_x(t-\tau)}d\tau-D_y\,\delta_y\int_0^t\frac{\partial^4 w}{\partial y^4}e^{-\delta_y(t-\tau)}d\tau-\\
&-4D_{xy}\,\delta_{xy}\int_0^t\frac{\partial^4 w}{\partial x^2\partial y^2}e^{-\delta_{xy}(t-\tau)}d\tau-D_\mu\left(\delta_x\int_0^t\frac{\partial^4 w}{\partial x^4}e^{-\delta_x(t-\tau)}d\tau+\delta_y\int_0^t\frac{\partial^4 w}{\partial y^4}.e^{-\delta_y(t-\tau)}d\tau\right)+\\
&+\frac{\partial^2}{\partial t^2}\left(\rho\, h\, w-\frac{\partial \Phi_x}{\partial x}-\frac{\partial \Phi_y}{\partial y}\right)=p(x,y,t)\,,
\end{aligned} \tag{14}$$

where $D_\mu=D_x\,\mu_{yx}=D_y\,\mu_{xy}$ and $p(x,y,t)=F_0\,T_F(t)\,X(x_F)\,Y(y_F)$.

As an example, we outline the solution of rectangular Kirchhoff or Rayleigh thin plate from fig. 1 having simply supported edges.

In this case we may assume the solution of equation (14) in the form

$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W(t)\,X(x)\,Y(y)\,, \tag{15}$$

where $X(x)=\sin(\alpha_m x)$, $Y(y)=\sin(\beta_n y)$ and $\alpha_m=\dfrac{m\pi}{a}$, $\beta_n=\dfrac{n\pi}{b}$.

The function $W(t)$ represents the time dependence of displacement w. Substituting solution (15) into equation (14) and after some lengthy rearrangements of equations for $W(t)$ [7], we obtain

$$A_1 \frac{d^2W(t)}{dt^2} + A_2 W(t) - \int_0^t W(t) \left[A_3 e^{-\delta_x (t-\tau)} + A_4 e^{-\delta_y (t-\tau)} + A_5 e^{-\delta_{xy} (t-\tau)} \right] d\tau = A_6 T_F(t) , \quad (16)$$

where

$$\begin{aligned} A_1 &= 1, \\ A_2 &= \left[D_x \alpha_m^4 + D_y \beta_n^4 + 2\left(D_\mu + 2D_{xy}\right)\alpha_m^2 \beta_n^2 \right] (\rho h A)^{-1}, \\ A_3 &= \delta_x \left(D_x \alpha_m^4 + D_\mu \alpha_m^2 \beta_n^2 \right) (\rho h A)^{-1}, \\ A_4 &= \delta_y \left(D_y \beta_n^4 + D_\mu \alpha_m^2 \beta_n^2 \right) (\rho h A)^{-1}, \\ A_5 &= 4 D_{xy} \delta_{xy} \alpha_m^2 \beta_n^2 (\rho h A)^{-1}, \\ A_6 &= \frac{4 p_{mn}}{a b \rho h A} F_0 \end{aligned} \quad (17)$$

and $A = 1 + \Psi_{mn}$ and $\Psi_{mn} = 0$ for the Kirchhoff model,

or $\Psi_{mn} = \frac{h^2}{12}\left(\alpha_m^2 + \beta_n^2\right)$ for the Rayleigh model.

The function p_{mn} is defined as $p_{mn} = \int_0^a \int_0^b X(x_F) Y(y_F) X(x) Y(y)\, dx\, dy$.

Using the Laplace transformation with new parameter s we get the integral transform of unknown function w

$$\overline{W}(s) = A_6 \overline{T}_F(s) \overline{F}(s),$$

where

$$\overline{F}(s) = \frac{s^3 + s^2 a_2 + s a_1 + a_0}{\sum_{i=0}^{5} b_{5-i} s^{5-i}} . \quad (18)$$

