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Single variable refined beam theories for the bending, buckling and free vibration of homogenous beams

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Abstract

In this paper, single variable beam theories taking into account effect of transverse shear deformation are developed and applied for the bending, buckling and free vibration analysis of thick isotropic beams. The most important feature of the present beam theories is that unlike any other higher order theory, the proposed class of theories contains only one unknown variable and does not require shear correction factor. The displacement field of the present theories is built upon the classical beam theory. The theories account for parabolic distribution of transverse shear stress using constitutive relations, satisfying the traction free conditions at top and bottom surfaces of the beam. Governing differential equation and boundary conditions of these theories are obtained using the principle of virtual work. Results obtained for the displacements, stresses, fundamental frequencies and critical buckling loads of simply supported isotropic solid beams are compared with those obtained by other theories to validate the accuracy of the present theories.

Keywords: thick beam, shear deformation, single variable beam theory, bending, buckling, vibration

1. Introduction

Thick beams are basically two dimensional problems of elasticity theory. Reduction of this problem to the one dimensional approximate problem for its analysis has always been the main objective of researchers.

It is wellknown that classical beam theory (CBT) is based on Euler-Bernoulli hypothesis that the plane sections perpendicular to the neutral axis before deformation remain plane and perpendicular to the neutral axis after deformation. The CBT neglects the effects of transverse shear deformation due to which, it is applied to thin beam only. It underestimates the values of displacements and overestimates the frequencies and buckling loads in case of thick or moderately thick beams.

To overcome the limitations of CBT, first order shear deformation theory (FSDT) is developed. The theory is also known as the Timoshenko beam theory [22]. The theory is based on the hypothesis that the plane sections perpendicular to the neutral axis before deformation remain plane, but not necessarily perpendicular to the neutral axis after deformation. FSDT considers the linear variation of midplane displacement. In this theory transverse shear strain distribution is assumed to be constant through the beam thickness and thus requires problem dependent shear correction factor to appropriately represent the strain energy of shear deformation.

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The limitations of CBT and FSDT led to the development of higher order shear deformation theories for the analysis of thick beams considering the effect of transverse shear deformation. Levinson [10], Krishna Murty [9] and Heyliger and Reddy [7] presented a parabolic shear deformation theory for the static and dynamic analysis of thick beams. Ghugal [3] has extended this theory, including the transverse normal strain and transverse shear effects for the bending and vibration of isotropic beams. A trigonometric shear deformation theory taking into account effect of transverse shear deformation for shear flexible beams is presented by Ghugal and Shimpi [6]. Ghugal and Sharma [5] extended the hyperbolic shear deformation theory of Soldatos [20] for the bending analysis of isotropic beams with various boundary conditions. Sayyad and Ghugal [15] have developed trigonometric shear and normal deformation theory considering the effects of transverse shear and normal deformations for the bending of isotropic and laminated beams. Sayyad and Ghugal [16] also developed a new hyperbolic theory representing the combined effect of shear and bending rotations on flexural analysis of thick beams. Karama et al. [8] have developed exponential shear deformation theory for the bending, buckling and vibration of thick beam. Sayyad [14] developed unified beam theory for the bending and free vibration analysis of isotropic beams in which parabolic, trigonometric, hyperbolic and exponential functions are used in terms of thickness coordinates to represent the effect of transverse shear deformation. Ghugal [4] was the first to develop the parabolic shear deformation theory including one unknown for the bending and vibration of isotropic beams. Recently, Sayyad et al. [17–19] presented refined shear deformation theories for the bending analysis of isotropic, laminated composite and sandwich beams. The displacement fields and number of unknowns involved in various shear deformation theories are summerized in this paper (see Table 1).

From the above literature, it is pointed out that the minimum number of unknowns in the higher order theories reported in the literature in last few decades is two. Therefore, in the present study a new class of refined shear deformation theories is presented which involves only one unknown satisfying the traction free boundary conditions at top and bottom surfaces of the beams. To prove the efficiency of the proposed theories, they are applied for the bending, buckling and free vibration analysis of simply supported uniform isotropic solid beams of rectangular cross-section.

2. Theoretical formulations

2.1. Beam under consideration

Consider a prismatic beam of length L and rectangular cross-section ($b \times h$) as shown in Fig. 1. The loading $q(x)$ is assumed in the vertical plane of symmetry of the cross-section and directed transversely to the beam. The beam occupies the region $0 \leq x \leq L$, $-b/2 \leq y \leq b/2$, $-h/2 \leq z \leq h/2$ in a Cartesian coordinate system. The downward z-direction is taken as positive.

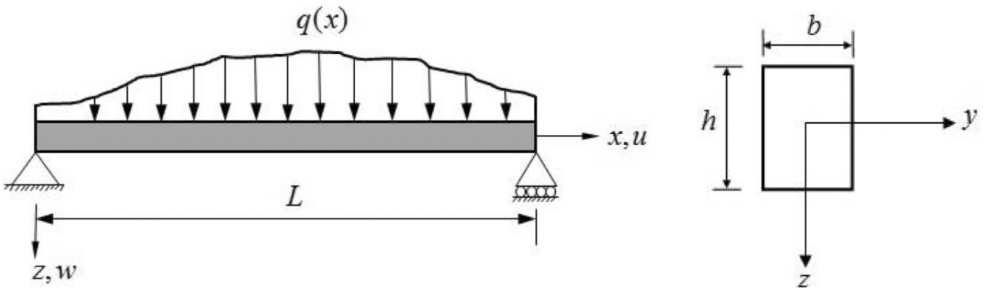


Fig. 1. A simply supported beam of rectangular cross-section

Table 1. The displacement field and the number of unknowns involved in various beam theories

