FAKULTA APLIKOVANÝCH VĚD ZÁPADOČESKÉ UNIVERZITY V PLZNI

Navier-Stokesovy rovnice a související problémy

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disertační práce
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v oboru **Aplikovaná matematika**

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Navier-Stokes equations and related problems

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Plzeň, 21. června, 2017.

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Dr. Matteo Caggio

Abstrakt

Disertační práce je věnována studiu matematických problémů Navierových - Stokesových rovnic v kontextu rigorózního matematického odvození modelů a jejich matematické analýzy. Zejména je práce zaměřena na problematiku singulárních limit v mechanice tekutin pro stlačitelné tekutiny (režim malého Machova čísla, velkého Reynoldsova čísla, redukce dimenze) a problematice regularity pro nestlačitelné tekutiny.

Klíčová slova

Navierovy-Stokesovy rovnice, stlačitelné tekutiny, Navierovy-Stokesovy-Fourierovy rovnice, singulární limity, slabé řešení, silné řešení, Eulerovy rovnice, teorie regularity, nestlačitelné tekutiny, anisotropní Lebesgueovy prostory.

Abstract

The present thesis is devoted to the study of mathematical problems related to the Navier-Stokes equations in the context of mathematical rigorous derivation of models and their analysis. In particular we deal with the problem of singular limits in fluid mechanics for compressible fluids (low Mach number limit and high Reynolds number limit, reduction of dimension) and the problem of global regularity for incompressible fluids.

Keywords

Navier-Stokes equations, compressible fluids, Navier-Stokes-Fourier equations, singular limits, weak solutions, strong solutions, Euler equations, regularity theory, incompressible fluids, anisotropic Lebesgue spaces.

Estratto

Il presente lavoro di tesi è dedicato allo studio di problematiche legate alle equazioni di Navier-Stokes nel contesto della derivazione rigorosa di modelli e della loro analisi. In particolare ci occuperemo dei problemi relativi ai limiti singolari nella meccanica dei fluidi comprimibili (limite di bassi numeri di Mach e alti numeri di Reynolds, riduzione di dimensione) e del problema della regolarità globale per fluidi incomprimibili.

Parole chiave

Equazioni di Navier-Stokes, fluidi comprimibili, equazioni di Navier-Stokes-Fourier, problemi ai limiti singolari, soluzioni deboli, soluzioni forti, equazioni di Eulero, teoria della regolarità, fluidi incomprimibili, spazi di Lebesgue anisotropi.

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Preface

Navier-Stokes equations is a challenging problem in mathematical analysis. During the years several authors have faced different problems related to these equations. Some of these problems concern variations of the Navier-Stokes equations depending on the properties of the fluid and the presence of external forces. The present work deals with the so-called problem of singular limit in fluid mechanics for compressible fluids and the problem of global regularity for an incompressible fluid. The following articles are the results of this work:

- Guo Z., M. Caggio, Z. Skalák, Regularity criteria for the Navier-Stokes equations based on one component of velocity, *Nonlinear Analysis: Real World Application*, **35**, 379-396, 2017.

- Caggio M., Š. Nečasová, Inviscid incompressible limit for rotating fluids, *to appear in Nonlinear Analysis.*

- Ducomet B., M. Caggio, Š. Nečasová, M. Pokorný, The rotating Navier-Stokes-Fourier system on thin domains, *submitted in Acta Appl. Math; available on arXiv:1606.01054v1.*

Contents

Chapter 1

Introduction

The present work is devoted to the study of mathematical problems related to models describing the dynamics of fluids.

A fluid is a continuous medium whose state is characterized by its velocity, pressure and density fields, and possibly other relevant fields (for example temperature).

Most of the fluid dynamics results have been obtained starting from the Navier-Stokes equations. These equations have many variations depending on the properties of the fluid itself, for example compressibility, thermoconductivity, viscosity, etc., and on the forces acting on the fluid, for example the centrifugal force, the Coriolis force, the gravity force etc. (see Nazarenko [79]).

Two kind of problems will be under consideration: the problem of singular limits for compressible fluids and the problem of global regularity for incompressible fluids.

1.1 The problem of singular limits for compressible fluids

The problem of singular limits for compressible fluids can be presented in the following way. One starts from a system of equations describing the motion of a kind of fluid. After a scale analysis the system presents several characteristic parameters whose asymptotic behavior determines a change in the fluid phenomenology and consequently, at least at a formal level, a different system of equations compared to the starting one. The singular limit problem requires to show that the solution of the starting system converges to the solution of the limit (or target) system when these parameters tend to zero or infinity in some sense.

In the following we would like to briefly describe the problems we will deal with, postponing a deeper analysis to the next chapters.

1.1.1 The inviscid incompressible limit for compressible barotropic fluids

The motion of a compressible barotropic fluid is described by means of two unknown fields: the density $\varrho = \varrho(x,t)$ and the velocity $\mathbf{u} = \mathbf{u}(x,t)$ of the fluid, functions of the spatial position $x \in \mathbb{R}^3$ and the time $t \in \mathbb{R}$, and satisfying the following Navier-Stokes system of equations. The continuity equation reads

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0. \tag{1.1.1}$$

The momentum equation is

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x S (\nabla_x \mathbf{u}) + \varrho \mathbf{f}, \tag{1.1.2}$$

with the stress tensor given by the following relation

$$S = S(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathrm{I} \right) + \eta \, \mathrm{div}_x \mathbf{u} \mathrm{I}, \quad \mu > 0, \quad \eta \geq 0. \tag{1.1.3}$$

The system above presents two parameters: the shear viscosity coefficient μ and the bulk viscosity coefficient η. The scalar function p is the pressure, given function of the density, and $\varrho \mathbf{f}$ represents an external forcing.

For each physical quantity X present in the Navier-Stokes system (1.1.1) - (1.1.3), we introduce its characteristic value X_{char} and replace X with its dimensionless analogue X/X_{char}. As a result, we obtain the scaled version of the compressible Navier-Stokes system

$$[\mathcal{S}r] \, \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1.4}$$

$$[\mathcal{S}r] \, \partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{[\mathcal{M}a]^2} \nabla_x p(\varrho) = \frac{1}{[\mathcal{R}e]} \mathrm{div}_x S + \frac{1}{[\mathcal{F}r]^2} \varrho \mathbf{f}. \tag{1.1.5}$$

The above system presents several characteristic numbers. The Strouhal number

$$[\mathcal{S}r] = \frac{length_{char}}{time_{char} velocity_{char}}.$$

The Strouhal number plays a role in oscillating, non-steady flows, as the Kármán vortex street. It is often defined as

$$[\mathcal{S}r] = \frac{fL}{U},$$

where f is the frequency of vortex shedding in the wake of von Kármán, L is the characteristic length of the body invested by the flow and U is the characteristic velocity of the flow investing body. The Mach number

$$[\mathcal{M}a] = \frac{velocity_{char}}{\sqrt{pressure_{char}/density_{char}}}.$$

The Mach number is the ratio of the characteristic velocity of the flow to the speed of the sound in the fluid. Low Mach number limit characterizes incompressibility. The Reynolds number

$$[\mathcal{R}e] = \frac{density_{char} velocity_{char} length_{char}}{viscosity_{char}}.$$

The Reynolds number is the ratio of the inertial to the viscous forces in the fluid. High Reynolds number is attributed to turbulent flows. The Froude number

$$[\mathcal{F}r] = \frac{velocity_{char}}{\sqrt{length_{char} frequency_{char}}}.$$

The Froude number is the ratio of the flow inertia to the external field. The latter in many applications simply due to gravity.

Redefining the Reynolds number and the Mach number in terms of a non-negative parameter ε, namely $\mathcal{R}e := \varepsilon^{-1}$ and $\mathcal{M}a := \varepsilon$, and setting the other characteristic numbers equal to one, the inviscid incompressible limit aims to show the convergence $\mathbf{u} \to \mathbf{v}$ and $\varrho \to 1$, for $\varepsilon \to 0$, where $\mathbf{v}$ is the solution of the incompressible Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \quad \mathrm{div}_x \mathbf{v} = 0 \tag{1.1.6}$$

and $\mathbf{u}$ is the solution of the compressible Navier-Stokes system. Indeed, in the high Reynolds number limit the viscosity of fluid becomes negligible and in the low Mach number limit the fluid becomes incompressible. The inviscid and/or incompressible limit problem was investigated by several authors in similar and different contexts: in bounded, unbounded or expanding domains, in presence of external forces and for barotropic or heat conductive fluids. For more details we refer to the works of Bardos and Nguyen [2], Feireisl [39], Feireisl and Novotný [44], Feireisl, Jin and Novotný [46], Feireisl, Nečasová and Sun [47], Lions and Masmoudi [72] (see also [73, 74]), Masmoudi [75], Sueur [104] and references therein.

In the context described above, we will deal with the *inviscid incompressible limit for a compressible barotropic fluid in a "fast" rotating frame occupying the whole space* $\mathbb{R}^3$. More precisely, we would like to show the convergence of the solution of the compressible Navier-Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1.7}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) = -\frac{1}{\varepsilon^2} \nabla_x p(\varrho) + \varepsilon \mathrm{div}_x S(\nabla_x \mathbf{u}) - (\varrho \mathbf{u} \times \omega), \tag{1.1.8}$$

$$S = S(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbf{I} \right) + \eta \, \mathrm{div}_x \mathbf{u} \mathbf{I}, \quad \mu > 0, \quad \eta \geq 0. \tag{1.1.9}$$

towards the solution of the rotating incompressible Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \times \omega + \nabla_x \Pi = 0, \quad \mathrm{div}_x \mathbf{v} = 0, \tag{1.1.10}$$

for large values of the angular velocity $\omega = [0, 0, 1]$, namely "fast" rotating frame. Above, the shear viscosity coefficient μ and the bulk viscosity coefficient η are assumed to be constant. The quantity $(\varrho \mathbf{u} \times \omega)$ represents the Coriolis force. The effect of the centrifugal force is neglected. This is a standard simplification adopted, for instance, in models of atmosphere or astrophysics (see [54, 55, 56]).

