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Vibration modes of a single plate with general boundary conditions

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Abstract

This paper deals with free flexural vibration modes and natural frequencies of a thin plate with general boundary conditions — a simply supported plate connected to its surroundings with torsional springs. Vibration modes were derived on the basis of the Rajalingham, Bhat and Xistris approach. This approach was originally used for a clamped thin plate, so its adaptation was needed. The plate vibration function was usually expressed as a single partial differential equation. This partial differential equation was transformed into two ordinary differential equations that can be solved in the simpler way. Theoretical background of the computations is briefly described. Vibration modes of the supported plate with torsional springs are presented graphically and numerically for three different values of stiffness of torsional springs.

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Keywords: thin plate, vibration modes, natural frequencies

1. Introduction

Vibration modes of a single rectangular plate with classical boundary conditions have been solved by many authors. Leissa [5] published basic review of plate vibration. Exact solution of free vibration modes and natural frequencies can be found only for a simply supported plate. In other cases, approximate natural frequencies and modes can be obtained by approximate methods, e.g. the Rayleigh-Ritz method or the Galerkin method. The Kantorovich method was used by Laura and Cortinez [3, 4], and other authors, e.g. by Bhat [1], to reduce the partial differential equation into ordinary differential equations. As characteristic functions, some authors used beam characteristic functions, e.g. Leissa [5], plate characteristic functions, e.g. Rajalingham [10], and orthogonal polynomials, e.g. Mundkur et al. [6].

Rajalingham, Bhat and Xistris [9] suggested a method for the computation of vibration modes and natural frequencies for a single clamped plate. They used the energetic approach and the variational method to reduce the partial differential equation into two ordinary differential equations. Further, these equations take into account a dependence of the solution in the direction x on the solution in the direction y and vice versa. Both symmetric and antisymmetric solutions of the ordinary differential equations are applied.

All authors solved vibration modes and natural frequencies of a plate with clamped or simply supported edges. These boundary conditions cannot be achieved in real constructions. Moreover, they are difficult to reach in an experiment. Plates are joined to its surroundings with more or less stiff connections.

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The plate, whose vibration modes are mentioned in the present paper, is simply supported on its edges and connected with torsional springs to its neighbourhood. Different values of stiffness of these springs simulate different boundary conditions. Zero stiffness corresponds to a simply supported plate, whereas very high stiffness comes close to the case of a clamped plate.

The derivation of plate governing equations of free vibration modes is introduced. The solution of these equations is suggested and expressed. The computations of vibration modes are performed for three different values of stiffness. The resulting vibration modes are presented graphically and numerically.

2. Theoretical background

2.1. Description of a single plate

The considered plate has dimensions L_x, L_y and thickness h. The thickness is much smaller than other dimensions. The plate is simply supported on its edges and connected with torsional springs to its neighbourhood, Fig. 1. The boundary conditions are zero deflections and moment equilibrium on the edges

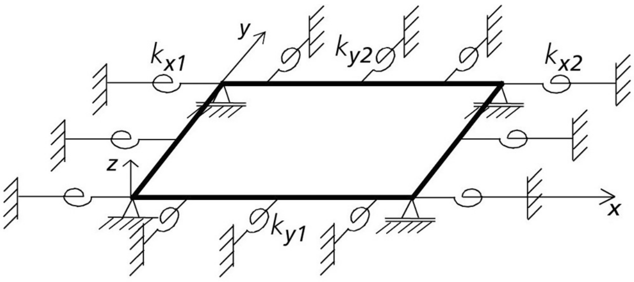


Fig. 1. Thin plate supported with torsional springs

$$w_x(0) = 0, \qquad w_y(0) = 0, \tag{1}$$

$$w_x(L_x) = 0, \qquad w_y(L_y) = 0, \tag{2}$$

$$\left.\frac{\mathrm{d}^2 w_x}{\mathrm{d}x^2}\right|_0 - c_{x1} \left.\frac{\mathrm{d}w_x}{\mathrm{d}x}\right|_0 = 0, \qquad \left.\frac{\mathrm{d}^2 w_y}{\mathrm{d}y^2}\right|_0 - c_{y1} \left.\frac{\mathrm{d}w_y}{\mathrm{d}y}\right|_0 = 0, \tag{3}$$