Theory	**Displacement field**	**Unknowns**
CBT	$u(x,z,t) = -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x}$, $w(x,t) = w_0(x,t)$.	One
Ghugal [4]	$u(x,z,t) = -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} - \frac{(1+\mu)h^2}{4}z\left[1 - \frac{4}{3}\left(\frac{z}{h}\right)^2\right]\frac{\mathrm{d}^3w_0(x,t)}{\mathrm{d}x^3}$, $w(x,t) = w_0(x,t)$.	One
Timoshenko [22]	$u(x,z,t) = -z\phi(x,t)$, $w(x,t) = w_0(x,t)$.	Two
Heyliger and Reddy [7]	$u(x,z,t) = z\left[\phi(x,t) - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi(x,t) + \frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x}\right)\right]$, $w(x,t) = w_0(x,t)$.	Two
Ghugal and Shimpi [6]	$u(x,z,t) = -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} + \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\phi(x,t)$, $w(x,t) = w_0(x,t)$.	Two
Soldatos [20]	$u(x,z,t) = -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} + \left[z\cosh\left(\frac{1}{2}\right) - h\sinh\left(\frac{z}{h}\right)\right]\phi(x,t)$, $w(x,t) = w_0(x,t)$.	Two
Karama et al. [8]	$u(x,z,t) = -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} + z\exp\left[-2\left(\frac{z}{h}\right)^2\right]\phi(x,t)$, $w(x,t) = w_0(x,t)$.	Two
Sayyad and Ghugal [16]	$u(x,z,t) = -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} + \left[z\cosh\left(\frac{1}{2}\right) - h\sinh\left(\frac{z}{h}\right)\right]\left(\phi + \frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x}\right)$, $w(x,t) = w_0(x,t)$.	Two
Ghugal [3]	$u(x,z,t) = z\left[\phi(x,t) - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi(x,t) + \frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x}\right)\right]$, $w(x,z,t) = w_0(x,t) + \left[1 - 4\left(\frac{z}{h}\right)^2\right]\psi(x,t)$.	Three
Sayyad and Ghugal [15]	$u(x,z,t) = -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} + \frac{h}{\pi}\sin\frac{\pi x}{h}\phi(x,t)$, $w(x,z,t) = w_0(x,t) + \frac{h}{\pi}\cos\frac{\pi z}{h}\psi(x,t)$.	Three
Zenkour [23]	$u(x,z,t) = -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} + \left[h\sinh\left(\frac{z}{h}\right) - \frac{4}{3}\frac{z^3}{h^2}\cosh\left(\frac{1}{2}\right)\right]\frac{\mathrm{d}\phi(x,t)}{\mathrm{d}x}$, $w(x,z,t) = w_0(x,t) + \frac{1}{12}\left[\cosh\left(\frac{z}{h}\right) - \frac{4z^2}{h^2}\cosh\left(\frac{1}{2}\right)\right]\phi(x,t)$.	Two
Thai et al. [21]	$u(x,z,t) = -\left(z - \frac{4z^3}{3h^2}\right)\beta(x,t) - \frac{4z^3}{3h^2}\phi(x,t)$, $w(x,t) = w_0(x,t)$.	Three
Mantari et al. [11]	$u(x,z,t) = -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} + \left[\sin\left(\frac{\pi z}{h}\right)\mathrm{e}^{m\cos\left(\frac{\pi z}{h}\right)} + \frac{m\pi z}{h}\right]\phi(x,t)$, $w(x,t) = w_0(x,t)$.	Two

Table 1. *Continued*

Theory	Displacement field	Unknowns
Mantari et al. [12]	$u(x,z,t) =$ $-z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} + \left[\tan(mz) - mz\sec^2\left(\frac{mh}{2}\right)\right]\phi(x,t)$, $w(x,t) = w_0(x,t)$.	Two
Carrera et al. [1]	Carrera's Unified Formulation (CUF) $u = F_t u_t + F_b u_b + F_r u_r, \quad r = 2,\dots,N$ $\sigma_n = F_t\sigma_{nt} + F_b\sigma_{nb} + F_r\sigma_{nr}, \quad r = 2,\dots,N$	Depending on value of r

w_0 = Displacement of neutral axis in z-direction
ϕ = Shear slope at neutral axis corresponding to shear deformation
ψ = Shear slope at neutral axis corresponding to normal deformation (thickness stretching)

2.2. *Assumptions made in theoretical formulation*

Assumptions made in the theoretical formulation of proposed theory are as follows:

1. The axial displacement u in x-direction consists of two parts:
   (a) A displacement component analogous to displacement of classical beam theory.
   (b) Displacement component due to transverse shear deformation, which is assumed to be parabolic, sinusoidal and exponential in nature with respect to thickness coordinate.
2. There is no relative motion in the y-direction at any points in the cross-section of the beam.
3. The transverse displacement w in the z-direction is assumed to be a function of x and t only.
4. One-dimensional constitutive law is used.

2.3. *Kinematics*

Based on the above assumptions, the displacement field of the proposed beam theories is given as:

$$\begin{aligned} u(x,z,t) &= -z\frac{\mathrm{d}w_0(x,t)}{\mathrm{d}x} - \frac{(1+\mu)h^2}{4}f(z)\frac{\mathrm{d}^3w_0(x,t)}{\mathrm{d}x^3}, \\ w(x,t) &= w_0(x,t), \end{aligned} \tag{1}$$

where u and w are the axial and transverse displacements of the beam center line in x- and z-directions, respectively. The functions $f(z)$ assigned according to the shearing stress distribution through the thickness of the beam as given below:

Parabolic shear deformation theory (PSDT): $f(z) = \left[z - \frac{4}{3}\frac{z^3}{h^2}\right]$.

Trigonometric shear deformation theory (TSDT): $f(z) = \frac{h}{\pi}\sin\frac{\pi z}{h}$.

Exponential shear deformation theory (ESDT): $f(z) = z\exp\left[-2\left(\frac{z}{h}\right)^2\right]$.

The theory presented by Ghugal [4] is the special case of present displacement field. The normal strain ε_x corresponding to the displacement in x-direction and the shear strain γ_{zx} associated with the proposed beam theories are as follows:

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} = -z\frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2} - \frac{(1+\mu)h^2}{4} f(z)\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^4}, \tag{2}$$

$$\gamma_{zx} = \frac{\mathrm{d}u}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{(1+\mu)h^2}{4}\left[\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right]\frac{\mathrm{d}^3 w_0}{\mathrm{d}x^3}. \tag{3}$$

One dimensional constitutive law is used to obtain bending and transverse shear stresses. These stresses are as follows:

$$\sigma_x = E\varepsilon_x = E\left[-z\frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2} - \frac{(1+\mu)h^2}{4} f(z)\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^4}\right], \tag{4}$$

$$\tau_{zx} = G\gamma_{zx} = -G\frac{(1+\mu)h^2}{4}\left[\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right]\frac{\mathrm{d}^3 w_0}{\mathrm{d}x^3}, \tag{5}$$

where E is Young's modulus, μ is Poisson's ratio and G is shear modulus.

2.4. Governing differential equation of proposed beam theories

By using the relations (2)–(5) of strains and stresses and the principle of virtual work, variationally consistent governing differential equation and associated boundary conditions are obtained. The dynamic version of the principle of virtual work when applied to the beam leads to:

$$\begin{aligned}\int_0^L \int_A (\sigma_x \delta\epsilon_x + \tau_{xz}\delta\gamma_{xz})\,\mathrm{d}A\,\mathrm{d}x + \rho\int_0^L \int_A \left(\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}\delta u + \frac{\mathrm{d}^2 w}{\mathrm{d}t^2}\delta w\right)\mathrm{d}A\,\mathrm{d}x - \\ \int_0^L q(x)\delta w\,\mathrm{d}x - \int_0^L N_{xx}^0 \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}\delta w}{\mathrm{d}x}\,\mathrm{d}x = 0,\end{aligned} \tag{6}$$

where the symbol δ denotes the variational operator, A is the cross-sectional area, ρ denotes the density of material and N_{xx}^0 is the axial compressive force. Substitution of strains from (2) and (3) into (6) leads to the following equation in terms of moments M^b, M^s and shear force V resultants