The analysis will be based on the work of Caggio and Nečasová [7]. The problem is a particular case of the Masmoudi [75] result where we will use a different technique (see the discussion below).

The technique to reach the convergence will be based on the so-called *relative energy method* in the framework of the *relative energy inequality*. The relative energy inequality was introduced by Dafermos [20] in the context of the Second Law of Thermodynamics. In the fluid context, it was introduced by Germain [51]. Afterwards, the method was developed by Feireisl, Novotný and co-workers in the framework of the problem of singular limits in fluid mechanics (see for example Feireisl and Novotný [41], [43], Feireisl, Jin and Novotný [45] and Feireisl, Novotný and Sun [50] and references therein). In the following we describe briefly the method, leaving the technical details to the next chapters. The basic idea is to introduce a relative energy functional. This functional plays the role of measuring the stability of two solutions. One with more regularity compared to the other one. In our context, the two solutions will be the weak solution of the Navier-Stokes system and the classical solution of the Euler system respectively. Next, along with the relative energy functional, a relative energy inequality has to be derived. This last will give us the possibility to reach the convergence in terms of a Gronwall type inequality.

The compressibility of the fluid allows the propagation of acoustic waves described by the acoustic system related to the Navier-Stokes model. The acoustic waves have to decay in the incompressible limit. Therefore, the analysis requires a technique in order to ensure this decay. In the whole space is common to use the so-called *dispersive estimates* (see Desjardins and Grenier [22], Feireisl and Novotný [42], Masmoudi [75], Schochet [95] and Strichartz [103]). We will introduce the acoustic system and the dispersive estimates during our analysis.

1.1.2 The dimension reduction limit for compressible heat conducting fluids

The motion of an heat conducting compressible fluid is described by means of three unknown fields: the density $\varrho = \varrho(x,t)$, the velocity field $\mathbf{u} = \mathbf{u}(x,t)$ and the temperature $\vartheta = \vartheta(x,t)$ of the fluid, functions of the spatial position $x \in \mathbb{R}^3$ and the time $t \in \mathbb{R}$, and satisfying the following Navier-Stokes-Fourier system of equations. The continuity equation reads

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0. \tag{1.1.11}$$

The momentum equation is

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x S(\vartheta, \nabla_x \mathbf{u}) + \varrho \mathbf{f}. \tag{1.1.12}$$

with the stress tensor given by the following relation

$$S(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}. \tag{1.1.13}$$

The entropy equation is

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_x (\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(S(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right), \tag{1.1.14}$$

with

$$\mathbf{q} = -\kappa\left(\vartheta\right)\nabla_x \vartheta. \tag{1.1.15}$$

In the system above the shear viscosity coefficient $\mu\left(\vartheta\right)$, the bulk viscosity coefficient $\eta\left(\vartheta\right)$ and the heat conductivity coefficient $\kappa\left(\vartheta\right)$ are functions of the temperature. The scalar functions $p(\varrho,\vartheta)$ and $s(\varrho,\vartheta)$ are the pressure and the entropy respectively, functions of the density and the temperature, and $\varrho\mathbf{f}$ represents an external forcing.

In analogy with the arguments presented before, we can obtain the scaled version of the compressible Navier-Stokes-Fourier system

$$[\mathcal{S}r]\,\partial_t \varrho + \mathrm{div}_x\left(\varrho\mathbf{u}\right) = 0, \tag{1.1.16}$$

$$\begin{aligned}
&[\mathcal{S}r]\,\partial_t\left(\varrho\mathbf{u}\right) + \mathrm{div}_x\left(\varrho\mathbf{u}\otimes\mathbf{u}\right) + \left[\frac{1}{\mathcal{M}a^2}\right]\nabla_x p(\varrho,\vartheta) \\
&\quad = \left[\frac{1}{\mathcal{R}e}\right]\mathrm{div}_x S\left(\vartheta,\nabla_x\mathbf{u}\right) + \left[\frac{1}{\mathcal{F}r^2}\right]\varrho\mathbf{f},
\end{aligned} \tag{1.1.17}$$

$$\begin{aligned}
&\partial_t\left(\varrho s\left(\varrho,\vartheta\right)\right) + \mathrm{div}_x\left(\varrho s\left(\varrho,\vartheta\right)\mathbf{u}\right) + \left[\frac{1}{\mathcal{P}e}\right]\mathrm{div}_x\left(\frac{\mathbf{q}\left(\vartheta,\nabla_x\vartheta\right)}{\vartheta}\right) \\
&\quad = \frac{1}{\vartheta}\left(\left[\frac{\mathcal{M}a^2}{\mathcal{R}e}\right]S\left(\vartheta,\nabla_x\mathbf{u}\right):\nabla_x\mathbf{u} - \left[\frac{1}{\mathcal{P}e}\right]\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right).
\end{aligned} \tag{1.1.18}$$

where the Péclet number $[\mathcal{P}e]$ is defined as follows

$$[\mathcal{P}e] = \frac{pressure_{char}velocity_{char}length_{char}}{heat\,conductivity_{char}temperature_{char}}.$$

Similarly to Reynolds number, high Péclet number corresponds to low heat conductivity of the fluid that may be attributed to turbulent flows.

Redefining the Froude number in terms of a non-negative parameter ϵ, namely $\mathcal{F}r = \epsilon^\beta$, with β arbitrary non-negative number, and setting the other characteristic numbers equal to one, the dimension reduction limit aims to show the convergence $[\varrho,\mathbf{u},\vartheta] \to [r,\mathbf{w},\Theta]$, for $\epsilon \to 0$, where the couple $[\varrho,\mathbf{u},\vartheta]$ is the solution of the three-dimensional Navier-Stokes-Fourier system and the couple $[r,\mathbf{w},\Theta]$ is the solution of the corresponding two-dimensional system.

Indeed, in the low Froude number limit the gravitational effects become predominant forcing the fluid to a two-dimensional dynamics.

The analysis will be based on the work of Ducomet, Caggio, Nečasová and Pokorný [25] and it aims the extension of the result of Feireisl, Novotný and co-workers [1].

Remark 1. For the sake of clarity, in the presence of gravity force, the system describing an heat conducting fluid is given by the Navier-Stokes-Fourier-Poisson system of equations.

Remark 2. It is possible to read ϵ as follows

$$\epsilon = \frac{l}{L}.$$

Here, l is the horizontal length and L the vertical length. Consequently, the limit can be also seen, more easily, in terms of a pure geometric reduction.

In the context describe above, we will deal with the *dimension reduction limit for a compressible heat conducting fluid in a rotating frame occupying a bounded domain in* $\mathbb{R}^3$ *where the external forcing is given by the gravity force.* More precisely, we consider a fluid confined in a straight layer $\Omega_\epsilon = \omega \times (0, \epsilon)$ where ω is a two-dimensional domain. We rescale to a fix domain as follows

$$(x_h, \epsilon x_3) \in \Omega_\epsilon \rightarrow (x_h, x_3) \in \Omega,$$

where $x_h = (x_1, x_2) \in \omega$ and $x_3 \in (0,1)$. Above, we denoted

$$\nabla_\epsilon = \left(\nabla_h, \frac{1}{\epsilon}\partial_{x_3}\right), \quad \nabla_h = (\partial_{x_1}, \partial_{x_2}), \tag{1.1.19}$$

$$\mathrm{div}_\epsilon \mathbf{u} = \mathrm{div}_h \mathbf{u}_h + \frac{1}{\epsilon}\partial_{x_3} u_3, \quad \mathbf{u}_h = (u_1, u_2), \quad \mathrm{div}_h \mathbf{u}_h = \partial_{x_1} u_1 + \partial_{x_2} u_2, \tag{1.1.20}$$

$$\triangle_\epsilon = \partial^2_{x_1 x_1} + \partial^2_{x_2 x_2} + \frac{1}{\epsilon^2}\partial^2_{x_3 x_3}. \tag{1.1.21}$$

The continuity equation reads now as follow

$$\partial_t \varrho + \mathrm{div}_\epsilon (\varrho \mathbf{u}) = 0, \tag{1.1.22}$$

the momentum equation is

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_\epsilon (\varrho \mathbf{u} \otimes \mathbf{u}) + \varrho \mathbf{u} \times \boldsymbol{\chi} + \nabla_\epsilon p(\varrho, \vartheta)$$

$$= \mathrm{div}_\epsilon S(\vartheta, \nabla_\epsilon \mathbf{u}) + \epsilon^{-2\beta} \varrho \nabla_\epsilon \phi + \varrho \nabla_\epsilon |x \times \boldsymbol{\chi}|^2, \tag{1.1.23}$$

the entropy equation is

$$\partial_t (\varrho s(\varrho, \vartheta)) + \mathrm{div}_\epsilon (\varrho s(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_\epsilon \left(\frac{\mathbf{q}(\vartheta, \nabla_\epsilon \vartheta)}{\vartheta}\right)$$

$$= \frac{1}{\vartheta}\left(S(\vartheta, \nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_\epsilon \vartheta) \cdot \nabla_\epsilon \vartheta}{\vartheta}\right), \tag{1.1.24}$$

with

$$S(\vartheta, \nabla_\epsilon \mathbf{u}) = \mu(\vartheta)\left(\nabla_\epsilon \mathbf{u} + \nabla_\epsilon^t \mathbf{u} - \frac{2}{3}\mathrm{div}_\epsilon \mathbf{u} \mathbf{I}\right) + \eta(\vartheta)\,\mathrm{div}_\epsilon \mathbf{u} \mathbf{I} \tag{1.1.25}$$

and

$$\mathbf{q} = -\kappa(\vartheta) \nabla_\epsilon \vartheta. \tag{1.1.26}$$

The quantities $\varrho \mathbf{u} \times \boldsymbol{\chi}$ and $\varrho \nabla_\epsilon |x \times \boldsymbol{\chi}|^2$ represent the Coriolis force and the centrifugal force respectively with $\boldsymbol{\chi} = [0,0,1]$ angular velocity and

$$\nabla_\epsilon |x \times \boldsymbol{\chi}|^2 = \left(\nabla_h |x \times \boldsymbol{\chi}|^2, 0\right) = \frac{(x_1, x_2, 0)}{\sqrt{x_1^2 + x_2^2}}.$$