Solving the algebraic equation

$$\sum_{i=0}^{5} b_{5-i} s^{5-i} = 0 , \quad (19)$$

where

$$\begin{aligned} a_2 &= \delta_x + \delta_y + \delta_{xy}, & b_5 &= 1, & b_4 &= a_2, \\ a_1 &= \delta_x\delta_y + \delta_y\delta_{xy} + \delta_{xy}\delta_x, & b_3 &= a_1 + A.a_2, & b_2 &= a_0 + A_2 a_2, \\ a_0 &= \delta_x\delta_y\delta_{xy}, & b_1 &= A_2 a_1, & b_0 &= A_2 a_0, \end{aligned}$$

we find five roots of polynomial in the denominator of relation (18). Three different types of roots exist depending on coefficients b_{5-i} which are functions of material parameters ρ, E, G, λ, η, μ and plate dimensions $a \times b \times h$:

1. two complex conjugate roots $s_{1,2} = \beta_1 \pm i\,\omega_1$, $s_{3,4} = \beta_2 \pm i\,\omega_2$ and one real root $s_5 = \beta_3$.

2. one complex conjugate root $s_{1,2} = \beta \pm i\omega$ and three real roots s_3, s_4, s_5.

3. five real roots.

With respect to supposed physical behaviour of vibrating plate, we will deal with the first case. In this case, the equation (18) can be rewritten in the form

$$\overline{F}(s) = \overline{F}_1(s) + \overline{F}_2(s) + \overline{F}_3(s), \tag{20}$$

where $\overline{F}_1(s) = \dfrac{C_1 s + D_1}{s^2 + p_1 s + q_1}$, $\overline{F}_2(s) = \dfrac{C_2 s + D_2}{s^2 + p_2 s + q_2}$, $\overline{F}_3(s) = \dfrac{D_3}{s + \beta_3}$

and the coefficients p_i and q_i are defined as $p_i = 2\beta_i$ and $q_i = \omega_i^2$.

Unknown coefficients C_1, C_2, D_1, D_2 and D_3 can be found using the equality of relations (18) and (20) that leads to following system of linear algebraic equations

$$C_1 + C_2 = 0,$$

$$C_1 p_2 + C_2 p_1 + (C_1 + C_2)\beta_3 + D_1 + D_2 = 1,$$

$$C_1 q_2 + C_2 p_1 + D_1 p_2 + D_2 p_1 + (C_1 p_2 + C_2 p_1 + D_1 + D_2)\beta_3 + D_3(q_1 + q_2 + p_1 p_2) = a_2,$$

$$(C_1 q_2 + C_2 q_1 + D_1 p_2 + D_2 p_1)\beta_3 + D_1 q_2 + D_2 q_1 + D_3(q_1 p_2 + q_2 p_1) = a_1,$$

$$(D_1 q_2 + D_2 q_1)\beta_3 + q_1 q_2 = a_0.$$

The Laplace transform of the unknown function $\overline{W}(s)$ is then expressed by relation

$$\overline{W}(s) = A_6 \overline{T}_F(s)\left[\overline{F}_1(s) + \overline{F}_2(s) + \overline{F}_3(s)\right].$$

The original function $W(t)$ can be obtained by inverse transformation using convolution theorem and then it gives

$$W(t) = \frac{4 p_{mn} F_0}{a\, b\, \rho\, h\,(1 + \Psi_{mn})} \int_0^t T_F(\tau) \left\{ \sum_{i=1}^{2} \left[e^{-\beta_i (t-\tau)} \left(C_i \cos\omega_i (t-\tau) + \frac{D_i - C_i \beta_i}{\omega_i} \sin\omega_i (t-\tau) \right] + \right.\right.$$

$$\left. + D_3 e^{-\alpha_3 (t-\tau)} \right\} d\tau. \tag{21}$$

Introducing equation (21) into (15), we get the final relation for displacement w

$$w(x,y,t) = \frac{4F_0}{a\, b\, \rho\, h} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_{mn}}{1 + \Psi_{mn}} \sin(\alpha_m x) \sin(\beta_n x) \int_0^t T_F(\tau) \left\{ \sum_{i=1}^{2} \left[e^{-\beta_i (t-\tau)} \left(C_i \cos\omega_i (t-\tau) + \right.\right.\right.$$

$$\left.\left. + \frac{D_i - C_i \beta_i}{\omega_i} \sin\omega_i (t-\tau)) \right] + D_3\, e^{-\beta_3 (t-\tau)} \right\} d\tau. \tag{22}$$