$$\begin{aligned}-\int_0^L M^b \frac{\mathrm{d}^2\delta w_0}{\mathrm{d}x^2}\,\mathrm{d}x - \frac{(1+\mu)h^2}{4}\int_0^L M^s\frac{\mathrm{d}^4\delta w_0}{\mathrm{d}x^4}\,\mathrm{d}x - \frac{(1+\mu)h^2}{4}\int_0^L V\frac{\mathrm{d}^3\delta w_0}{\mathrm{d}x^3}\,\mathrm{d}x + \\ I_0\int_0^L \frac{\mathrm{d}^3 w_0}{\mathrm{d}x\,\mathrm{d}t^2}\frac{\mathrm{d}\delta w_0}{\mathrm{d}x}\,\mathrm{d}x + I_1\int_0^L \frac{\mathrm{d}^3 w_0}{\mathrm{d}x\,\mathrm{d}t^2}\frac{\mathrm{d}^3\delta w_0}{\mathrm{d}x^3}\,dx + I_1\int_0^L \frac{\mathrm{d}^5 w_0}{\mathrm{d}x^3\,\mathrm{d}t^2}\frac{\mathrm{d}\delta w_0}{\mathrm{d}x}\,\mathrm{d}x + \\ I_2\int_0^L \frac{\mathrm{d}^5 w_0}{\mathrm{d}x^3\,\mathrm{d}t^2}\frac{\mathrm{d}^3\delta w_0}{\mathrm{d}x^3}\,\mathrm{d}x + I_3\int_0^L \frac{\mathrm{d}^2 w_0}{\mathrm{d}t^2}\delta w_0\mathrm{d}x - \int_0^L q(x)\,\delta w_0\,\mathrm{d}x - \int_0^L N_{xx}^0\frac{\mathrm{d}w_0}{\mathrm{d}x}\frac{\mathrm{d}\delta w_0}{\mathrm{d}x}\,\mathrm{d}x = 0,\end{aligned} \tag{7}$$

where moment resultants M^b, M^s and shear force resultant V are as follows

$$M^b = \int_A \sigma_x z\,\mathrm{d}A, \quad M^s = \int_A \sigma_x f(z)\,\mathrm{d}A \quad \text{and} \quad V = \int_A \tau_{xz}\frac{\mathrm{d}f(z)}{\mathrm{d}z}\,\mathrm{d}A. \tag{8}$$

Integrating the equation (7) by parts, and collecting the coefficients of δw_0, the following governing differential equation is obtained:

$$\frac{\mathrm{d}^2 M^b}{\mathrm{d}x^2} + \frac{(1+\mu)h^2}{4}\frac{\mathrm{d}^4 M^s}{\mathrm{d}x^4} - \frac{(1+\mu)h^2}{4}\frac{\mathrm{d}^3 V}{\mathrm{d}x^3} + I_2\frac{\mathrm{d}^8 w_0}{\mathrm{d}x^6\,\mathrm{d}t^2} + 2I_1\frac{\mathrm{d}^6 w_0}{\mathrm{d}x^4\,\mathrm{d}t^2} + \\ I_0\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^2\,\mathrm{d}t^2} - I_3\frac{\mathrm{d}^2 w_0}{\mathrm{d}t^2} + q(x) + N_{xx}^0\frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2} = 0. \tag{9}$$

Substituting M^b, M^s and V from (8) into (9) leads to the following governing differential equation

$$C_0\frac{\mathrm{d}^8 w_0}{\mathrm{d}x^8} + (2B_0 - D_0)\frac{\mathrm{d}^6 w_0}{\mathrm{d}x^6} + A_0\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^4} - \\ I_2\frac{\mathrm{d}^8 w_0}{\mathrm{d}x^6\,\mathrm{d}t^2} - 2I_1\frac{\mathrm{d}^6 w_0}{\mathrm{d}x^4\,\mathrm{d}t^2} - I_0\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^2\,\mathrm{d}t^2} + I_3\frac{\mathrm{d}^2 w_0}{\mathrm{d}t^2} = q(x) + N_{xx}^0\frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2}, \tag{10}$$

where A_0, B_0, C_0 and D_0 are stiffness coefficients and I_0, I_1, I_2 and I_3 are inertia terms as defined below:

$$\begin{aligned} A_0 &= E\int_A z^2\,\mathrm{d}A, \quad B_0 = \frac{E(1+\mu)h^2}{4}\int_A zf(z)\,\mathrm{d}A, \\ C_0 &= \frac{E(1+\mu)^2h^4}{16}\int_A [f(z)]^2\,\mathrm{d}A, \quad D_0 = \frac{G(1+\mu)^2h^4}{16}\int_A \left[\frac{\mathrm{d}f(z)}{\mathrm{d}z}\right]^2\mathrm{d}A, \\ I_0 &= \rho\int_A z^2\,\mathrm{d}A, \quad I_1 = \frac{\rho(1+\mu)h^2}{4}\int_A zf(z)\,\mathrm{d}A, \\ I_2 &= \frac{\rho(1+\mu)^2h^4}{16}\int_A [f(z)]^2\,\mathrm{d}A, \quad I_3 = \rho\int_A \mathrm{d}A. \end{aligned} \tag{11}$$

The associated boundary conditions at $x = 0$ and $x = L$ are as follows:

Either $$\frac{\mathrm{d}M^b}{\mathrm{d}x} + \frac{(1+\mu)h^2}{4}\frac{\mathrm{d}^3 M^s}{\mathrm{d}x^3} - \frac{(1+\mu)h^2}{4}\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} - \\ N_{xx}^0\frac{\mathrm{d}w_0}{\mathrm{d}x} + I_2\frac{\mathrm{d}^7 w_0}{\mathrm{d}x^5\,\mathrm{d}t^2} + 2I_1\frac{\mathrm{d}^5 w_0}{\mathrm{d}x^3\,\mathrm{d}t^2} + I_0\frac{\mathrm{d}^3 w_0}{\mathrm{d}x\,\mathrm{d}t^2} = 0 \quad \text{or} \quad w_0 \quad \text{is prescribed.} \tag{12}$$

Either $$M^b + \frac{(1+\mu)h^2}{4}\frac{\mathrm{d}M^s}{\mathrm{d}x^2} - \frac{(1+\mu)h^2}{4}\frac{\mathrm{d}V}{\mathrm{d}x} - \\ I_2\frac{\mathrm{d}^6 w_0}{\mathrm{d}x^4\,\mathrm{d}t^2} - I_1\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^2\,\mathrm{d}t^2} = 0 \quad \text{or} \quad \frac{\mathrm{d}w_0}{\mathrm{d}x} \quad \text{is prescribed.} \tag{13}$$

Either $$\frac{(1+\mu)h^2}{4}V - \frac{(1+\mu)h^2}{4}\frac{\mathrm{d}M^s}{\mathrm{d}x} + \\ I_2\frac{\mathrm{d}^5 w_0}{\mathrm{d}x^3\,\mathrm{d}t^2} + I_1\frac{\mathrm{d}^3 w_0}{\mathrm{d}x\,\mathrm{d}t^2} = 0 \quad \text{or} \quad \frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2} \quad \text{is prescribed.} \tag{14}$$

Either $$\frac{(1+\mu)h^2}{4}M^s = 0 \quad \text{or} \quad \frac{\mathrm{d}^3 w_0}{\mathrm{d}x^3} \quad \text{is prescribed.} \tag{15}$$

3. Validation of proposed beam theories

For the purpose of demonstrating the validity of the proposed beam theories, the following examples are considered.