The gravitational force is expressed by $\varrho \nabla_\epsilon \phi$ where the potential ϕ satisfies the Poisson's equation

$$-\triangle_\epsilon \phi = 4\pi G(\alpha\varrho + (1-\alpha)g) \quad \text{in} \quad (0,T)\times\Omega. \tag{1.1.27}$$

Here, G is the Newton constant and α a positive parameter. The first contribution on the right-hand side of the relation (3.0.6) corresponds to self-gravitation while in the second one g is a given function modeling the external gravitational effects. Here and hereafter, we assume that the function ϱ is extended by zero outside of Ω. Supposing further that g is such that the integral below converges, we have

$$\phi(t,x) = G\int_{\mathbb{R}^3} K(x-y)\left(\alpha\varrho(t,y) + (1-\alpha)g(y)\right)\mathrm{d}y,$$

where $K(x-y) = \frac{1}{|x-y|}$ and the parameter α may take the values 0 or 1. For $\alpha = 0$ the gravitation only acts as an external field, for $\alpha = 1$ only the self-gravitation is present. Since we also work with $\nabla_\epsilon \phi(t,x)$, we have to further assume that

$$\int_{\mathbb{R}^3} \nabla_\epsilon K(x-y)\left(\alpha\varrho(t,y) + (1-\alpha)g(y)\right)\mathrm{d}y < \infty.$$

In particular, the gravitational force is given by the following relation (see [25] and [26])

$$\begin{aligned}
\nabla_\epsilon \phi(t,x) = \epsilon \int_{\Omega_\epsilon} \alpha\varrho(t,\xi) \frac{(x_1-\xi_1, x_2-\xi_2, \epsilon(x_3-\xi_3))}{\left(|x_h-\xi_h|^2 + \epsilon^2|x_3-\xi_3|^2\right)^{3/2}} d\xi \\
+ \int_{\mathbb{R}^3} (1-\alpha)g(y) \frac{(x_1-y_1, x_2-y_2, \epsilon(x_3-y_3))}{\left(|x_h-y_h|^2 + \epsilon^2|x_3-y_3|^2\right)^{3/2}} dy \\
= \epsilon\alpha\mathbf{\Phi}_1 + (1-\alpha)\mathbf{\Phi}_2.
\end{aligned} \tag{1.1.28}$$

In our analysis we will distinguish two cases with respect to the behavior of the Froude number, namely $\mathcal{F}r = \sqrt{\epsilon}$ for $\beta = 1/2$ and $\mathcal{F}r = 1$ for $\beta = 0$. According to the choice of the Froude number, we have to consider the correct form of the gravitational potential. In the former the self-gravitation, namely $\alpha = 1$, and in the latter the external gravitation force, namely $\alpha = 0$. In the latter, we could also include the self-gravitation, it would, however disappeared after the limit passage. Taking $\mathcal{F}r = \sqrt{\epsilon}$ for $\beta = 1/2$ the momentum equation reads as follow

$$\begin{aligned}
\partial_t(\varrho\mathbf{u}) + \mathrm{div}_\epsilon(\varrho\mathbf{u}\otimes\mathbf{u}) + \varrho\mathbf{u}\times\boldsymbol{\chi} + \nabla_\epsilon p(\varrho,\vartheta) \\
= \mathrm{div}_\epsilon S(\vartheta, \nabla_\epsilon\mathbf{u}) + \varrho\mathbf{\Phi}_1 + \varrho\nabla_\epsilon|x\times\boldsymbol{\chi}|^2.
\end{aligned} \tag{1.1.29}$$

While, taking $\mathcal{F}r = 1$ for $\beta = 0$, we have

$$\begin{aligned}
\partial_t(\varrho\mathbf{u}) + \mathrm{div}_\epsilon(\varrho\mathbf{u}\otimes\mathbf{u}) + \varrho\mathbf{u}\times\boldsymbol{\chi} + \nabla_\epsilon p(\varrho,\vartheta) \\
= \mathrm{div}_\epsilon S(\vartheta, \nabla_\epsilon\mathbf{u}) + \varrho\mathbf{\Phi}_2 + \varrho\nabla_\epsilon|x\times\boldsymbol{\chi}|^2.
\end{aligned} \tag{1.1.30}$$

For $\mathcal{F}r = \sqrt{\epsilon}$ and $\beta = 1/2$, the corresponding two-dimensional momentum equation reads as follows

$$r\partial_t \mathbf{w} + r\mathbf{w}\cdot\nabla_h \mathbf{w} + \nabla_h p(r,\Theta) + r(\mathbf{w}\times\boldsymbol{\chi})$$
$$= \mathrm{div}_h S(\Theta, \nabla_h \mathbf{w}) + r\nabla_h \phi_h + r\nabla_h |x\times\boldsymbol{\chi}|^2, \tag{1.1.31}$$

with the formula

$$\phi_h(t,x_h) = \int_\omega \frac{r(t,y_h)}{|x_h - y_h|} dy_h \tag{1.1.32}$$

and

$$S_h(\Theta, \nabla_h \mathbf{w}) = \mu\left(\nabla_h \mathbf{w} + \nabla_h^t \mathbf{w} - \mathrm{div}_h \mathbf{w}\right) + \left(\eta + \frac{\mu}{3}\right)\mathrm{div}_h \mathbf{w} \mathrm{I}_h \tag{1.1.33}$$

where I_h is the unit tensor in $\mathbb{R}^{2\times 2}$ in the domain $(0,T)\times\omega$. While, for $\mathcal{F}r = 1$ and $\beta = 0$, we have

$$\phi_h(t,x_h) = G\int_{\mathbb{R}^3} \frac{g(y)}{\sqrt{|x_h - y_h|^2 + y_3^2}} dy. \tag{1.1.34}$$

As in the previous discussion, the technique to reach the convergence will be based on the relative energy method in order to show the convergence of the weak solution of the three-dimensional Navier-Stokes-Fourier system to the classical solution of the corresponding two-dimensional system. In particular, we will follow the framework developed in [43]. The main point of the analysis will be the treatment of the gravitational force.

From a phenomenological point of view, this limit concerns the rigorous derivation of the equations describing astrophysical objects called accretion disk which are thin structures observed in various places in the universe. These disks are indeed three-dimensional but their thickness is usually much smaller than their extension, therefore they are often modeled as two-dimensional structures. Indeed, if a massive object attracts matter distributed around it through Newtonian gravitation in presence of an angular momentum, this matter is not accreted isotropically around the central object but forms a thin disk around it. For further details we refer to the work of Choudhuri [17], Montesinos Armijo [78], Ogilvie [87], Pierens [89], Pringle [90] and Shore [96].

1.2 The problem of global regularity for incompressible fluids

The motion of an incompressible fluid is described by means of its velocity field $\mathbf{u} = \mathbf{u}(x,t)$, functions of the spatial position $x \in \mathbb{R}^3$ and the time $t \in \mathbb{R}$, and satisfying the following Navier-Stokes system of equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla_x \mathbf{u} - \mu\Delta_x \mathbf{u} + \nabla_x p = \mathbf{f}, \quad \mathrm{div}_x \mathbf{u} = 0. \tag{1.2.1}$$

In the system above μ is the shear viscosity coefficient. The scalar function p is the pressure, functions of the spatial position $x \in \mathbb{R}^3$ and the time $t \in \mathbb{R}$, and $\mathbf{f}$ represents a given external forcing.

An open problem in applied analysis concerns the global regularity of the solution of the Navier-Stokes equations in the whole space $\mathbb{R}^3$. Over the years, several authors have faced the problem (see, for example, [18], [19], [24], [63],

[64], [66], [70], [71], [92], [102], [107], [108], [109]). It is known that for the initial data $\mathbf{u}_0 \in L^2_\sigma$ (solenoidal functions in L^2) the problem (1.2.1) possesses at least one global weak solution $\mathbf{u}$ satisfying the energy inequality

$$||\mathbf{u}(t)||_2^2/2 + \int_0^t ||\nabla_x \mathbf{u}(\tau)||_2^2 \, d\tau \le ||\mathbf{u}_0||_2^2/2 \tag{1.2.2}$$

for every $t \ge 0$ (see [53], [67] and [93]). Such solutions are called Leray-Hopf solutions.

More precisely (see [93]), given $\mathbf{u}_0 \in L^2_\sigma$, a weak solution of (1.2.1) on $[0,T)$ is a function $\mathbf{u} \in L^\infty\left(0,T;L^2_\sigma\right) \cap L^2\left(0,T;W^{1,2}\right)$ such that

$$\int_0^T (\mathbf{u}, \varphi_t) - (\nabla \mathbf{u}, \nabla \varphi) - ((\mathbf{u} \cdot \nabla)\,\mathbf{u}, \varphi) = -(\mathbf{u}_0, \varphi) \tag{1.2.3}$$

for every $\varphi \in \mathcal{D}\left([0,T), \mathbb{R}^3\right)$, the set of all functions in $C_0^\infty\left([0,T), \mathbb{R}^3\right)$ that are also divergence free, and the following existence Theorem holds (see [93]).

Theorem 3. *For any $\mathbf{u}_0 \in L^2_\sigma$ there exists at least one weak solution of (1.2.1). This solution is weakly continuous into L^2, namely for any $\mathbf{v} \in L^2$,*

$$\lim_{t \to t_0} (\mathbf{u}(t), \mathbf{v}) = (\mathbf{u}(t_0), \mathbf{v})$$

for all $t_0 \in [0,T)$, and in addition it satisfies the energy inequality (1.2.2) for every $t \in [0,T)$. Moreover, $\mathbf{u}(t) \to \mathbf{u}_0$ in L^2 as $t \to 0$.

Remark 4. Above we used $(\cdot,\cdot)$ to denote the inner product in L^2.

Nevertheless, the uniqueness, regularity, and continuous dependence on initial data for weak solutions are still open problems ([10]).