$$\left.\frac{\mathrm{d}^2 w_x}{\mathrm{d}x^2}\right|_{Lx} - c_{x2} \left.\frac{\mathrm{d}w_x}{\mathrm{d}x}\right|_{Lx} = 0, \qquad \left.\frac{\mathrm{d}^2 w_y}{\mathrm{d}y^2}\right|_{Ly} - c_{y2} \left.\frac{\mathrm{d}w_y}{\mathrm{d}y}\right|_{Ly} = 0, \tag{4}$$

where w_x stands for displacement in the x-direction depending on space; w_y stands for displacement in the y-direction depending on space; $c_{ij} = k_{ij}/EJ$, $i = x, y$, $j = 1, 2$; k_{ij} stands for torsional stiffness between inertia and plate; E stands for Young modulus, J stands for quadratic moment of the cross section.

The flexural rigidity D is defined as (e.g. Norton et al. [7])

$$D = \frac{Eh^3}{12(1-\mu^2)}, \tag{5}$$

where μ stands for Poisson constant.

2.2. Plate vibration equations

The Kirchhoff's theory for thin plates is applied. The governing equations of the plate considered in this paper are derived using energetic approach. According to the principle of Hamilton, the system is in equilibrium when (e.g. Brepta et al. [2])

$$\int_{t_1}^{t_2} \delta(L)\,\mathrm{d}t = 0, \tag{6}$$

where L is the Lagrange function defined as the difference between kinetic E_k and potential energy E_p

$$L = E_k - E_p. \tag{7}$$

The kinetic energy of the plate is given as follows

$$E_k = \frac{1}{2}\int_0^{Ly}\int_0^{Lx} \rho_s h \left(\frac{\partial W}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}y \tag{8}$$

and the potential energy

$$E_p = \frac{1}{2}D\int_0^{Ly}\int_0^{Lx}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2\mu\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2(1-\mu)\left(\frac{\partial^2 W}{\partial x \partial y}\right)\right]\mathrm{d}x\,\mathrm{d}y, \tag{9}$$

where ρ_s stands for weight of an unit area plate; W stands for displacement depending on space and time.

Using the separation variable method, the steady harmonic free flexural vibration can be expressed by the following equation

$$W(x,y,t) = w(x,y)\mathrm{e}^{\mathrm{j}\omega t} = w(x,y)T(t) = w_x(x)w_y(y)T(t), \tag{10}$$

where w stands for displacement depending on space; j stands for imaginary unit; ω is angular frequency; t stands for time. By insertion (10) into (9) and (8) into (6), it is possible to obtain two fourth order ordinary differential equations

$$\frac{\mathrm{d}^4 w_x}{\mathrm{d}x^4} + \frac{2(B^{(00)} - (1-\mu)b^{(01)})}{B^{(00)}}\frac{\mathrm{d}^2 w_x}{\mathrm{d}x^2} - \left(\Omega^2\frac{\rho_s h}{D} - \frac{B^{(04)} - b^{(03)} + b^{(12)}}{B^{(00)}}\right)w_x = 0, \tag{11}$$

$$\frac{\mathrm{d}^4 w_y}{\mathrm{d}y^4} + \frac{2(A^{(00)} - (1-\mu)a^{(01)})}{A^{(00)}}\frac{\mathrm{d}^2 w_y}{\mathrm{d}y^2} - \left(\Omega^2\frac{\rho_s h}{D} - \frac{A^{(04)} - a^{(03)} + a^{(12)}}{A^{(00)}}\right)w_y = 0, \tag{12}$$

where Ω stands for natural angular frequency and

$$A^{(rs)} = \int_0^{Lx}\frac{\mathrm{d}^r w_x}{\mathrm{d}x^r}\frac{\mathrm{d}^s w_x}{\mathrm{d}x^s}\,\mathrm{d}x, \tag{13}$$

$$B^{(rs)} = \int_0^{Ly}\frac{\mathrm{d}^r w_y}{\mathrm{d}y^r}\frac{\mathrm{d}^s w_y}{\mathrm{d}y^s}\,\mathrm{d}y, \tag{14}$$

$$a^{(rs)} = \left[\frac{\mathrm{d}^r w_x}{\mathrm{d}x^r}\frac{\mathrm{d}^s w_x}{\mathrm{d}x^s}\right]_0^{Lx}, \tag{15}$$

$$b^{(rs)} = \left[\frac{\mathrm{d}^r w_y}{\mathrm{d}y^r}\frac{\mathrm{d}^s w_y}{\mathrm{d}y^s}\right]_0^{Ly}. \tag{16}$$