3.1. Bending analysis of simply supported isotropic beams

In this example, bending analysis of a simply supported isotropic beam subjected to various static loadings as shown in Fig. 2 (a)–(c) is considered. The transverse load $q(x)$ given by (16) is acting on the top surface of the beam, i.e. $z = -h/2$,

$$q(x) = \sum_{m=1,3,5,\ldots}^{\infty} q_m \sin \frac{m\pi x}{L}, \tag{16}$$

where m is the positive integer and q_m are the coefficients of Fourier expansion as given in Table 2. The quantity $q_0(x)$ in this table denotes the maximum intensity of load (see Fig. 2).

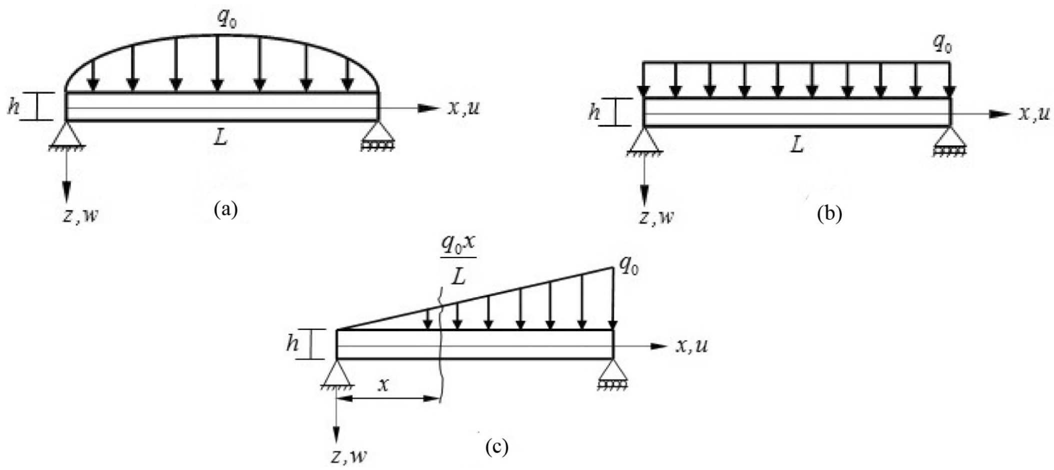


Fig. 2. A simply supported isotropic beam subjected to (a) a sinusoidal load (b) a uniformly distributed load (c) a linearly varying load

Table 2. Coefficient of Fourier expansion q_m for various static loadings

Load	Coefficient of Fourier expansion
Sinusoidal load	$q_m = q_0$ for $m = 1$
Uniformly distributed load	$q_m = \begin{cases} \dfrac{4q_0}{m\pi} & \text{for } m = 1, 3, 5, \ldots \\ 0 & \text{for } m = 2, 4, 6, \ldots \end{cases}$
Linearly varying load	$q_m = \begin{cases} \dfrac{2q_0}{m\pi} \cos m\pi & \text{for } m = 1, 3, 5, \ldots \\ 0 & \text{for } m = 2, 4, 6, \ldots \end{cases}$

According to Navier the following is the solution form for $w_0(x)$ that satisfies the simply supported boundary conditions exactly:

$$w_0(x) = \sum_{m=1,3,5,\ldots}^{\infty} w_m \sin\frac{m\pi x}{L}, \tag{17}$$

where w_m are the unknown coefficients. The governing equation for bending analysis of beams can be obtained by discarding time dependent terms and axial compressive force N_{xx}^0 from (10) as follows:

$$C_0\frac{\mathrm{d}^8 w_0}{\mathrm{d}x^8} + (2B_0 - D_0)\frac{\mathrm{d}^6 w_0}{\mathrm{d}x^6} + A_0\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^4} = q(x). \tag{18}$$

Substituting solution form from (17) and the load $q(x)$ from (16) into the governing equation (18) one can obtain

$$w_m = \frac{q_m}{\left[C_0\dfrac{m^8\pi^8}{L^8} - (2B_0 - D_0)\dfrac{m^6\pi^6}{L^6} + A_0\dfrac{m^4\pi^4}{L^4}\right]}. \tag{19}$$

Having obtained the value of w_m one can determine displacements and stresses using the relations (1)–(5).

3.2. *Buckling analysis of simply supported isotropic beams*

In this example, the efficiency of the proposed beam theories is checked for the buckling analysis of isotropic beams. A simply supported beam subjected to an axial compressive forces F_0 as shown in Fig. 3 is considered.

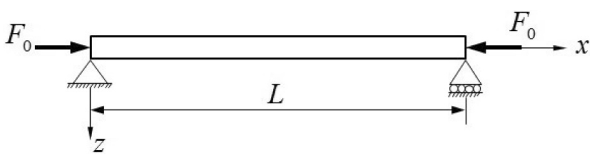


Fig. 3. A simply supported isotropic beam subjected to an axial compressive force

The governing equation of the proposed beam theories in the case of static buckling is obtained by discarding time dependent terms and setting $q(x) = 0$ and $N_{xx}^0 = -F_0$ in (10). The equation becomes

$$C_0\frac{\mathrm{d}^8 w_0}{\mathrm{d}x^8} + (2B_0 - D_0)\frac{\mathrm{d}^6 w_0}{\mathrm{d}x^6} + A_0\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^4} + F_0\frac{\mathrm{d}^2 w_0}{\mathrm{d}x^2} = 0. \tag{20}$$

Substituting solution scheme of $w_0(x)$ from (17) into (20) one can obtain F_0:

$$F_0 = \frac{\left[C_0\dfrac{m^8\pi^8}{L^8} - (2B_0 - D_0)\dfrac{m^6\pi^6}{L^6} + A_0\dfrac{m^4\pi^4}{L^4}\right]}{\dfrac{m^2\pi^2}{L^2}}. \tag{21}$$