If $\mathbf{u}_0 \in W^{1,2}_\sigma$ (solenoidal functions from the standard Sobolev space $W^{1,2}$), then strong solutions exist for a short interval of time whose length depends on the physical data of the initial-boundary value problem. Moreover, this strong solution is unique in the larger class of weak solutions ([19], [63], [102], [107]). In fact, a strong solution is a weak solution with the additional regularity ([93])

$$\mathbf{u} \in L^\infty\left(0,T;W^{1,2}\right) \cap L^2\left(0,T;W^{2,2}\right).$$

From the pioneer works of Prodi [91] and of Serrin [98], many results were presented in providing sufficient conditions for the global regularity (see for example Chae and Lee [13], Constantin [18], Doering and Gibbon [24], Ladyzhenskaya [63, 64], Lemarié-Rieussett [66], Lions [70, 71], Sohr [102] and Temam [107, 108, 109] and references therein).

Some of these conditions provide regularity criteria for the velocity field (see for example Escauriaza, Seregin and Šverák [31], Fabes, Jones and Riviere [32] and Serrin [98]): if a Leray-Hopf weak solution $\mathbf{u}$ satisfies

$$\mathbf{u} \in L^r(0,T;L^s(\mathbb{R}^3)) \quad \text{for} \quad \text{some} \quad \frac{2}{r} + \frac{3}{s} \le 1, \quad 3 \le s \le \infty$$

then $\mathbf{u}$ is regular.

Others involve analogous criteria for the pressure (see for example Berselli [5], Berselli and Galdi [6], Cao and Titi [9], Kukavica [59], Seregin and Šverák [97], Zhou [115]): if the pressure p satisfies

$$p \in L^r(0,T;L^s(\mathbb{R}^3)) \quad \text{for} \quad \text{some} \quad \frac{2}{r}+\frac{3}{s} \le 2, \quad s > \frac{3}{2}$$

or

$$\nabla p \in L^r(0,T;L^s(\mathbb{R}^3)) \quad \text{for} \quad \text{some} \quad \frac{2}{r}+\frac{3}{s} \le 3, \quad 1 \le s \le \infty$$

then $\mathbf{u}$ is regular.

An analogical situation occurs for $\nabla \mathbf{u}$. It was proved in [3] that $\mathbf{u}$ is regular if

$$\nabla \mathbf{u} \in L^r(0,T;L^s(\mathbb{R}^3))$$

where $s \in (3/2,\infty)$ and

$$\frac{2}{r}+\frac{3}{s} = 2.$$

Still others state sufficient conditions for regularity in terms of the vorticity (see for example Beirao da Veiga [4]): if the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ of a Leray-Hopf weak solution $\mathbf{u}$ belongs to the space

$$L^r(0,T;L^s(\mathbb{R}^3)) \quad \text{for} \quad \text{some} \quad \frac{2}{r}+\frac{3}{s} \le 2, \quad s > 1$$

then $\mathbf{u}$ is regular. The result above concerns the regularity of the solution $\mathbf{u}$ when conditions are imposed on all the components of the vorticity vector. Chae and Choe [12] obtained regularity by imposing the conditions

$$\omega_j \in L^r\left(0,T;L^s\right), \quad j=1,2, \quad \text{for} \quad \text{some} \quad \frac{2}{r}+\frac{3}{s} \le 2, \quad s \in (3/2,\infty)$$

namely, on only two components of the vorticity vector, while the problem with one vorticity component is an outstanding open problem.

1.2.1 Regularity criteria in terms of one velocity component

The above mentioned criteria are based on the entire velocity vector or on the entire gradient. In the last two decades many authors have studied the regularity criteria where additional conditions were imposed only on some velocity components or on some items of the velocity and pressure gradients. The first contribution in this direction was done by Neustupa and Penel [81]. After, over the years, several authors have obtained important results in that direction (see for example Kukavica and Ziane [61], Zhou and Pokorný [116], [117] and reference therein).

In the context described, we are interested in criteria based on only one velocity component. More specifically, we will study criteria based on u_3, ∇u_3 and $\nabla^2 u_3$, and prove, for example, that the condition

$$\nabla u_3 \in L^\beta(0,T;L^p),$$

where $p \in (2,\infty]$ and

$$2/\beta + 3/p = 7/4 + 1/(2p),$$

yields the regularity of $\mathbf{u}$ on $(0, T]$.

The analysis will be based on the work of Guo, Caggio and Skalák [52] in the framework of anisotropic Lebesgue spaces.

The anisotropic Lebesgue spaces framework seems to be convenient for our purposes, since it differentiates between different directions. It can be useful in the situations where regularity conditions are imposed only on one velocity component. Indeed, in Theorems 38 - 42 we will see that the use of the anisotropic Lebesgue spaces framework can improve some results from the literature.

Chapter 2

Inviscid incompressible limit for rotating fluids

We consider the scaled compressible Navier-Stokes system for a barotropic rotating fluid in the whole space $\mathbb{R}^3$ already mentioned in Introduction. The continuity equation reads

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.0.1}$$

the momentum equation is

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) = -\frac{1}{\varepsilon^2} \nabla_x p(\varrho) + \varepsilon \mathrm{div}_x S(\nabla_x \mathbf{u}) - (\varrho \mathbf{u} \times \omega), \tag{2.0.2}$$

with the stress tensor given by the following relation

$$S = S(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathrm{I} \right) + \eta \, \mathrm{div}_x \mathbf{u} \mathrm{I}, \quad \mu > 0, \quad \eta \geq 0. \tag{2.0.3}$$

The system is supplemented by the initial conditions

$$\varrho (x, 0) = \varrho_0 (x) , \ \mathbf{u} (x, 0) = \mathbf{u}_0 (x) \tag{2.0.4}$$

and by the following far field conditions for the density and the velocity field

$$\lim_{|x| \to \infty} \varrho(x, t) = 1, \quad \lim_{|x| \to \infty} \mathbf{u}(x, t) = 0. \tag{2.0.5}$$

The first relation in (2.0.5) means the mass of the fluid is infinite.

As mentioned in the previous chapter, we want to show that the weak solution of the Navier-Stokes system converges to the classical solution of the corresponding rotating incompressible Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \times \omega + \nabla_x \Pi = 0, \quad \mathrm{div}_x \mathbf{v} = 0, \tag{2.0.6}$$

for large values of ω, namely "fast" rotating frame.

2.1 Weak and classical solutions

In the following, we introduce the definition of weak solutions for the compressible Navier-Stokes system (2.0.1 - 2.0.3) and we discuss the global-in-time existence. In particular, we define the so-called bounded energy weak solution (see [38], [48] and [86]) and we discuss the global-in-time existence. Then, we discuss the global existence of the classical solution of the incompressible Euler system (2.0.6). For the discussion on weak solutions we will consider an arbitrary open set $\Omega \subset \mathbb{R}^3$.

The introduction of the bounded energy weak solution is motivated by the following discussion. In [21] it was shown the existence of weak solutions to the compressible Navier-Stokes equations on unbounded domain satisfying the differential form of the energy inequality (and consequently the integral form) for a barotropic fluid with finite mass. While the existence of weak solutions for a fluid with infinite mass *remains an open question*. Weak solutions satisfying the differential form of the energy inequality are usually termed finite energy weak solutions (see [2], [45], [49], [62] and [86]), while weak solutions satisfying the integral form of the energy inequality are usually termed bounded energy weak solutions (see [38], [48] and [86]).

Because our analysis will be performed in the whole space $\mathbb{R}^3$ under the condition that the mass of the fluid is infinite (see relation 2.0.5), we have to use the integral form of the energy inequality and consequently to deal with bounded energy weak solutions.

2.1.1 Bounded energy weak solutions

Multiplying (formally) the equation (2.0.2) by $\mathbf{u}$ and integrating by parts, we deduce the energy inequality in its integral form

$$E(T) + \varepsilon \int_0^T \int_\Omega S\left(\nabla_x \mathbf{u}\right) : \nabla_x \mathbf{u} \, \mathrm{d}x\mathrm{d}t \leq E_0 \tag{2.1.1}$$

where the total energy E is given by the formula

$$E = E\left[\varrho, \mathbf{u}\right](t) = \int_\Omega \frac{1}{2}\varrho \left|\mathbf{u}\right|^2 + \frac{H(\varrho)}{\varepsilon^2}\mathrm{d}x, \tag{2.1.2}$$

with E_0 the initial energy, and

$$H(\varrho) = \int_1^\varrho \frac{p\left(z\right)}{z^2} dz \tag{2.1.3}$$

the Helmholtz free energy (see [41] and [86]).

Remark 5. Here and hereafter the Helmholtz free energy will have the following form (see Novotný and Straškraba [86]):

$$H(\varrho) = \frac{1}{\gamma - 1}\left(\varrho^\gamma - \gamma\varrho + \gamma - 1\right).$$

The parameter γ is the adiabatic index or heat capacity ratio.

Now, we define the so-called bounded energy weak solution of the compressible Navier-Stokes system (2.0.1 - 2.0.3) (see Feireisl, Novotný and Petzeltová [48] and Novotný and Straškraba [86]).

Definition 6. (Bounded energy weak solution) Let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set. We say that $[\varrho, \mathbf{u}]$ is a bounded energy weak solution of the compressible Navier-Stokes system (2.0.1 - 2.0.3) in the time-space cylinder $(0,T) \times \Omega$ if

$$\varrho \in L^{\infty}\left((0,T), L^{\gamma}_{loc}(\Omega)\right), \quad \varrho \geq 0 \quad \text{a.e. in} \quad (0,T) \times \Omega,$$

$$H(\varrho) \in L^{\infty}((0,T), L^1(\Omega)),$$

$$\mathbf{u} \in L^2\left((0,T), \left(D_0^{1,2}(\Omega)\right)^3\right), \quad \varrho |\mathbf{u}|^2 \in L^{\infty}\left((0,T), L^1(\Omega)\right).$$

The continuity equation (2.0.1) holds in $\mathcal{D}'((0,T)\times\Omega)$. The momentum equation (2.0.2) holds in $(\mathcal{D}'((0,T)\times\Omega))^3$. The energy inequality (2.1.1) holds for a.a. $t \in (0,T)$ with E defined by

$$E = \int_{\Omega} \frac{1}{2} \frac{|\varrho \mathbf{u}|^2}{\varrho} 1_{\{x;\varrho>0\}} + \frac{H(\varrho)}{\varepsilon^2} \mathrm{d}x \tag{2.1.4}$$

and E_0 defined by

$$E_0 = \int_{\Omega} \frac{1}{2} \frac{|\varrho_0 \mathbf{u}_0|^2}{\varrho_0} 1_{\{x;\varrho_0>0\}} + \frac{H(\varrho_0)}{\varepsilon^2} \mathrm{d}x. \tag{2.1.5}$$

Remark 7. Here, the space $D_0^{1,2}(\Omega)$ is a completion of $\mathcal{D}(\Omega)$, the space of smooth functions compactly supported in Ω, with respect to the norm

$$\|\mathbf{u}\|^2_{D_0^{1,2}(\Omega)} = \int_{\Omega} |\nabla \mathbf{u}|^2 \,\mathrm{d}x.$$

Now, the following theorem concerns with the global-in-time existence of the bounded energy weak solution (see [38] and [48]).