The derivation of equations (11)–(16) is beyond the scope of this article and readers are kindly referred to the paper [9]. Equations (11) and (12) can be rewritten as follows

$$\frac{\mathrm{d}^4 w_x}{\mathrm{d}x^4} + Y_b\cdot\frac{\mathrm{d}^2 w_x}{\mathrm{d}x^2} - Y_c\cdot w_x = 0, \tag{17}$$

$$\frac{\mathrm{d}^4 w_y}{\mathrm{d}y^4} + X_b\cdot\frac{\mathrm{d}^2 w_y}{\mathrm{d}y^2} - X_c\cdot w_y = 0, \tag{18}$$

where

$$Y_b = \frac{2(B^{(00)} - (1-\mu)b^{(01)})}{B^{(00)}}\frac{\mathrm{d}^2 w_x}{\mathrm{d}x^2}, \qquad Y_c = \Omega^2 \frac{\rho_s h}{D} - \frac{B^{(04)} - b^{(03)} + b^{(12)}}{B^{(00)}}, \tag{19}$$

$$X_b = \frac{2(A^{(00)} - (1-\mu)a^{(01)})}{A^{(00)}}\frac{\mathrm{d}^2 w_y}{\mathrm{d}y^2}, \qquad X_c = \Omega^2 \frac{\rho_s h}{D} - \frac{A^{(04)} - a^{(03)} + a^{(12)}}{A^{(00)}}. \tag{20}$$

The solution of equations (17) and (18) can have a symmetric (S) and antisymmetric (A) form

$$w_x = \left\{ \begin{array}{ll} \frac{\cosh\left(\left(x-\frac{L_x}{2}\right)p_2\right)}{\cosh\left(\frac{L_x}{2}p_2\right)} - \frac{\cos\left(\left(x-\frac{L_x}{2}\right)p_1\right)}{\cos\left(\frac{L_x}{2}p_1\right)} & \text{(S)} \\ \frac{\sinh\left(\left(x-\frac{L_x}{2}\right)p_2\right)}{\sinh\left(\frac{L_x}{2}p_2\right)} - \frac{\sin\left(\left(x-\frac{L_x}{2}\right)p_1\right)}{\sin\left(\frac{L_x}{2}p_1\right)} & \text{(A)} \end{array} \right\}, \tag{21}$$

$$w_y = \left\{ \begin{array}{ll} \frac{\cosh\left(\left(y-\frac{L_y}{2}\right)q_2\right)}{\cosh\left(\frac{L_y}{2}q_2\right)} - \frac{\cos\left(\left(y-\frac{L_y}{2}\right)q_1\right)}{\cos\left(\frac{L_y}{2}q_1\right)} & \text{(S)} \\ \frac{\sinh\left(\left(y-\frac{L_y}{2}\right)q_2\right)}{\sinh\left(\frac{L_y}{2}q_2\right)} - \frac{\sin\left(\left(y-\frac{L_y}{2}\right)q_1\right)}{\sin\left(\frac{L_y}{2}q_1\right)} & \text{(A)} \end{array} \right\}. \tag{22}$$

This form of solution is very suitable for the clamped thin plate. But for the plate supported with torsional springs, this form of solution has not been proved to be useful. Therefore, the "full solution" was used instead of the former one

$$w_x = C_1 \mathrm{e}^{\mathrm{j}p_1 x} + C_2 \mathrm{e}^{-\mathrm{j}p_1 x} + C_3 \mathrm{e}^{p_2 x} + C_4 \mathrm{e}^{-p_2 x}, \tag{23}$$

$$w_y = D_1 \mathrm{e}^{\mathrm{j}q_1 y} + D_2 \mathrm{e}^{-\mathrm{j}q_1 y} + D_3 \mathrm{e}^{q_2 y} + D_4 \mathrm{e}^{-q_2 y}, \tag{24}$$

which can be written as

$$w_x = A_1 \sin(p_1 x) + A_2 \cos(p_1 x) + A_3 \sinh(p_2 x) + A_4 \cosh(p_2 x), \tag{25}$$

$$w_y = B_1 \sin(q_1 y) + B_2 \cos(q_1 y) + B_3 \sinh(q_2 y) + B_4 \cosh(q_2 y), \tag{26}$$

where C_i, D_i, $i = 1, 2, 3, 4$ are integration constants. The constants A_i, B_i are given as follows

$$A_1 = C_1 - C_2, \quad A_2 = -C_1 - C_2, \quad A_3 = C_3 - C_4, \quad A_4 = C_3 + C_4, \tag{27}$$

$$B_1 = D_1 - D_2, \quad B_2 = -D_1 - D_2, \quad B_3 = D_3 - D_4, \quad B_4 = D_3 + D_4. \tag{28}$$