3.3. Free vibration analysis of simply supported isotropic beams

In this section, the proposed beam theories are applied for the free vibration analysis of isotropic beams. The governing equation for free vibration analysis is obtained by setting transverse load $q(x)$ and axial compressive forces F_0 to zero in (10). The equation becomes

$$C_0\frac{\mathrm{d}^8 w_0}{\mathrm{d}x^8} + (2B_0 - D_0)\frac{\mathrm{d}^6 w_0}{\mathrm{d}x^6} + A_0\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^4} - I_2\frac{\mathrm{d}^8 w_0}{\mathrm{d}x^6\,\mathrm{d}t^2} - \\ 2I_1\frac{\mathrm{d}^6 w_0}{\mathrm{d}x^4\,\mathrm{d}t^2} - I_0\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^2\,\mathrm{d}t^2} + I_3\frac{\mathrm{d}^2 w_0}{\mathrm{d}t^2} = 0. \tag{22}$$

Following the Navier's solution procedure, closed form solution to the displacement variable w_0 satisfing boundary conditions of flexural vibration can be expressed in the following form

$$w_0(x,t) = \sum_{m=1}^{\infty} w_m \sin\frac{m\pi x}{L}\sin\omega_m t, \tag{23}$$

where w_m is the amplitude of translation and ω_m is the fundamental frequency of m-th mode of vibration. Substitution of this solution form into (22) results in the following standard eigenvalue problem

$$[K - \omega^2 M]w_m = 0, \tag{24}$$

where

$$K = C_0\frac{m^8\pi^8}{L^8} - (2B_0 - D_0)\frac{m^6\pi^6}{L^6} + A_0\frac{m^4\pi^4}{L^4}, \tag{25}$$

$$M = I_2\frac{m^6\pi^6}{L^6} - 2I_1\frac{m^4\pi^4}{L^4} + I_0\frac{m^2\pi^2}{L^2} + I_3\frac{m^2\pi^2}{L^2}. \tag{26}$$

From the solution of (24), the fundamental frequencies for all modes of vibration can be obtained.

4. Numerical results and discussions

The numerical results are obtained for the beam made up of isotropic material with Young's modulus $E = 210$ GPa and Poisson's ratio $\mu = 0.3$. The displacements, stresses, fundamental frequencies and critical buckling loads are presented in the following non-dimensional form:

$$\bar{u}\left(L, \pm\frac{h}{2}\right) = \frac{uEbh^2}{q_0L^3}, \qquad \bar{w}\left(\frac{L}{2}, 0\right) = \frac{10wEbh^3}{q_0L^4}, \qquad \bar{\sigma}_x\left(\frac{L}{2}, \pm\frac{h}{2}\right) = \frac{b\sigma_x}{q_0}, \tag{27}$$

$$\bar{\tau}_{zx}(0,0) = \frac{b\tau_{zx}}{q_0}, \qquad \bar{\omega} = \omega_m\left(\frac{L^2}{h}\right)\sqrt{\frac{\rho}{E}}, \qquad N_{cr} = \frac{F_0L^2}{Eh^3}.$$

The percentage errors in results of a particular theory are shown in Tables 3–6. These errors are determined with respect to the results of exact elasticity solution as

$$\%\ \text{error} = \frac{\text{value by a particular theory} - \text{value by exact elasticity solution}}{\text{value by exact elasticity solution}} \times 100. \tag{28}$$

4.1. Bending analysis of simply supported isotropic beams

In this section, the results obtained by using present parabolic (PSDT), trigonometric (TSDT) and exponential (ESDT) shear deformation theories are compared and discussed with the corresponding results of CBT of Bernoulli-Euler, FSDT of Timoshenko [22], higher order shear deformation theory (HSDT) of Heyliger and Reddy [7] and the exact elasticity solution given by Pagano [13].

The non-dimensional displacements and stresses of isotropic beam subjected to a sinusoidal load are presented in Table 3. The obtained results are compared with the results generated by using other theories and exact solution available in the literature. The displacements and stresses are obtained for $L/h = 4$ and 10. Table 3 also includes the % error in numerical results predicted by various theories with respect to the exact solution. From Table 3 it is observed that the axial displacement predicted by present theories is in excellent agreement with the exact solution whereas HSDT overestimates the same. The present theories predict accurate value of transverse displacement as compared to other theories whereas CBT shows lower values of transverse displacement and independent of aspect ratio. The bending stresses obtained by using present theories are in good agreement with that of exact solutions. The transverse shear stress can be obtained using the constitutive relation τ_{zx}^{CR} and by integrating the equilibrium equation of the theory of elasticity. The transverse shear stress predicted by constitutive relation is higher than the exact solution, whereas it is in excellent agreement with the exact solution when obtained using equilibrium equation. The through thickness distributions of axial displacement, bending stress and transverse shear stress *via* constitutive relation for $L/h = 4$ are shown in Fig. 4 (a)–(c), respectively.

Tables 4 and 5 show the comparison of displacements and stresses of isotropic beams subjected to uniformly distributed and linearly varying loads, respectively. From the examination of these tables it is pointed out that the present results are found to agree well with those of exact solutions. In both the cases transverse shear stress is obtained accurately *via* constitutive

Table 3. The comparison of displacements and stresses of a simply supported isotropic beam subjected to a sinusoidal load

L/h	Source	$\bar{u}$	% Error	$\bar{w}$	% Error	$\bar{\sigma}_x$	% Error	$\bar{\tau}_{zx}^{CR}$	% Error	$\bar{\tau}_{zx}^{EE}$	% Error
4	PSDT	0.775	0.780	1.424	0.921	9.745	−2.139	2.208	16.211	1.839	−3.211
	TSDT	0.781	1.560	1.425	0.992	9.816	−1.426	2.209	16.263	1.850	−2.632
	ESDT	0.784	1.951	1.421	0.709	9.856	−1.024	2.203	15.947	1.844	−2.947
	HSDT	0.794	3.251	1.429	1.276	9.986	0.281	1.906	0.316	1.897	−0.158
	FSDT	0.774	0.650	1.430	1.347	9.727	−2.320	1.273	−33.000	1.910	0.526
	CBT	0.774	0.650	1.232	−12.680	9.727	−2.320	–	–	1.910	0.526
	Exact	0.769	0.000	1.411	0.000	9.958	0.000	1.900	0.000	1.900	0.000
10	PSDT	1.942	0.622	1.263	0.318	61.027	0.181	4.897	2.770	4.766	0.021
	TSDT	1.944	0.725	1.263	0.318	61.076	0.261	4.896	2.749	4.769	0.084
	ESDT	1.945	0.777	1.263	0.318	61.125	0.341	4.895	2.728	4.768	0.063
	HSDT	1.943	0.674	1.263	0.318	61.052	0.222	4.773	0.168	4.769	0.084
	FSDT	1.935	0.259	1.264	0.397	60.793	−0.204	3.183	−33.200	4.775	0.210
	CBT	1.935	0.259	1.232	−2.145	60.793	−0.204	–	–	4.775	0.210
	Exact	1.930	0.000	1.259	0.000	60.917	0.000	4.765	0.000	4.765	0.000