Theorem 8. *(Global-in-time existence of bounded energy weak solution) Let $\Omega \subset \mathbb{R}^3$ be an arbitrary open set. Let the pressure p be given by a general constitutive law satisfying*

$$p \in C^1[0,\infty), \quad p(0) = 0, \quad \frac{1}{a}\varrho^{\gamma-1} - b \leq p'(\varrho) \leq a\varrho^{\gamma-1} + b, \quad \textit{for all} \quad \varrho > 0 \tag{2.1.6}$$

with

$$a > 0, \quad b \geq 0, \quad \gamma > \frac{3}{2}.$$

Let the initial data ϱ_0, $\mathbf{u}_0$ satisfy

$$\varrho_0 \in L^1(\Omega), \quad H(\varrho_0) \in L^1(\Omega), \quad \varrho_0 \geq 0 \quad \textit{a.e. in} \quad \Omega,$$

$$\varrho_0 \mathbf{u}_0 \in \left(L^1(\Omega)\right)^3 \quad \textit{such that} \quad \frac{|\varrho_0 \mathbf{u}_0|^2}{\varrho_0} 1_{\{x;\varrho_0>0\}} \in L^1(\Omega)$$

$$\textit{and such that} \quad \varrho_0 \mathbf{u}_0 = 0 \quad \textit{whenever} \quad x \in \{\varrho_0 = 0\}. \tag{2.1.7}$$

Then the problem (2.0.1 - 2.0.3) admits at least one bounded energy weak solution $[\varrho, \mathbf{u}]$ *on* $(0,T)\times\Omega$ *in the sense of Definition 6. Moreover* $[\varrho, \mathbf{u}]$ *satisfy the energy inequality (2.1.1).*

Remark 9. The first existence result for problem (2.0.1 - 2.0.3) was obtained by Lions [71] on condition that $\Omega \subset \mathbb{R}^3$ is a domain with smooth and compact boundary and that $p(\varrho) \approx \varrho^\gamma$ with $\gamma \geq \frac{9}{5}$. This result was relaxed to $\gamma > \frac{3}{2}$ by Feireisl, Novotný and Petzeltová [49] on condition that Ω is a bounded smooth domain. Existence for certain classes of unbounded domains was shown in Novotný and Straškraba [86] (see also Lions [71]).

Remark 10. The existence result in Feireisl [38] and Feireisl, Novotný and Petzeltová [48] holds in the presence of the Coriolis force (see for example Feireisl and Novotný [44] and Feireisl, Jin and Novotný [46] and reference therein).

2.1.2 Classical solutions

For the solvability of the system (2.0.6) with the initial data $\mathbf{v}(0) = \mathbf{v}_0$, we report the following result (see Takada [105]):

Theorem 11. *Let* $s \in \mathbb{R}$ *satisfy* $s > \frac{3}{2} + 1$. *Then, for* $0 < T < \infty$ *and* $\mathbf{v}_0 \in W^{s,2}\left(\mathbb{R}^3\right)$ *satisfying* $div_x \mathbf{v}_0 = 0$, *there exists a positive parameter* $\Omega_0 = \Omega_0(s, T, \|\mathbf{v}_0\|_{W^{s,2}})$ *such that if* $|\omega| \geq \Omega_0$ *then the system (2.0.6) possesses a unique classical solution* $\mathbf{v}$ *satisfying*

$$\mathbf{v} \in C\left([0,T]\,; W^{s,2}(\mathbb{R}^3;\mathbb{R}^3)\right),$$

$$\partial_t \mathbf{v} \in C\left([0,T]\,; W^{s-1,2}(\mathbb{R}^3;\mathbb{R}^3)\right),$$

$$\nabla\Pi \in C\left([0,T]\,; W^{s,2}(\mathbb{R}^3;\mathbb{R}^3)\right). \tag{2.1.8}$$

Remark 12. The global existence stated above was proved by Kho, Lee and Takada [57] for the initial data in $W^{s,2}\left(\mathbb{R}^3\right)$ with $s > 7/2$.

Remark 13. Theorem 11 deals with inviscid flows in a rotating frame under the condition of fast rotation. In terms of scale analysis (see Nazarenko [79]), if we define by U and L the characteristic velocity and length scale of the fluid, we can estimate the order of magnitude of the non-linear term and the rotational term in the equation (2.0.6) as follows

$$\mathbf{v}\cdot\nabla\mathbf{v} \sim O\left(\frac{U^2}{L}\right), \tag{2.1.9}$$

$$\mathbf{v}\times\omega \sim O\left(\Omega U\right), \tag{2.1.10}$$

where

$$\omega \sim O\left(\Omega\right) \sim O\left(\frac{U}{L}\right), \tag{2.1.11}$$

with Ω characteristic angular velocity. Comparing (2.1.9) and (2.1.10), we have

$$\frac{U}{L} \sim \Omega. \tag{2.1.12}$$

Fast rotation implies

$$\frac{U}{\Omega L} \ll 1 \tag{2.1.13}$$

and we can neglect the non-linear term in (2.0.6), obtaining

$$\partial_t \mathbf{v} + \mathbf{v} \times \omega + \nabla_x \Pi = 0, \quad \mathrm{div}_x \mathbf{v} = 0. \tag{2.1.14}$$

These are linear equations. In other words, fast rotation leads to averaging mechanism that weakens the nonlinear effects. This of course prevents singularity allowing the life span of the solution to extend (see Chemin, Desjardines, Gallagher and Grenier [16] and references therein).

2.2 Acoustic waves

In the following, we introduce the acoustic system related to the equations (2.0.1) and (2.0.2). Then, we briefly discuss the acoustic energy introducing appropriate energy estimates. Finally, we discuss the decay of acoustic waves in the limit of Mach number tends to zero introducing the dispersive estimate mentioned before.

We assume the perturbation of the density of the first order and small compared to the given ambient fluid density. Therefore, we can write the acoustic system related to the equations (2.0.1) and (2.0.2) by the following linear relations (see Feireisl and Novotný [41], Feireisl, Nečasová and Sun [47] and Lighthill [68, 69]):

$$\varepsilon \partial_t s + \triangle \Psi = 0, \quad \varepsilon \partial_t \nabla \Psi + a \nabla_x s = 0, \; a = p'(1) > 0, \tag{2.2.1}$$

with the initial data

$$s(0) = \varrho_0^{(1)}, \quad \nabla_x \Psi(0) = \nabla_x \Psi_0 = \mathbf{u}_0 - \mathbf{v}_0 \tag{2.2.2}$$

where $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$ and $\mathbf{H}$ denotes the Helmholtz projection into the space of solenoidal functions and Ψ is a potential. Here, s is defined as the change in density for a given ambient fluid density. In other words, the density perturbation. The sound velocity squared is represented by a. For more detail physical discussion concerning acoustics, we refer to the book of Falkovich [33] and Landau-Lifshitz [65].

2.2.1 Energy and dispersive estimates

The total change in energy of the fluid caused by the acoustic wave is given by the integral

$$\int_{\mathbb{R}^3} \left(\frac{1}{2} a \left| s \right|^2 + \frac{1}{2} \left| \nabla_x \Psi \right|^2 \right) \mathrm{d}x, \tag{2.2.3}$$

where the integrand may be regarded as the density of sound energy (see Landau-Lifshitz [65]). It is easy to verify (see Landau-Lifshitz [65]) that the density of sound energy is conserved in time, namely

$$\left[\int_{\mathbb{R}^3}\left(\frac{1}{2}a\,|s|^2+\frac{1}{2}\,|\nabla_x\Psi|^2\right)(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=T}=0. \tag{2.2.4}$$

In addition, we have the following energy estimates (see Feireisl and Novotný [42])

$$\|\nabla_x\Psi\,(t,\cdot)\|_{W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)}+\|s\,(t,\cdot)\|_{W^{k,2}(\mathbb{R}^3)}$$

$$\leq c\left(\|\nabla_x\Psi_0\|_{W^{k,2}(\mathbb{R}^3;\mathbb{R}^3)}+\left\|\varrho_0^{(1)}\right\|_{W^{k,2}(\mathbb{R}^3)}\right),\ k=0,1,..., \tag{2.2.5}$$

for any $t>0$. Instead, concerning the decay of the acoustic waves in the incompressible limit, the following dispersive estimates hold

$$\|\nabla_x\Psi\,(t,\cdot)\|_{W^{k,p}(\mathbb{R}^3;\mathbb{R}^3)}+\|s\,(t,\cdot)\|_{W^{k,p}(\mathbb{R}^3)}$$

$$\leq c(1+\frac{t}{\varepsilon})^{-\left(\frac{1}{q}-\frac{1}{p}\right)}\left(\|\nabla_x\Psi_0\|_{W^{k,q}(\mathbb{R}^3;\mathbb{R}^3)}+\left\|\varrho_0^{(1)}\right\|_{W^{k,q}(\mathbb{R}^3)}\right), \tag{2.2.6}$$

$$2\leq p\leq\infty,\ \frac{1}{p}+\frac{1}{q}=1,\ k=0,1,....$$

For the purpose of our analysis and the use of the estimates (2.2.5) and (2.2.6), it is convenient to regularize the initial data (2.2.2) in the following way

$$\varrho_0^{(1)}=\varrho_{0,\eta}^{(1)}=\chi_\eta\star\left(\psi_\eta\varrho_0^{(1)}\right),\quad \nabla_x\Psi_0=\nabla_x\Psi_{0,\eta}=\chi_\eta\star\left(\psi_\eta\nabla_x\Psi_0\right),\quad \eta>0, \tag{2.2.7}$$

where $\{\chi_\eta\}$ is a family of regularizing kernels and $\psi_\eta\in C_0^\infty(\mathbb{R}^3)$ are standards cut-off functions. Consequently, the acoustic system possesses a (unique) smooth solution $[s,\Psi]$ and the quantities $\nabla_x\Psi$ and s are compactly supported in $\mathbb{R}^3$ (see Feireisl and Novotný [42]).

