# Spektrum sigma-delta modulace

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# Spectrum of the Sigma Delta Modulation

*Abstract* – **The objective of this paper was describe spectrum of the sigma delta modulation and their normalization to the Power Spectral Density (PSD)** .

*Keywords – Power Spectrum; Power Spectral Density; Sigma Delta Modulator.*

## I. MOTIVATION

To validate an analog-to-digital conversion we first need to answer questions:(i) what is a converter resolution? (ii) what is an accuracy of the converter? The resolution can be expressed by the number of discrete levels over the range of an analog input signal, and the smallest change in the value of an input signal results in the change of digital output. If modulators are validated separately, without the digital part of a Sigma-Delta ($\Sigma\Delta$) converter, we can hardly speak about resolutions (there is no digital world at the output). In our case, when the modulator is validated separately, an accuracy of the modulator is investigated by the comparison of the signal power and the total noise power in band of interest. Because the signal and the total noise power should be extracted from the spectrum, attention is focused on analysis in the frequency domain.

## II. FREQUENCY SPECTRUM OF THE MODULATORS

A Sigma-Delta Modulator ($\Sigma\Delta$M) is a noise-shaping converter - the quantization noise is pushed away from the desired frequency band - thus the attention should be focused on the frequency domain representation of the modulators output stream.

### *A. Discrete Fourier Transform and Periodogram*

Spectral properties can be obtained by Dicrete Fourier Tranform (DFT), where a vector of N numbers $x_k,\ k = \{0, ..., N-1\}$ is transformed into a vector of N complex numbers $y_m,\ m = \{0, ..., N-1\}$. To obtain DFT, algorithms called Fast Fourier Transform (FFT) is used due his efficiency. The FFT embedded in ELDO offers two type of normalization: (i) multiplying by $2/NBPT$, default option and (ii) no normalization. In case of the latter, the definition of FFT can be written:

$$Y_m = \sum_{k=0}^{N-1} x_k e^{-2\pi i \frac{mk}{N}}, \quad m = 0, ..., N-1, \tag{1}$$

Input data of a FFT algorithm, $x_k$, are purely real (reference voltages of Digital to Analog Converter (DAC)), what result in symmetry $Y_{N-m} = Y_m^*$, where $Y_m^*$ denotes complex conjugation. Assume an example of testbench, where number of points $NBPT = 4096$ (is even), then $Y_0$ and $Y_{NBPT/2}$ are real and the upper half of the vector, $\{Y_{NBPT/2+1}\ , ... ,$

$Y_{NBPT-1}\}$, in never computed. The result of FFT is vector, $\{Y_0, \dots, Y_{NBPT-1}\}$ of data composed by the real part and the imaginary part:

$$\begin{aligned} Y_{f_m} &= \mathrm{Re}(Y_{f_m}); \quad f_m = 0, ..., NBPT/2 \\ Y_{f_m} &= \mathrm{Im}(Y_{f_m}); \quad f_m = NBPT/2 + 1, ..., NBPT - 1 \end{aligned} \tag{2}$$

The magnitude of the vector obtained by the FFT algorithm can be rewritten:

$$\boldsymbol{Y_{mag}} = \sqrt{\mathrm{Re}(\boldsymbol{Y_{f_m}})^2 + \mathrm{Im}(\boldsymbol{Y_{f_m}})^2} \tag{3}$$

Basically, the spectral estimation problem can be expressed by the formulation:*"From a finite record of stationary data sequence, estimate how the total power is distributed over frequency"*[1]. There are several different convention to normalized Power Spectrum (PS) which is also called PSD.