HSDT: Heyliger and Reddy [7], FSDT: Timoshenko [22], Exact: Pagano [13]

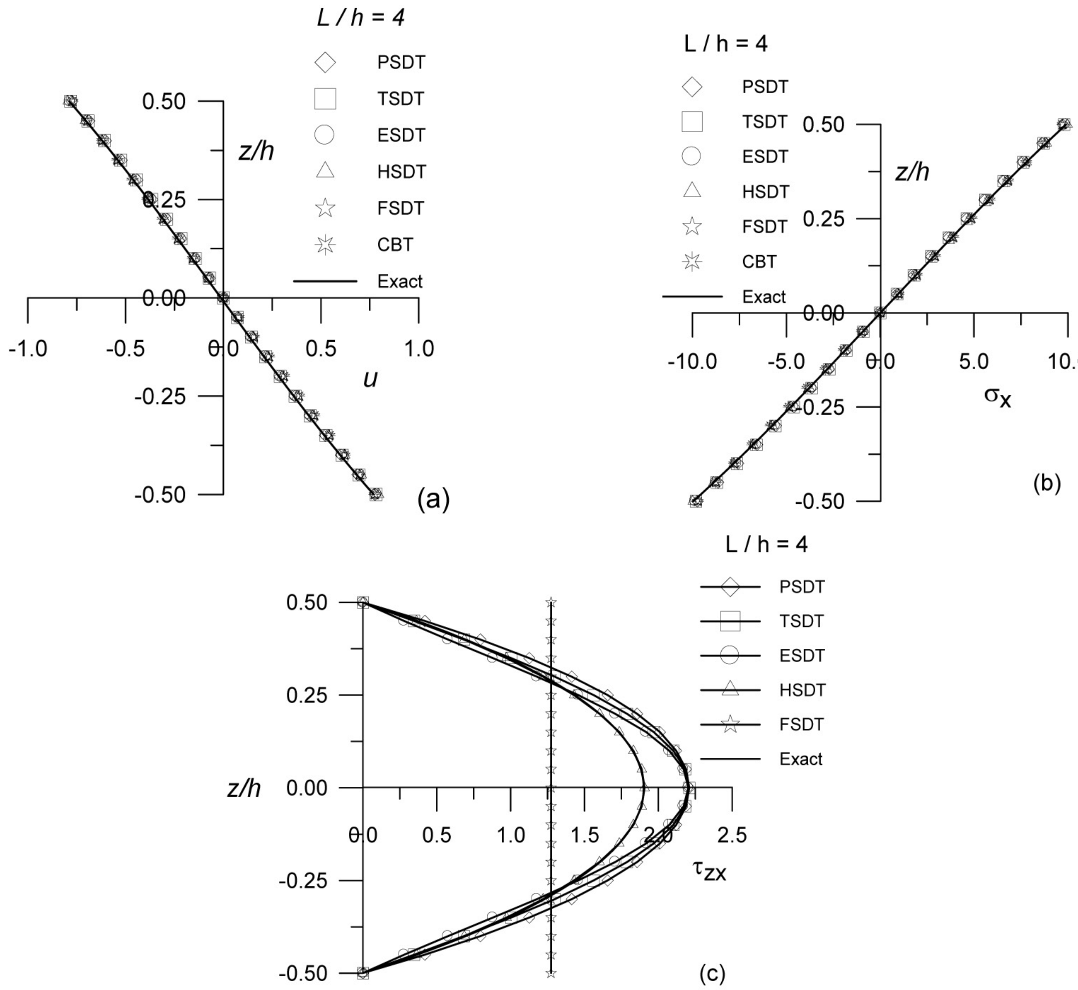


Fig. 4. Through thickness distributions of (a) axial displacement $\bar{u}$, (b) bending stress $\bar{\sigma}_x$ and (c) transverse shear stress τ_{zx}^{CR} of a simply supported isotropic beam subjected to a sinusoidal load

relation. Through thickness distributions of axial displacement, bending stress and shear stress for isotropic beams subjected to uniformly distributed and linearly varying loads are shown in Fig. 5 (a)–(c) and Fig. 6 (a)–(c), respectively.

4.2. Buckling analysis of simply supported isotropic beams

In this section, the efficiency of present theories (PSDT, TSDT, ESDT) is examined for buckling response of isotropic beams due to axial compressive force. Since the exact elasticity solution is not available for the buckling analysis of isotropic beams, the present results are compared and discussed with the corresponding results generated by using CBT of Bernoulli-Euler, FSDT of Timoshenko [22] and HSDT of Heyliger and Reddy [7]. The critical buckling load is obtained when the beam is subjected to an axial compressive force F_0. The numerical results are presented in Table 6 for $L/h = 4$, 10 and 100. From Table 6 it is pointed out that critical buckling load predicted by present theories and HSDT of Heyliger and Reddy [7] is more or less same whereas CBT predict the higher value of the critical buckling load. Fig. 7 shows the variation of critical buckling load N_{cr} with respect to the ratio L/h.

Table 4. The comparison of displacements and stresses of a simply supported isotropic beam subjected to a uniformly distributed load

L/h	Source	$\bar{u}$	% Error	$\bar{w}$	% Error	$\bar{\sigma}_x$	% Error	$\bar{\tau}_{zx}^{CR}$	% Error
4	PSDT	0.985	−0.203	1.810	1.401	12.458	2.115	2.984	−0.533
	TSDT	0.993	0.608	1.810	1.401	12.545	2.828	2.998	−0.067
	ESDT	0.997	1.013	1.805	1.120	12.569	3.025	2.975	−0.833
	HSDT	1.031	4.457	1.806	1.176	12.263	0.516	2.908	−3.066
	FSDT	1.000	1.317	1.806	1.176	12.000	−1.639	1.969	−34.36
	CBT	1.000	1.317	1.563	−12.430	12.000	−1.639	–	–
	Exact	0.987	0.000	1.785	0.000	12.200	0.000	3.000	0.000
10	PSDT	2.505	0.602	1.601	0.188	75.244	0.059	7.491	−0.120
	TSDT	2.508	0.723	1.601	0.188	75.295	0.126	7.509	0.120
	ESDT	2.510	0.803	1.601	0.188	75.342	0.189	7.481	−0.253
	HSDT	2.512	0.883	1.602	0.250	75.268	0.090	7.361	−1.853
	FSDT	2.500	0.402	1.602	0.250	75.000	−0.266	4.922	−34.370
	CBT	2.500	0.402	1.563	−2.190	75.000	−0.266	–	–
	Exact	2.490	0.000	1.598	0.000	75.200	0.000	7.500	0.000

HSDT: Heyliger and Reddy [7], FSDT: Timoshenko [22], Exact: Pagano [13]