2.3. *Solution of* p_1, p_2, q_1, q_2

The coefficients p_1, p_2, q_1, q_2 are obtained from four equations. The first and second equation can be derived from (11), (12), (17), (18) using the Vieta's formula and assuming roots in the form $\pm p_2$, $\pm \mathrm{j} \cdot p_1$

$$\frac{\mathrm{d}^4 w_x}{\mathrm{d}x^4} - (p_2^2 - p_1^2)\frac{\mathrm{d}^2 w_x}{\mathrm{d}x^2} - p_2^2 p_1^2 w_x = 0. \tag{29}$$

The comparison of coefficients from (17) and (29) yields

$$p_2^2 - p_1^2 = -Y_b, \tag{30}$$

$$q_2^2 - q_1^2 = -X_b, \tag{31}$$

$$p_2^2 - p_1^2 = -\frac{2(B^{(00)} - (1-\mu)b^{(01)})}{B^{(00)}}, \tag{32}$$

$$q_2^2 - q_1^2 = -\frac{2(A^{(00)} - (1-\mu)a^{(01)})}{A^{(00)}}. \tag{33}$$

The third and fourth equation can be obtained from the boundary conditions. Equations (32) and (33) are inserted into boundary conditions (1)–(4). For the x-direction, the derived equations are as follows

$$\begin{aligned} w_x(0) &= 0 = A_2 + A_4, && (34)\\ w_x(L_x) &= 0 = A_1\sin(p_1L_x) + A_2\cos(p_1L_x) + \\ &\quad A_3\sinh(p_2L_x) + A_4\cosh(p_2L_x), && (35)\\ \left.\frac{\mathrm{d}^2w_x}{\mathrm{d}x^2}\right|_0 - c_{x1}\left.\frac{\mathrm{d}w_x}{\mathrm{d}x}\right|_0 &= 0 = -A_2\cdot p_1^2 - A_4\cdot p_2^2 - c_{x1}\cdot(A_1\cdot p_1 + A_3\cdot p_2), && (36)\\ \left.\frac{\mathrm{d}^2w_x}{\mathrm{d}x^2}\right|_{Lx} - c_{x2}\left.\frac{\mathrm{d}w_x}{\mathrm{d}x}\right|_{Lx} &= 0 = -A_1\cdot p_1^2\cdot\sin(p_1L_x) - A_2\cdot p_1^2\cdot\cos(p_1L_x) + \\ &\quad A_3\cdot p_2^2\cdot\sinh(p_2L_x) + A_4\cdot p_2^2\cdot\cosh(p_2L_x) - \\ &\quad c_{x2}\cdot(A_1\cdot p_1\cdot\cos(p_1L_x) - A_2\cdot p_1\cdot\sin(p_1L_x) + \\ &\quad A_3\cdot p_2\cdot\cosh(p_2L_x) + A_3\cdot p_2\cdot\sinh(p_2L_x)). && (37)\end{aligned}$$

Two conditions arise from equations (36)–(37) and similarly for the y-direction

$$f_1(p_1, p_2) = 0, \quad (38)$$

$$f_2(q_1, q_2) = 0, \quad (39)$$

specifically

$$\begin{aligned} f_1 = \{&p_1\cdot[(p_1^2+p_2^2)\sinh(p_2L_x) - c_{x1}p_2(\cos(p_1L_x) - \cosh(p_2L_x))]\cdot \\ &[(p_1^2+p_2^2)(p_1\sin(p_1L_x) - c_{x2}\cos(p_1L_x)) + \\ &c_{x1}\cdot(p_1(-p_1\cos(p_1L_x) + c_{x2}\sin(p_1L_x)) + p_2(-p_2\cosh(p_2L_x) + c_{x2}\sinh(p_2L_x)))] + \\ &p_2[(p_1^2+p_2^2)\sin(p_1L_x) - c_{x1}p_1(\cos(p_1L_x) - \cosh(p_2L_x))]\cdot \\ &[(p_1^2+p_2^2)(p_2\sinh(p_2L_x) - c_{x2}\cosh(p_2L_x) - c_{x1}\cdot(p_1(-p_1\cos(p_1L_x) + c_{x2}\sin(p_1L_x)) + \\ &p_2(-p_2\cosh(p_2L_x) + c_{x2}\sinh(p_2L_x)))]\} = 0, \qquad (40)\end{aligned}$$