Then others required quantities can be derived:

Displacements components by substitution of eq. (22) into eq. (1), (2)

$$u(x,y,z,t) = -z \frac{\partial w}{\partial x} = -\frac{4 z F_0}{\rho\, h\, a\, b} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\alpha_m p_{mn}}{1 + \Psi_{mn}} \cos(\alpha_m x) \sin(\beta_n y) T_K(t),$$

$$v(x,y,z,t) = -z\frac{\partial w}{\partial y} = -\frac{4zF_0}{\rho\,h\,a\,b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\beta_n p_{mn}}{1+\Psi_{mn}}\sin(\alpha_m x)\cos(\beta_n y)T_K(t) \ .$$

Components of velocities

$$\dot{u}(x,y,z,t) = -z\frac{\partial^2 w}{\partial x\partial t} = -\frac{4z\,F_0}{\rho\,h\,a\,b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_m p_{mn}}{1+\Psi_{mn}}\cos(\alpha_m x)\sin(\beta_n y)\frac{\partial T_K(t)}{\partial t} \ ,$$

$$\dot{v}(x,y,z,t) = -z\frac{\partial^2 w}{\partial y\partial t} = -\frac{4zF_0}{\rho\,h\,a\,b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\beta_n p_{mn}}{1+\Psi_{mn}}\sin(\alpha_m x)\cos(\beta_n y)\frac{\partial T_K(t)}{\partial t} \ ,$$

$$\dot{w}(x,y,z,t) = \frac{\partial w}{\partial t} = \frac{4F_0}{\rho\,h\,a\,b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_{mn}}{1+\Psi_{mn}}\sin(\alpha_m x)\sin(\beta_n y)\frac{\partial T_K(t)}{\partial t} \ .$$

Where convolution integral

$$T_K(t) = \int_0^t T_F(\tau)\left\{\sum_{i=1}^{2}\left[e^{-\beta_i(t-\tau)}\left(C_i\cos\omega_i(t-\tau) + \frac{D_i - C_i\beta_i}{\omega_i}\sin\omega_i(t-\tau)\right)\right] + D_3\,e^{-\beta_3(t-\tau)}\right\}d\tau \ .$$

Stress components $\sigma_x(x, y, z, t)$, $\sigma_y(x, y, z, t)$, $\tau_{xy}(x, y, z, t)$, can be derived by substitution of eq. (22) into eq. (7)

$$\sigma_x(x,y,z,t) = \frac{z\,E_x}{1-\mu_x\,\mu_y}\frac{4F_0}{a\,b\,h\,\rho}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\alpha_m^2 + \mu_y\beta_n^2)p_{mn}}{1+\Psi_{mn}}\sin(\alpha_m x)\sin(\beta_n y)\cdot$$

$$\cdot\left[T_K(t) - \delta_x\int_0^t T_K(\tau)e^{-\delta_x(t-\tau)}d\tau\right],$$

$$\sigma_y(x,y,z,t) = \frac{z\,E_y}{1-\mu_x\,\mu_y}\frac{4F_0}{a\,b\,h\,\rho}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\mu_x\alpha_m^2 + \beta_n^2)p_{mn}}{1+\Psi_{mn}}\sin(\alpha_m x).\sin(\beta_n y)\cdot$$

$$\cdot\left[T_K(t) - \delta_y.\int_0^t T_K(\tau)e^{-\delta_y(t-\tau)}d\tau\right],$$

$$\tau_{xy}(x,y,z,t) = -\frac{8zGF_0}{a\,b\,h\,\rho}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\alpha_m\beta_n p_{mn}}{1+\Psi_{mn}}\cos(\alpha_m x)\cos(\beta_n y)\cdot$$

$$\cdot\left[T_K(t) - \delta_{xy}\int_0^t T_K(\tau)e^{-\delta_{xy}(t-\tau)}d\tau\right].$$

3. Conclusion

Results of this work, which lie in the solution of plate vibration for the case of the Maxwell material model, will help us in finding the solution for the case of the Zener model of standard viscoelastic solid.

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