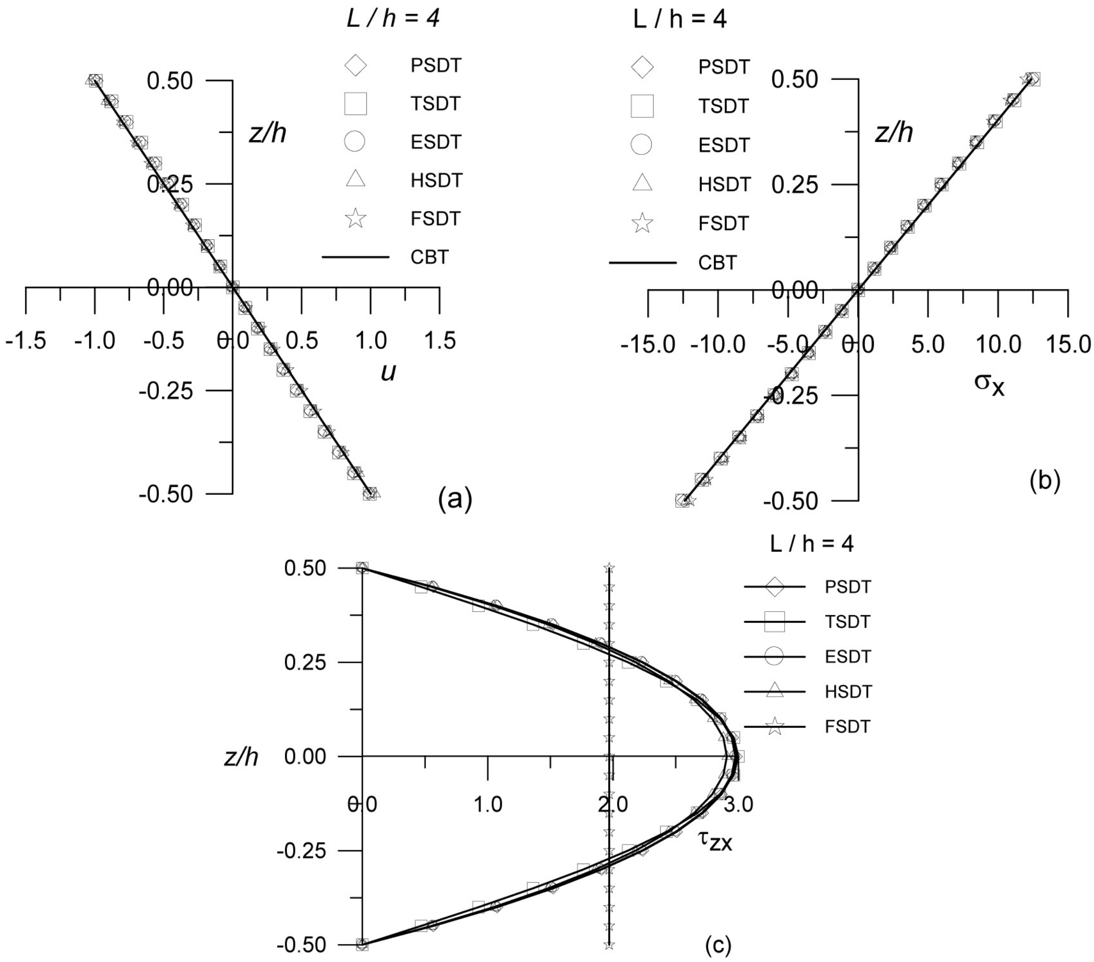


Fig. 5. Through thickness distributions of (a) axial displacement $\bar{u}$, (b) bending stress $\bar{\sigma}_x$ and (c) transverse shear stress τ_{zx}^{CR} of simply supported isotropic beam subjected to a uniformly distributed load

Table 5. The comparison of displacements and stresses of a simply supported isotropic beam subjected to a linearly varying load

L/h	Source	$\bar{u}$	% Error	$\bar{w}$	% Error	$\bar{\sigma}_x$	% Error	$\bar{\tau}_{zx}^{CR}$	% Error
4	PSDT	0.493	−0.203	0.905	1.401	6.229	2.115	1.492	−0.533
	TSDT	0.497	0.608	0.905	1.401	6.273	2.828	1.499	−0.067
	ESDT	0.499	1.013	0.903	1.120	6.285	3.025	1.488	−0.833
	HSDT	0.516	4.457	0.903	1.176	6.132	0.516	1.454	−3.066
	FSDT	0.500	1.317	0.903	1.176	6.000	−1.639	0.985	−34.360
	CBT	0.500	1.317	0.782	−12.430	6.000	−1.639	–	–
	Exact	0.494	0.000	0.893	0.000	6.100	0.000	1.500	0.000
10	PSDT	1.253	0.602	0.801	0.188	37.622	0.059	3.746	−0.120
	TSDT	1.254	0.723	0.801	0.188	37.648	0.126	3.755	0.120
	ESDT	1.255	0.803	0.801	0.188	37.671	0.189	3.741	−0.253
	HSDT	1.256	0.883	0.801	0.250	37.634	0.090	3.681	−1.853
	FSDT	1.250	0.402	0.801	0.250	37.500	−0.266	2.461	−34.370
	CBT	1.250	0.402	0.782	−2.190	37.500	−0.266	–	–
	Exact	1.245	0.000	0.799	0.000	37.600	0.000	3.750	0.000

HSDT: Heyliger and Reddy [7], FSDT: Timoshenko [22], Exact: Pagano [13]