$$\begin{aligned} f_2 = \{&q_1\cdot[(q_1^2+q_2^2)\sinh(q_2L_y) - c_{y1}q_2(\cos(q_1L_y) - \cosh(q_2L_y))]\cdot \\ &[(q_1^2+q_2^2)(q_1\sin(q_1L_y) - c_{y2}\cos(q_1L_y)) + \\ &c_{y1}\cdot(q_1(-q_1\cos(q_1L_y) + c_{y2}\sin(q_1L_y)) + q_2(-q_2\cosh(q_2L_y) + c_{y2}\sinh(q_2L_y)))] + \\ &q_2[(q_1^2+q_2^2)\sin(q_1L_y) - c_{y1}q_1(\cos(q_1L_y) - \cosh(q_2L_y))]\cdot \\ &[(q_1^2+q_2^2)(q_2\sinh(q_2L_y) - c_{y2}\cosh(q_2L_y) - c_{y1}\cdot(q_1(-q_1\cos(q_1L_y) + c_{y2}\sin(q_1L_y)) + \\ &q_2(-q_2\cosh(q_2L_y) + c_{y2}\sinh(q_2L_y)))]\} = 0. \qquad (41)\end{aligned}$$

The coefficients p_1, p_2, q_1, q_2 are computed from (31), (32), (39), (40) by the Newton's method using starting coefficients p_1, p_2, q_1, q_2 for each mode.

For comparison, the extra computations are carried out. The coefficients p_1, p_2, q_1, q_2 are evaluated using (37), (39) or (39), (40) and the conditions

$$p_1 = p_2, \quad (42)$$

$$q_1 = q_2 \quad (43)$$

instead of (31), (32). It means that equations (11) and (12) are independent and the solutions in the directions x and y are independent.

2.4. Natural frequencies

The natural angular frequencies Ω for known p_1, p_2, q_1, q_2 are given according to [9]

$$\Omega^2 = p_1^2 p_2^2 + q_1^2 q_2^2 - \frac{1}{2}(p_1^2 - p_2^2)(q_1^2 - q_2^2) - \frac{2\mu(1-\mu)a^{(01)}b^{(01)}}{A^{(00)}B^{(00)}}. \qquad (44)$$

This formula was originally derived for the clamped thin plate and was used for the considered plate.

3. Results

The computations were performed for a square plate ($L_x = L_y = 1$ m) with thickness $h = 1$ mm. The coefficient c_{ij} was considered 5, 10, 100 m^{-1} and was equal for all edges. The computations were performed using Matlab.

Results of computations are presented graphically and numerically for both types of evaluations: firstly using (32), (33) and secondly using (42), (43). Results for the first five vibration modes are depicted in Table 1. Displacements of the elastic plate are shown from their highest values (in red) to their lowest ones (in dark blue). The first 10 modes are expressed numerically in Tables 2–4 using p_1, p_2, q_1, q_2 and Ω.

Table 1. Vibration modes – comparison for three stiffnesses (horizontal axis – x direction, vertical axis – y direction, the highest values in red, the lowest values in dark blue)