Suppose sigma-delta bitstream as the signal (data length is $OSR\ f_s$) then the Fourier transform exists and the PSD can be obtained by two operations (i) autocorrelation + Fourier transform, (ii) Fourier transform + magnitude squared. In that case of later the estimation of PSD is called the periodogram and is defined by the $N/2+1$ frequency bins[2], as follows:

$$P(0) = \frac{{Y_{mag}}^2(0)}{NBPT^2},\ P(N/2) = \frac{{Y_{mag}}^2(N/2)}{NBPT^2} \tag{4}$$

$$P(f_m) = \frac{{Y_{mag}}^2(f_m) + {Y_{mag}}^2(N - f_m)}{NBPT^2}, \quad f_m = 1, ..., (NBPT/2) - 1, \tag{5}$$

In terms of the frequency domain analysis, there are several pitfalls (circumstances needed to be considered), which arise from a discrete spectrum representation of a sampled signal. The first problem is usually associated as a spectral leakage and the second issue is related to uncertainty. The reason for the leakage is that finite data sequence is equal to windowed data sequence by the square window function. Its Fourier transform has substantial components at high frequencies. To reduction spectral leakage, particular window can be used. As consequence, scaling and normalization is needed because a spectral spike and noise floor are depended on the window type and the window length.

In term of windowing, two parameters, Coherency Gain (CG) and Noise Gain (NG), can be defined[3]:

$$CG = \frac{1}{NBPT} \sum_{i=0}^{N-1} w(i), i = 0, \dots NBPT - 1, \tag{6}$$

$$NG = \frac{1}{NBPT} \sum_{i=0}^{N-1} w(i)^2, i = 0, \dots, NBPT - 1, \tag{7}$$

## *B. Normalization and Power Spectrum*

First, the result of the peridogram is correct only when rectangular windows is used. Due to spectral leakage, Hann window is preferred by the author and equations 4 and 5 can be rewritten:

$$PS_{rms}(f_m) = \frac{P(f_m)}{NG^2_{Hann}},\ f_m = 0, ..., (N/2) - 1, \tag{8}$$

This normalization of the PSD is sometimes called modified periodogram. In the section 2.1, periodogram was defined as the estimator of PS. But the question is in what sense is the estimator, $PS_{rms}(f_m)$, of the power spectrum "true" power spectrum of the sigma-delta bitstream?

The signal, located at frequency bins $\{f_{in} - f_{res},\, f_{in},\, f_{in} + f_{res}\}$ (Hann window is used), has the power $P_{sigRMS}$ equal to height of frequency bins:

$$P_{sigRMS} = \sum_{f_m = f_{in} - f_{res}}^{f_{in} + f_{res}} PS_{rms}(f_m), \tag{9}$$

One problem with this normalization is that noise floor is not displayed correct. Finite length data of the bitstream result in finite frequency resolution, $f_{res}$, and the power of the noise (this is also true for the signal) is accumulated in particular frequency bins. In other worlds, the power represented frequency bins $f_m = \{PS_{rms}(0), \ldots, PS_{rms}(N/2)\}$ are not equal to the continuous $PS_{rms}(f)$ at exactly $f_m$. We could expect the $PS_{rms}(m)$ to be some kind of average "true" continuous time power spectrum, $PS_{rms}(f)$, at frequency interval $\{f_m - f_{res}/2, f_m + f_{res}/2\}$ [2].

Roughly speaking, the frequency bins, $f_m$, has the width equal to frequency resolution, $f_{res}$, centered at $f_m$. To display PSD as some kind of continuous power spectrum, modified periodogram can be extended, divided by the with of a frequency bin equal to $f_{res} = f_s/NBPT$, as follows:

$$PSD_{rms}(f_m) = \frac{P(f_m)}{f_{res}\, NG_{Hann}} = \frac{P(f_m)\, NBPT}{f_s\, NG_{Hann}}, \tag{10}$$

Although, the vector produced by the equation 10 gives the discrete-time sequence of the frequency domain, it represent a continuous-time power spectrum width the resolution given by $f_{res}$.