2.3 Convergence analysis

For the purpose of the convergence analysis, we introduce the relative energy functional and the relative energy inequality associated to the system (2.0.1 - 2.0.3) already mentioned in the Introduction.

2.3.1 Relative energy inequality

The relative energy functional associated to the system (2.0.1 - 2.0.3) is given by the following relation

$$\mathcal{E}(\varrho,\mathbf{u}\mid r,\mathbf{U})=\int_{\mathbb{R}^3}\left[\frac{1}{2}\varrho\,|(\mathbf{u}-\mathbf{U})|^2\right.$$

$$\left.+\frac{1}{\varepsilon^2}\left(H\left(\varrho\right)-H'\left(r\right)\left(\varrho-r\right)-H\left(r\right)\right)\right]\mathrm{d}x \tag{2.3.1}$$

along with the relative energy inequality

$$[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})]_{t=0}^{t=T}$$

$$+\varepsilon \int_0^T \int_{\mathbb{R}^3} S\left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}\right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}\right) \mathrm{d}x\mathrm{d}t \leq \int_0^T \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U})\mathrm{d}t, \tag{2.3.2}$$

where the remainder $\mathcal{R}$ is expressed as follows

$$\mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) = \int_{\mathbb{R}^3} \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\right) \cdot (\mathbf{U} - \mathbf{u})\mathrm{d}x$$

$$+\varepsilon \int_{\mathbb{R}^3} S(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u})\mathrm{d}x$$

$$+\frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \left((r - \varrho)\, \partial_t H'(r) + \nabla_x H'(r) \cdot (r\mathbf{U} - \varrho \mathbf{u})\right) \mathrm{d}x$$

$$-\frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \left(p(\varrho) - p(r)\right) \mathrm{div}_x \mathbf{U} \mathrm{d}x.$$

$$+ \int_{\mathbb{R}^3} (\varrho \mathbf{u} \times \omega) \cdot (\mathbf{U} - \mathbf{u})\, \mathrm{d}x := I_1 + ... + I_5 \tag{2.3.3}$$

Here, r and $\mathbf{U}$ are sufficiently smooth functions such that

$$r > 0, \quad r - 1 \in C_c^\infty\left([0, T] \times \mathbb{R}^3\right), \quad \mathbf{U} \in C_c^\infty\left([0, T] \times \mathbb{R}^3; \mathbb{R}^3\right). \tag{2.3.4}$$

It can be shown (see Feireisl, Jin and Novotný [45] for different type of domains and boundary conditions) that any weak solution $[\varrho, \mathbf{u}]$ to the compressible Navier-Stokes system (2.0.1 - 2.0.3) satisfies the relative energy inequality for any pair of sufficiently smooth test functions r, $\mathbf{U}$ as in (2.3.4). The particular choice of $[r, \mathbf{U}]$ will be clarified later.

2.3.2 Main results

The following theorem is the main result of this chapter.

Theorem 14. *Let $M > 0$ be a constant. Let the pressure p satisfy*

$$p \in C^1\,[0, \infty) \cap C^3(0, \infty), \quad p(0) = 0, \quad \frac{1}{a}\varrho^{\gamma-1} - b \leq p'(\varrho) \leq a\varrho^{\gamma-1} + b, \tag{2.3.5}$$

for all $\varrho > 0$, with

$$a > 0, \quad b \geq 0, \quad \gamma > \frac{3}{2}.$$

Let the initial data $[\varrho_0, \mathbf{u}_0]$ for the Navier-Stokes system (2.0.1 - 2.0.3) be of the following form

$$\varrho(0) = \varrho_{0,\varepsilon} = 1 + \varepsilon \varrho_{0,\varepsilon}^{(1)}, \; \mathbf{u}(0) = \mathbf{u}_{0,\varepsilon}, \tag{2.3.6}$$

$$\left\| \varrho_{0,\varepsilon}^{(1)} \right\|_{L^2 \cap L^\infty(\mathbb{R}^3)} + \| \mathbf{u}_{0,\varepsilon} \|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \le M. \tag{2.3.7}$$

Let all the requirements of Theorem 11 be satisfied with the initial datum for the Euler system $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$. *Let* $[s, \Psi]$ *be the solution of the acoustic system (2.2.1) with the initial data (2.2.7). Then,*

$$\| \sqrt{\varrho} \left(\mathbf{u} - \mathbf{v} - \nabla \Psi \right)(t, \cdot) \|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^2$$

$$+ \left\| \frac{\varrho - 1}{\varepsilon}(t, \cdot) - s(t, \cdot) \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \frac{\varrho - 1}{\varepsilon^{2/\gamma}}(t, \cdot) - \frac{s(t, \cdot)}{\varepsilon^{(2/\gamma)-1}} \right\|_{L^\gamma(\mathbb{R}^3)}^\gamma$$

$$\le c \left(\| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^2 + \left\| \varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)} \right\|_{L^2(\mathbb{R}^3)}^2 \right), \quad t \in [0, T], \tag{2.3.8}$$

for any weak solutions $[\varrho, \mathbf{u}]$ *of the compressible Navier-Stokes system (2.0.1 - 2.0.3).*

Remark 15. The first relation in (2.3.6) refers to the first-order perturbation of the density, namely $\varepsilon \varrho_{0,\varepsilon}^{(1)}$, respect to the ambient fluid density settled equal one.

A consequence of the above Theorem is the following Corollary.

Corollary 16. *Let all the requirements of Theorem 14 be satisfied. Assume that*

$$\varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \quad \textit{in} \quad L^2(\mathbb{R}^3), \quad \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \quad \textit{in} \quad L^2(\mathbb{R}^3;\mathbb{R}^3) \quad \textit{when} \quad \varepsilon \to 0.$$

Then

$$\operatorname*{ess\ sup}_{t \in [0,T]} \| \sqrt{\varrho} \left(\mathbf{u} - \mathbf{v} \right)(t, \cdot) \|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^2 \to 0 \quad \textit{when} \quad \varepsilon \to 0,$$

$$\operatorname*{ess\ sup}_{t \in [0,T]} \| \varrho - 1 \|_{L^2(\mathbb{R}^3)}^2 \to 0 \quad \textit{when} \quad \varepsilon \to 0,$$

$$\operatorname*{ess\ sup}_{t \in [0,T]} \| \varrho - 1 \|_{L^\gamma(\mathbb{R}^3)}^\gamma \to 0 \quad \textit{when} \quad \varepsilon \to 0,$$

for any weak solutions $[\varrho, \mathbf{u}]$ *of the compressible Navier-Stokes system (2.0.1 - 2.0.3) and* $[r, \mathbf{U}]$ *sufficiently smooth test functions.*

2.3.3 Convergence

The following discussion is devoted to the proof of Theorem 14. Here and hereafter, the symbol c will denote a positive generic constant, independent by ε, usually found in inequalities, that will not have the same value when used in different parts in the analysis.

We start with the a priori bounds. In accordance with the energy inequality (2.1.1), we have

$$\operatorname*{ess\ sup}_{t \in [0,T]} \| \varrho(t, \cdot) \|_{L^\gamma \cap L^1(\mathbb{R}^3)} \le c(M), \tag{2.3.9}$$

$$\operatorname{ess}\sup_{t\in[0,T]} \|\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \leq c(M). \tag{2.3.10}$$

From (2.3.9) and (2.3.10), we obtain

$$\|\varrho\mathbf{u}(t,\cdot)\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)} = \|\sqrt{\varrho}\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)}$$

$$\leq \|\sqrt{\varrho}(t,\cdot)\|_{L^{2\gamma}(\mathbb{R}^3)} \|\sqrt{\varrho}\mathbf{u}(t,\cdot)\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}, \tag{2.3.11}$$

with

$$q = \frac{2\gamma}{\gamma+1}. \tag{2.3.12}$$

We conclude that

$$\operatorname{ess}\sup_{t\in[0,T]} \|\varrho\mathbf{u}(t,\cdot)\|_{L^q(\mathbb{R}^3;\mathbb{R}^3)} \leq c(M), \quad q = \frac{2\gamma}{\gamma+1}. \tag{2.3.13}$$

Moreover, introducing (see Germain [51])

$$I(\varrho,r) = H(\varrho) - H'(r)(\varrho - r) - H(r), \tag{2.3.14}$$

we observe that the map $\varrho \to I(\varrho,r)$ is, for any fixed $r>0$, a strictly convex function on $(0,\infty)$ with global minimum equal to 0 at $\varrho = r$, which grows at infinity with the rate ϱ^γ. Consequently, the integral $\int_{\mathbb{R}^3} I(\varrho,r)(t,x)\,\mathrm{d}x$ in (2.3.2) provides a control of $(\varrho - r)(t,\cdot)$ in L^2 over the sets $\{x : |\varrho - r|(t,x) < 1\}$ and in L^γ over the sets $\{x : |\varrho - r|(t,x) \geq 1\}$. So, for any r in a compact set $(0,\infty)$, there holds

$$I(\varrho,r) \approx |\varrho - r|^2 1_{\{|\varrho-r|<1\}} + |\varrho - r|^\gamma 1_{\{|\varrho-r|\geq 1\}}, \quad \forall \varrho \geq 0, \tag{2.3.15}$$

in the sense that $I(\varrho,r)$ gives an upper and lower bound in term of the right-hand side quantity (see Bardos and Nguyen [2], Feireisl, Novotný and Sun [50] and Sueur [104]). Indeed, is possible to show (see Bardos and Nguyen [2], Lemma 2.2) that for the quantity $I(\varrho,r)$ the following approximation holds

$$I(\varrho,r) \approx \varrho\,(H'(\varrho) - H'(r)) - r\,(\varrho - r)\,H''(r),$$

where the right-hand-side is of order $|\varrho - r|^2$ when $|\varrho - r| \leq 1$, and of order $|\varrho - r|^\gamma$ when $|\varrho - r| \geq 1$. Therefore, we have the following uniform bounds

$$\operatorname{ess}\sup_{t\in[0,T]} \left\| [(\varrho-1)(t,\cdot)]\, 1_{\{|\varrho-1|<1\}} \right\|_{L^2(\mathbb{R}^3)} \leq c(M)\varepsilon, \tag{2.3.16}$$

$$\operatorname{ess}\sup_{t\in[0,T]} \left(\left\| [(\varrho-1)(t,\cdot)]\, 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \right) \leq c(M)\varepsilon^{2/\gamma}, \tag{2.3.17}$$

where we have set $r = 1$ and $\mathbf{U} = 0$ in the relative energy inequality (2.3.2).