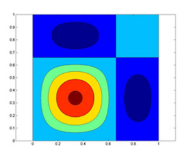
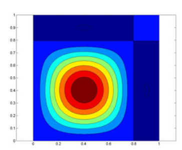
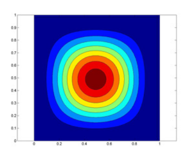
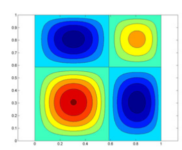
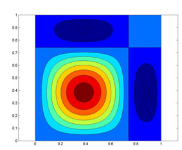
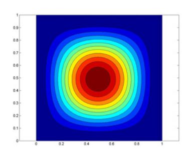
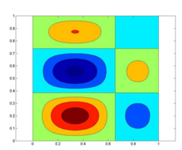
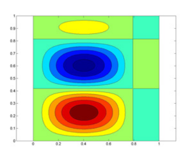
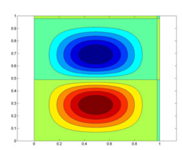
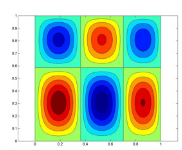
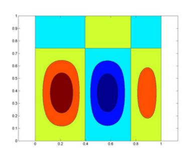
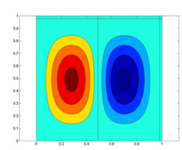
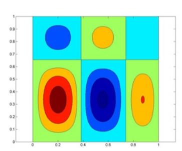
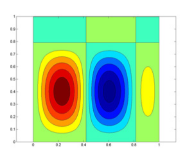
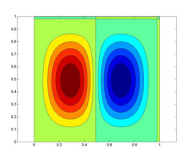
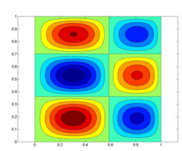
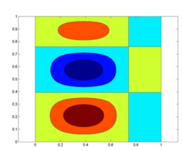
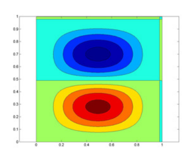
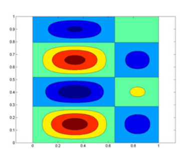
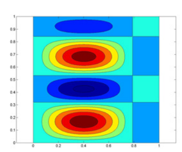
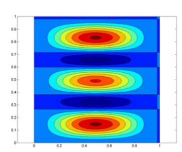
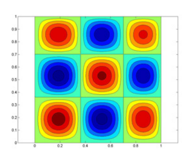
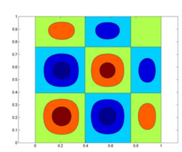
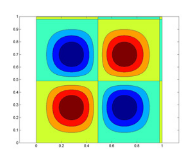
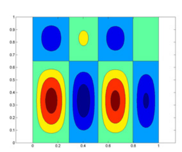
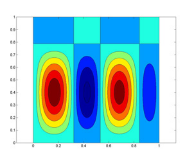
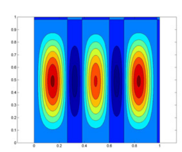
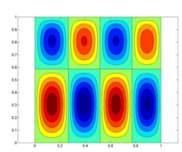
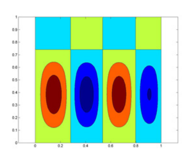
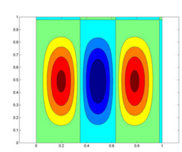
$p_1 \neq p_2, q_1 \neq q_2$			$p_1 = p_2, q_1 = q_2$		
$c = 5\ \mathrm{m}^{-1}$	$c = 10\ \mathrm{m}^{-1}$	$c = 100\ \mathrm{m}^{-1}$	$c = 5\ \mathrm{m}^{-1}$	$c = 10\ \mathrm{m}^{-1}$	$c = 100\ \mathrm{m}^{-1}$
					
					
					
					
					

Table 2. Vibration modes and natural frequencies Ω [rad $\cdot$ s^{-1}] – results for $c = 5$ m^{-1}

mode	$p_1 \neq p_2, q_1 \neq q_2$					$p_1 = p_2, q_1 = q_2$		
	p_1 [m^{-1}]	p_2 [m^{-1}]	q_1 [m^{-1}]	q_2 [m^{-1}]	Ω [rad $\cdot$ s^{-1}]	p_1 [m^{-1}]	q_1 [m^{-1}]	Ω [rad $\cdot$ s^{-1}]
1	5.991	2.994	9.282	3.165	12.617	5.945	5.945	49.980
2	9.282	3.165	5.991	2.994	12.617	5.945	9.281	93.098
3	6.008	2.814	6.008	2.814	13.212	9.281	5.945	93.098
4	5.977	3.161	12.500	3.605	23.326	9.281	9.281	121.805
5	12.500	3.604	5.977	3.161	23.326	5.945	12.486	159.865
6	9.280	3.268	9.280	3.268	31.732	12.486	5.945	159.865
7	5.966	3.312	15.665	3.881	35.368	9.281	12.486	178.118
8	15.665	3.881	5.966	3.312	35.368	12.486	9.281	178.118
9	9.278	3.381	12.499	3.666	47.401	12.486	12.486	220.491
10	12.499	3.666	9.278	3.666	47.401	5.945	15.657	247.675