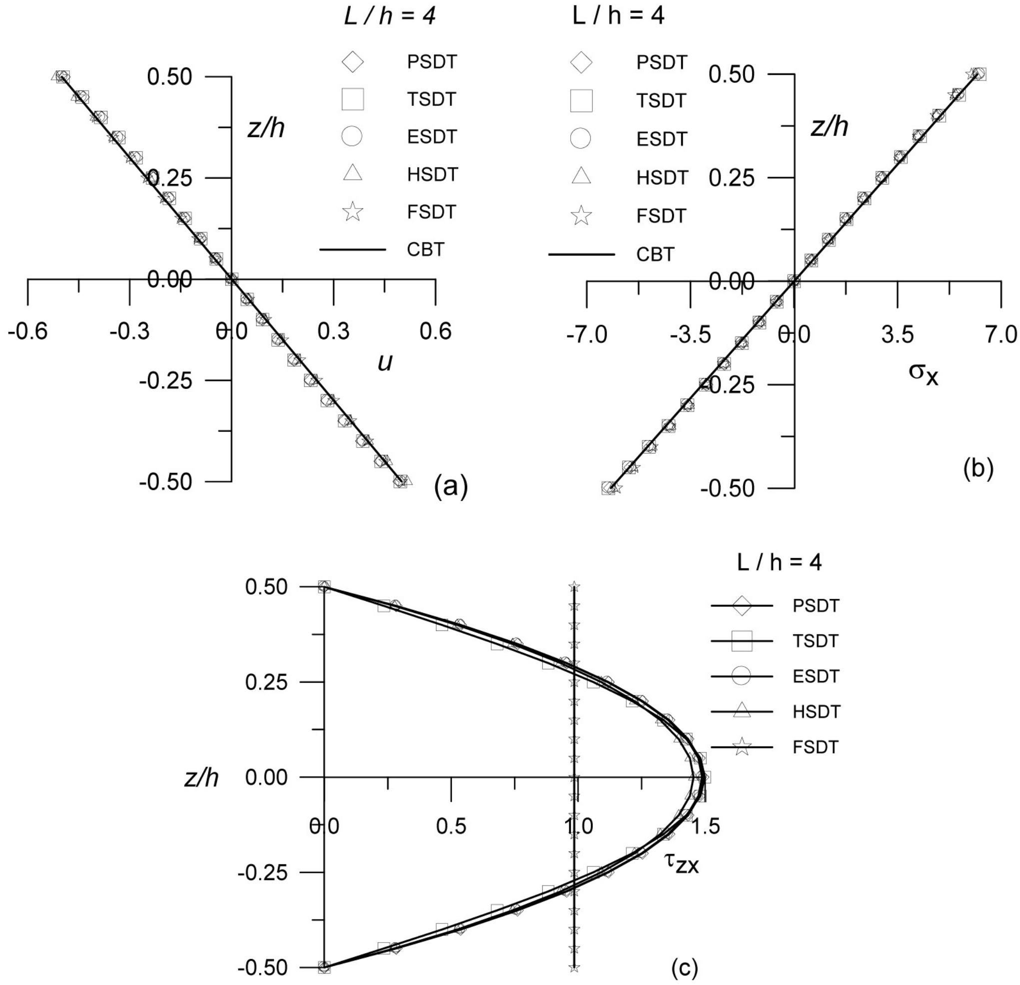
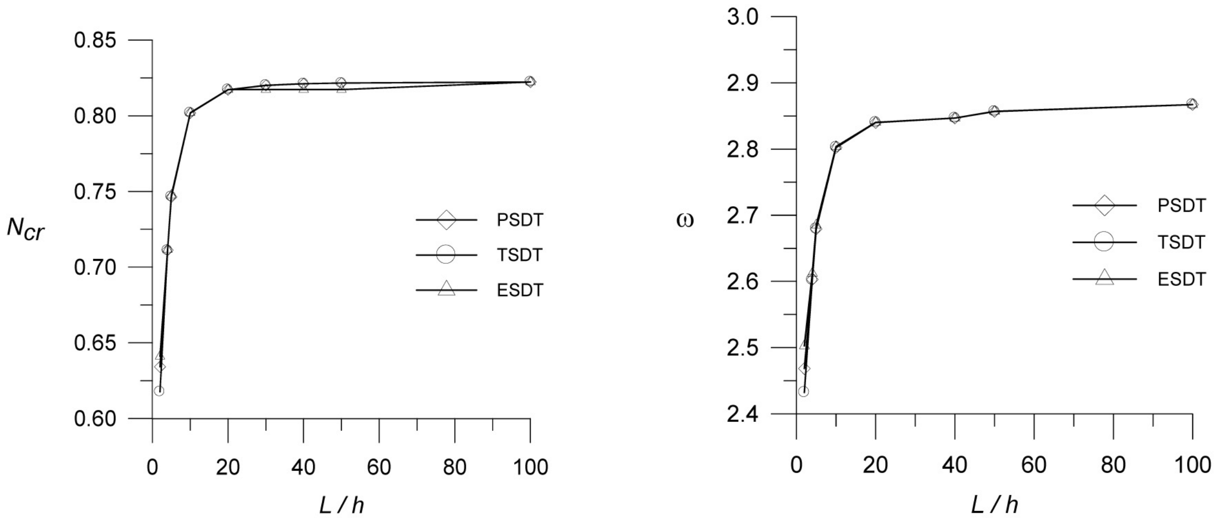


Fig. 6. Through thickness distributions of (a) axial displacement $\bar{u}$, (b) bending stress $\bar{\sigma}_x$ and (c) transverse shear stress τ_{zx}^{CR} of a simply supported isotropic beam subjected to a linearly varying load

Table 6. The comparison of non-dimensional critical buckling load and fundamental frequencies of a simply supported isotropic beam

	Critical buckling load (N_{cr})			Fundamental frequencies ($\bar{\omega}$)			
Source	$L/h = 4$	$L/h = 10$	$L/h = 100$	$L/h = 4$	% Error	$L/h = 10$	% Error
PSDT	0.711 2	0.801 8	0.822 3	2.603 0	0.034 5	2.802 2	−0.078 4
TSDT	0.711 0	0.801 9	0.822 3	2.602 1	0.000 0	2.802 4	−0.071 3
ESDT	0.711 6	0.802 0	0.822 3	2.612 1	0.384 3	2.803 8	−0.021 4
HSDT	0.709 0	0.801 9	0.822 0	2.596 0	−0.234 4	2.802 0	−0.085 6
FSDT	0.708 8	0.801 9	0.822 2	2.598 7	−0.130 7	2.802 7	−0.060 6
CBT	0.822 5	0.822 5	0.822 5	2.849 1	9.492 3	2.824 0	0.698 9
Exact	–	–	–	2.602 1	0.000 0	2.804 4	0.000 0

HSDT: Heyliger and Reddy [7], FSDT: Timoshenko [22], Exact: Cowper [2]



(a) (b)

Fig. 7. The variation of (a) non-dimensional critical buckling load N_{cr} and (b) fundamental frequency $\bar{\omega}$ with respect to the ratio L/h

4.3. Free vibration analysis of simply supported isotropic beams

In this section, the efficiency of present theories (PSDT, TSDT, ESDT) is checked for free vibration response of isotropic beam. Fundamental frequencies obtained using present theories are compared and discussed with the corresponding results obtained by using CBT of Bernoulli-Euler, FSDT of Timoshenko [22], Heyliger and Reddy [7] and exact elasticity solution provided by Cowper [2]. Fundamental frequencies obtained for $L/h = 4$ and 10 are presented in Table 6. From the Table 6 it is observed that the present theories are more accurate while predicting fundamental frequencies of an isotropic beam. The present TSDT predicts the exact value of the fundamental frequency for $L/h = 4$. As compared to the present theories, HSDT of Heyliger and Reddy [7] shows higher values of fundamental frequency. Variation of fundamental frequencies with respect to the ratio L/h is shown in Fig. 7.

5. Conclusions

In the present study, a new class of refined beam theories is presented which involves only one unknown variable similar to the classical beam theory. With this novelty of present theories, they are applied to the bending, buckling and the free vibration analysis of simply supported isotropic beams of rectangular cross-section. The theories are variationally consistent and do not require shear correction factor. From the numerical results and discussion, it is concluded that the present theories are more accurate while predicting bending, buckling and vibration responses of isotropic beams. The transverse shear stress predicted by using present theories is in excellent agreement with exact solution when obtained by using the constitutive relation.

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