Now, the basic idea is to apply (2.3.2) to $[r,\mathbf{U}] = [1+\varepsilon s, \mathbf{v} + \nabla_x \Psi]$. The particular choice of the test functions is motivate by the regularity of the solutions of the Euler (2.0.6) and acoustic (2.2.1) system. In the following, η will be fixed. For the initial data we have

$$[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](0) = \int_{\mathbb{R}^3} \frac{1}{2} \varrho_{0,\varepsilon} \left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x$$

$$+ \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left[H\left(1 + \varepsilon \varrho_{0,\varepsilon}^{(1)}\right) - \varepsilon H'\left(1 + \varepsilon \varrho_0^{(1)}\right)\left(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right) - H\left(1 + \varepsilon \varrho_0^{(1)}\right)\right] \mathrm{d}x, \tag{2.3.18}$$

where $\mathbf{u}_0 = \mathbf{H}[\mathbf{u}_0] + \nabla \Psi_0$. Given (2.3.6) and (2.3.7), for the first term on the right hand side of the equality (2.3.18), we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{1}{2} \varrho_{0,\varepsilon} \left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} \frac{1}{2} \left|1 + \varepsilon \varrho_{0,\varepsilon}^{(1)}\right| \left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} \frac{1}{2} \left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \frac{1}{2} \left|\varepsilon \varrho_{0,\varepsilon}^{(1)}\right| \left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x \\ &\leq \int_{\mathbb{R}^3} \frac{1}{2} \left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x + \varepsilon \left\|\varrho_{0,\varepsilon}^{(1)}\right\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \frac{1}{2} \left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x \\ &\leq c(M)\left(1 + \varepsilon\right) \left\|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}. \end{aligned} \tag{2.3.19}$$

For the second term on the right hand side of the equality (2.3.18), setting $a = 1 + \varepsilon \varrho_{0,\varepsilon}^{(1)}$ and $b = 1 + \varepsilon \varrho_0^{(1)}$, and observing that

$$H(a) = H(b) + H'(b)(a - b) + \frac{1}{2} H''(\xi)(a - b)^2, \;\; \xi \in (a, b),$$

$$\left|H(a) - H'(b)(a - b) - H(b)\right| \leq c\left|a - b\right|^2,$$

we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left[H\left(1 + \varepsilon \varrho_{0,\varepsilon}^{(1)}\right) - \varepsilon H'\left(1 + \varepsilon \varrho_0^{(1)}\right)\left(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right) - H\left(1 + \varepsilon \varrho_0^{(1)}\right)\right] \mathrm{d}x \\ &\leq c(M) \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left(\left|\varepsilon\left(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right)\right|^2\right) \mathrm{d}x \\ &\leq c(M) \left\|\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right\|^2_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{2.3.20}$$

Finally, we can conclude

$$[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](0) \leq c(M)[(1 + \varepsilon) \left\|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \left\|\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right\|^2_{L^2(\mathbb{R}^3)}].$$

Now, we decompose I_1 into

$$\int_0^T I_1 \mathrm{d}t = \int_0^T \int_{\mathbb{R}^3} \varrho\left[(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u})\right] \mathrm{d}x\mathrm{d}t$$

$$-\int_0^T\int_{\mathbb{R}^3}\varrho\nabla_x\mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\cdot(\mathbf{U}-\mathbf{u})\mathrm{d}x\mathrm{d}t. \tag{2.3.21}$$

For the second term on the right hand side of (2.3.21), thanks to the Sobolev imbedding theorem, the Minkowski inequality, (2.1.8) and the dispersive estimate (2.2.6), we have

$$\int_0^T\int_{\mathbb{R}^3}\varrho\nabla_x\mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\cdot(\mathbf{U}-\mathbf{u})\mathrm{d}x\mathrm{d}t$$

$$\leq\int_0^T\int_{\mathbb{R}^3}\varrho\left|\nabla_x\mathbf{U}\right|\cdot\left|(\mathbf{U}-\mathbf{u})\right|^2 dx\mathrm{d}t$$

$$\leq\int_0^T\mathcal{E}\left\|\nabla_x\mathbf{v}+\nabla_x^2\Psi\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t$$

$$\leq\int_0^T\mathcal{E}\left\|\nabla_x\mathbf{v}\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t+\int_0^T\mathcal{E}\left\|\nabla_x^2\Psi\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t$$

$$\leq c\int_0^T\mathcal{E}\mathrm{d}t \tag{2.3.22}$$

The first term on the right hand side of (2.3.21) can be rewritten as follows

$$\int_0^T\int_{\mathbb{R}^3}\varrho\left[(\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})\right]\mathrm{d}x\mathrm{d}t$$

$$=\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\cdot(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla_x\mathbf{v})\,\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\cdot\partial_t\nabla_x\Psi\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\otimes\nabla_x\Psi:\nabla_x\mathbf{v}\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\otimes\mathbf{v}:\nabla_x^2\Psi\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\cdot\nabla_x\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t. \tag{2.3.23}$$

In view of uniform bounds (2.3.13), (2.1.8) and dispersive estimate (2.2.6), the last three integrals can be estimated as follows

$$\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\otimes\nabla_x\Psi:\nabla_x\mathbf{v}\mathrm{d}x\mathrm{d}t=\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{v}+\nabla_x\Psi-\mathbf{u})\otimes\nabla_x\Psi:\nabla_x\mathbf{v}\mathrm{d}x\mathrm{d}t$$

$$=\int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{v})\otimes\nabla_x\Psi:\nabla_x\mathbf{v}\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_{\mathbb{R}^3}(\varrho\nabla_x\Psi)\otimes\nabla_x\Psi:\nabla_x\mathbf{v}\mathrm{d}x\mathrm{d}t$$

$$-\int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{u})\otimes\nabla_x\Psi:\nabla_x\mathbf{v}\mathrm{d}x\mathrm{d}t$$

$$\leq c\int_0^T\|\varrho\|_{L^1}\|\mathbf{v}\|_{L^\infty}\|\nabla_x\Psi\|_{L^\infty}\|\nabla_x\mathbf{v}\|_{L^\infty}\mathrm{d}t$$

$$+c\int_0^T\|\varrho\|_{L^1}\|\nabla_x\Psi\|_{L^\infty}\|\nabla_x\Psi\|_{L^\infty}\|\nabla_x\mathbf{v}\|_{L^\infty}\mathrm{d}t$$

$$+c\int_0^T\|\varrho\mathbf{u}\|_{L^{\frac{2\gamma}{\gamma+1}}}\|\nabla_x\Psi\|_{L^{\frac{2\gamma}{\gamma-1}}}\|\nabla_x\mathbf{v}\|_{L^\infty}\mathrm{d}t$$

$$\leq c(M)\left[\varepsilon\left(\log\left(\varepsilon+T\right)-\log\left(\varepsilon\right)\right)+\left(\varepsilon-\frac{\varepsilon^2}{\varepsilon+T}\right)+\left(\frac{\gamma\left(\varepsilon+T\right)\left(\frac{\varepsilon+T}{\varepsilon}\right)^{-1/\gamma}}{\gamma-1}-\frac{\gamma\varepsilon}{\gamma-1}\right)\right]. \tag{2.3.24}$$

Similarly to (2.3.24),

$$\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\otimes\mathbf{v}:\nabla_x^2\Psi\mathrm{d}x\mathrm{d}t=\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{v}+\nabla_x\Psi-\mathbf{u})\otimes\mathbf{v}:\nabla_x^2\Psi\mathrm{d}x\mathrm{d}t$$

$$\leq c(M)\left[\varepsilon\left(\log\left(\varepsilon+T\right)-\log\left(\varepsilon\right)\right)+\left(\varepsilon-\frac{\varepsilon^2}{\varepsilon+T}\right)+\left(\frac{\gamma\left(\varepsilon+T\right)\left(\frac{\varepsilon+T}{\varepsilon}\right)^{-1/\gamma}}{\gamma-1}-\frac{\gamma\varepsilon}{\gamma-1}\right)\right] \tag{2.3.25}$$

and

$$\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\cdot\nabla_x\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t=\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{v}+\nabla_x\Psi-\mathbf{u})\cdot\nabla_x\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t$$

$$\leq c(M)\left[\left(\varepsilon-\frac{\varepsilon^2}{\varepsilon+T}\right)+\left(\frac{\varepsilon}{2}-\frac{\varepsilon^3}{2\left(\varepsilon+T\right)^2}\right)+\left(\gamma\left(\varepsilon+T\right)\left(\frac{\varepsilon+T}{\varepsilon}\right)^{-1/\gamma}-\varepsilon\gamma\right)\right]. \tag{2.3.26}$$