Table 3. Vibration modes and natural frequencies Ω [rad $\cdot$ s^{-1}] – results for $c = 10$ m^{-1}

mode	$p_1 \neq p_2, q_1 \neq q_2$					$p_1 = p_2, q_1 = q_2$		
	p_1 [m^{-1}]	p_2 [m^{-1}]	q_1 [m^{-1}]	q_2 [m^{-1}]	Ω [rad $\cdot$ s^{-1}]	p_1 [m^{-1}]	q_1 [m^{-1}]	Ω [rad $\cdot$ s^{-1}]
1	5.768	2.525	5.768	2.525	7.891	5.221	5.221	38.550
2	5.710	2.730	9.073	2.750	8.665	5.221	8.857	83.052
3	9.073	2.750	5.710	2.730	8.665	8.857	5.221	83.052
4	5.660	2.911	12.393	3.168	11.005	8.857	8.857	101.946
5	12.393	3.168	5.660	2.911	11.005	5.221	12.245	152.387
6	5.619	3.065	15.580	3.382	22.516	12.245	5.221	152.387
7	15.580	3.382	5.619	3.065	22.516	8.857	12.245	169.210
8	9.065	2.866	9.065	2.866	37.210	12.245	8.857	169.210
9	5.584	3.120	18.771	3.682	39.332	12.245	12.245	212.028
10	18.771	3.682	5.584	3.120	39.332	5.221	18.708	351.037

Table 4. Vibration modes and natural frequencies Ω [rad $\cdot$ s^{-1}] – results for $c = 100$ m^{-1}

mode	$p_1 \neq p_2, q_1 \neq q_2$					$p_1 = p_2, q_1 = q_2$		
	p_1 [m^{-1}]	p_2 [m^{-1}]	q_1 [m^{-1}]	q_2 [m^{-1}]	Ω [rad $\cdot$ s^{-1}]	p_1 [m^{-1}]	q_1 [m^{-1}]	Ω [rad $\cdot$ s^{-1}]
1	5.633	2.322	5.633	2.322	2.166	4.734	4.734	31.699
2	5.350	2.991	18.566	3.029	10.839	4.734	7.866	65.804
3	18.566	3.029	5.350	2.991	10.839	7.866	4.734	65.804
4	5.545	2.526	8.797	2.404	15.131	7.866	7.866	87.494
5	8.797	2.404	5.545	2.526	15.131	4.734	11.020	123.489
6	5.467	2.711	12.184	2.727	16.285	11.020	4.734	123.489
7	12.184	2.727	5.467	2.711	16.285	7.866	11.020	136.288
8	5.405	2.859	15.310	2.796	17.670	11.020	7.866	136.288
9	15.310	2.796	5.405	2.859	17.670	11.020	11.020	171.741
10	5.304	3.104	21.696	3.080	21.703	4.734	14.177	202.242

4. Discussion and further work

Tables 1–3 show vibration modes of the square plate computed by the method suggested in this paper. Computed vibration modes fulfil our expectations and experience: the results for higher stiffness were close to the results for a clamped plate, whereas the results for lower stiffness were close to the results for a simply supported plate, see the paper by Phamová, Vampola [8]. The results obtained using conditions (42), (43) seem to correspond better with the reality. The comparison of results between (40), (41) and (42), (43) with another mathematical approach and experiment are intended. On the basis of the presented results, it can be stated that it is possible to use the suggested method for the evaluation of vibration modes of plates.

The obtained natural frequencies of a simply supported plate with torsional springs are not in agreement with our expectations. Equation (44) is not suitable for this type of boundary conditions. Unfortunately, the appropriate relation has not been found so far.

In future, there is intended the comparison of presented results with a different approach, e.g. the finite element method. Another relation for the computation of natural frequencies of the plate supported with torsional springs is necessary.

5. Conclusions

This paper presented the free flexural vibration modes of a plate with general boundary conditions. The plate vibration equation was reduced into two ordinary differential equations. The derivation of governing equations was indicated. The results were presented graphically and numerically. The described analytic-numerical method was able to achieve vibration modes that were in agreement with our expectations.

Acknowledgements

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