## III. TEST CASE OF THE SIGMA DELTA BITSTREAM

The example of normalized frequency spectrum of simulated output bit stream of a continuous-time modulator is shown in Figure I. Blue waveform represents normalization given 10 and green represents frequency bins of discrete power spectrum obtained 8. The modulator is oversampled by $OSR = 64$ with the sinusoidal input signal of $f_{in} = 250Hz$ and $A_{sin} = 0.5V$ amplitude. The spectrum is computed from $NBPT = 4096$ output samples. The input signal is found in frequency, $f_{in}$, equal to 250Hz and the effect of noise shaping is also clearly visible in the chart.

To validate normalization given by the equation 10, the power of the output bitstream in the time domain, $P_t$, must be equal to the total power in the frequency range $0, \ldots, f_s/2$, $P_f$,:

$$P_t = \frac{1}{T_{tran}} \int_0^{T_{tran}} (V_{out\_sdm}(t) - V_{CM})^2 dt =|_{V_{DDA}=2.5V} = 1.5625[W], \tag{11}$$

$$P_f = \int_0^{f_s} PSD_{rms}(f) df =|_{V_{DDA}=2.5V} = 1.5625[W], \tag{12}$$

The input sinusoidal signal with peak amplitude, $A_{sin} = 0.5$, has a power:

$$P_{sin}^{in} = \frac{1}{T_{tran}} \int_0^{T_{tran}} (A_{sin} sin(\varphi + j\omega_{in} t))^2 dt \;|_{A_{sin}=0.5V} = 125[mW], \tag{13}$$

The signal power of an input sinusoidal signal, $A_{sin}\,sin(\varphi + 2\pi f_{sin})$, stored in $\Sigma\Delta$ bitstream, is distributed in the frequency range $f_{sin}-2\,f_{res}, \ldots, f_{sin}+2\,f_{res}$ of the modified periodogram, equation 10, and we can write:

$$P_{sin}^{sd} = \sum_{f_m=f_{in}-f_{res}}^{f_{in}+f_{res}} PS_{rms}(f_m) \approx \int_{f_{sin}-2f_{res}}^{f_{sin}+2f_{res}} PSD_{rms}(f)df = 124.64mW \qquad (14)$$

## IV. Conclusion

The signal power extracted from the normalized spectrum $P_{sin}^{sd} = 124,6\,mW$ correspond to value of power in the time domain, $P_{sin} = 125[mW]$. The difference is caused by the transfer function of a loop filter of the modulator.

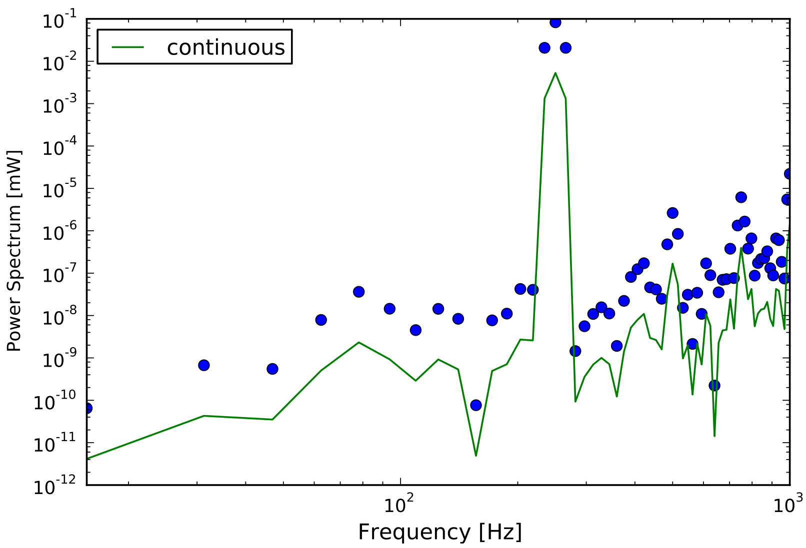


***Figure I.*** **Normalized power spectral density.**

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