Using (2.0.6), for the first term of (2.3.23), we have

$$\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\cdot\left(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla_x\mathbf{v}\right)\mathrm{d}x\mathrm{d}t$$

$$=-\int_0^T\int_{\mathbb{R}^3}\varrho(\mathbf{U}-\mathbf{u})\cdot\nabla_x\Pi\mathrm{d}x\mathrm{d}t-\int_0^T\int_{\mathbb{R}^3}(\mathbf{U}-\mathbf{u})\cdot\left(\omega\times\varrho\mathbf{v}\right)\mathrm{d}x\mathrm{d}t$$

$$= \int_0^T \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla_x \Pi \mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{R}^3} \varrho \mathbf{U} \cdot \nabla_x \Pi \mathrm{d}x\mathrm{d}t - \int_0^T \int_{\mathbb{R}^3} (\mathbf{U} - \mathbf{u}) \cdot (\omega \times \varrho \mathbf{v}) \,\mathrm{d}x\mathrm{d}t. \tag{2.3.27}$$

Regarding the first integral on the right hand side of (2.3.27), as a consequence of the estimate (2.3.13), we have

$$\varrho \mathbf{u} \to \mathbf{w} \text{ weakly-(*) in } L^\infty \left(0, T; L^{2\gamma/\gamma+1}(\mathbb{R}^3; \mathbb{R}^3)\right), \tag{2.3.28}$$

where $\mathbf{w}$ denotes the weak limit of the composition. Now, taking the limit in the weak formulation of the continuity equation

$$\varepsilon \int_0^T \int_{\mathbb{R}^3} \left(\frac{\varrho - 1}{\varepsilon}\right) \partial_t \varphi \mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \varrho \mathbf{u} \nabla_x \varphi \mathrm{d}x\mathrm{d}t = 0 \tag{2.3.29}$$

for sufficiently smooth φ, thanks to the estimate (2.3.16) and (2.3.17), we deduce that

$$\int_0^T \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla_x \varphi \mathrm{d}x\mathrm{d}t = 0 \tag{2.3.30}$$

when $\varepsilon \to 0$. We may infer that

$$\int_0^T \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla_x \Pi \mathrm{d}x\mathrm{d}t \to \int_0^T \int_{\mathbb{R}^3} \mathbf{w} \cdot \nabla_x \Pi \mathrm{d}x\mathrm{d}t = 0. \tag{2.3.31}$$

For the second integral on the right hand side of (2.3.27), we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^3} \varrho \mathbf{U} \cdot \nabla_x \Pi \mathrm{d}x\mathrm{d}t \right| &\leq \left| \int_0^T \int_{\mathbb{R}^3} (\varrho - 1) \cdot \mathbf{U} \cdot \nabla_x \Pi \mathrm{d}x\mathrm{d}t \right| \\ &+ \left| \int_0^T \int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla_x \Pi \mathrm{d}x\mathrm{d}t \right|. \end{aligned} \tag{2.3.32}$$

For the first integral on the right-hand side of (2.3.32), thanks to (2.1.8), the estimate (2.2.6) and the uniform bounds (2.3.16) and (2.3.17), we have

$$\int_0^T \int_{\mathbb{R}^3} (\varrho - 1) \cdot \mathbf{U} \cdot \nabla_x \Pi \mathrm{d}x\mathrm{d}t$$

$$\leq c\varepsilon \int_0^T \left\| [\frac{\varrho - 1}{\varepsilon}] 1_{\{|\varrho - 1| < 1\}} \right\|_{L^2(\mathbb{R}^3)} \cdot \|\mathbf{v} + \nabla_x \Psi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \cdot \|\nabla_x \Pi\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \,\mathrm{d}t$$

$$\leq c\varepsilon \int_0^T \left\| [\frac{\varrho - 1}{\varepsilon}] 1_{\{|\varrho - 1| < 1\}} \right\|_{L^2(\mathbb{R}^3)} \cdot \|\mathbf{v}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \cdot \|\nabla_x \Pi\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \,\mathrm{d}t$$

$$+ c\varepsilon \int_0^T \left\| [\frac{\varrho - 1}{\varepsilon}] 1_{\{|\varrho - 1| < 1\}} \right\|_{L^2(\mathbb{R}^3)} \cdot \|\nabla_x \Psi\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \cdot \|\nabla_x \Pi\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \,\mathrm{d}t \leq c(M)\varepsilon \tag{2.3.33}$$

and

$$\int_0^T \int_{\mathbb{R}^3} (\varrho - 1) \cdot \mathbf{U} \cdot \nabla_x \Pi \mathrm{d}x \mathrm{d}t$$

$$\leq c \int_0^T \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \left\| (\mathbf{v} + \nabla_x \Psi) \cdot \nabla_x \Pi \right\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c \int_0^T \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \left\| \mathbf{v} \cdot \nabla_x \Pi \right\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$+ c \int_0^T \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \left\| \nabla_x \Psi \cdot \nabla_x \Pi \right\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t. \tag{2.3.34}$$

Thanks to the following interpolation inequalities

$$\left\| \nabla_x \Psi \cdot \nabla_x \Pi \right\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \leq \left\| \nabla_x \Psi \cdot \nabla_x \Pi \right\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \left\| \nabla_x \Psi \cdot \nabla_x \Pi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1-\frac{\gamma-1}{\gamma}}$$

$$\leq \left\| \nabla_x \Psi \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \left\| \nabla_x \Pi \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \left\| \nabla_x \Psi \cdot \nabla_x \Pi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma}$$

$$\leq c(M) \left\| \nabla_x \Psi \cdot \nabla_x \Pi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \leq c(M) \left\| \nabla_x \Psi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \cdot \left\| \nabla_x \Pi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma}$$

$$\leq c(M) \left\| \nabla_x \Psi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma}, \tag{2.3.35}$$

$$\left\| \mathbf{v} \cdot \nabla_x \Pi \right\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \leq \left\| \mathbf{v} \cdot \nabla_x \Pi \right\|_{L^1(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \left\| \mathbf{v} \cdot \nabla_x \Pi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1-\frac{\gamma-1}{\gamma}}$$

$$\leq \left\| \mathbf{v} \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \left\| \nabla_x \Pi \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma-1}{\gamma}} \left\| \mathbf{v} \cdot \nabla_x \Pi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma}$$

$$\leq c \left\| \mathbf{v} \cdot \nabla_x \Pi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \leq c \left\| \mathbf{v} \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \cdot \left\| \nabla_x \Pi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \leq c, \tag{2.3.36}$$

and the estimate (2.2.6), for the integral in (2.3.34) we have,

$$\int_0^T \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \left\| \mathbf{v} \cdot \nabla_x \Pi \right\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$+ \int_0^T \left\| [\varrho - 1] 1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \cdot \left\| \nabla_x \Psi \cdot \nabla_x \Pi \right\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c(M) \varepsilon^{2/\gamma} + c(M) \varepsilon^{2/\gamma} \int_0^T \left\| \nabla_x \Psi \right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{1/\gamma} \mathrm{d}t$$

$$\leq c(M) \varepsilon^{2/\gamma} + c(M) \varepsilon^{2/\gamma} \left(\frac{\gamma (\varepsilon + T) \left(\frac{\varepsilon + T}{\varepsilon} \right)^{-1/\gamma}}{\gamma - 1} - \frac{\gamma \varepsilon}{\gamma - 1} \right). \tag{2.3.37}$$

For the second integral on the right-hand side of (2.3.32), we have

$$\int_0^T \int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla_x \Pi \mathrm{d}x \mathrm{d}t = \int_0^T \int_{\mathbb{R}^3} \mathbf{v} \cdot \nabla_x \Pi \mathrm{d}x \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \nabla_x \Psi \cdot \nabla_x \Pi \mathrm{d}x \mathrm{d}t. \quad (2.3.38)$$

Performing integration by parts in the first term on the right-hand side of (2.3.38), we have

$$\int_0^T \int_{\mathbb{R}^3} \mathrm{div}_x \mathbf{v} \cdot \Pi \mathrm{d}x \mathrm{d}t = 0$$

thanks to incompressibility condition $\mathrm{div}_x \mathbf{v} = 0$. For the second term on the right-hand side of (2.3.38) using integration by parts and the acoustic equation (2.2.1), we have

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} \nabla_x \Psi \cdot \nabla_x \Pi \mathrm{d}x \mathrm{d}t &= - \int_0^T \int_{\mathbb{R}^3} \triangle \Psi \cdot \Pi \mathrm{d}x \mathrm{d}t \\ &= \varepsilon \int_0^T \int_{\mathbb{R}^3} \partial_t s \cdot \Pi \mathrm{d}x \mathrm{d}t \\ &= \varepsilon \left[\int_{\mathbb{R}^3} s \cdot \Pi \mathrm{d}x \right]_{t=0}^{t=T} - \varepsilon \int_0^T \int_{\mathbb{R}^3} s \cdot \partial_t \Pi \mathrm{d}x \mathrm{d}t, \end{aligned} \quad (2.3.39)$$

that it goes to zero for $\varepsilon \to 0$. For the second term of (2.3.23), we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \varrho (\mathbf{U} - \mathbf{u}) \cdot \partial_t \nabla_x \Psi \mathrm{d}x \mathrm{d}t \\ &= - \int_0^T \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \partial_t \nabla_x \Psi \mathrm{d}x \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \varrho \mathbf{v} \cdot \partial_t \nabla_x \Psi \mathrm{d}x \mathrm{d}t \\ &\quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} \varrho \partial_t \left| \nabla_x \Psi \right|^2 \mathrm{d}x \mathrm{d}t, \end{aligned} \quad (2.3.40)$$

where

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} \varrho \mathbf{v} \cdot \partial_t \nabla_x \Psi \mathrm{d}x \mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^3} (\varrho - 1) \mathbf{v} \cdot \partial_t \nabla_x \Psi \mathrm{d}x \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \mathbf{v} \cdot \partial_t \nabla_x \Psi \mathrm{d}x \mathrm{d}t. \end{aligned} \quad (2.3.41)$$

We use the acoustic equation (2.2.1) to rewrite the first term above as follows

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} (\varrho - 1) \mathbf{v} \cdot \partial_t \nabla_x \Psi \mathrm{d}x \mathrm{d}t \\ &= -a \int_0^T \int_{\mathbb{R}^3} \frac{\varrho - 1}{\varepsilon} \mathbf{v} \cdot \nabla_x s \mathrm{d}x \mathrm{d}t, \end{aligned} \quad (2.3.42)$$

where, thanks to (2.1.8), (2.2.6), (2.3.16) and (2.3.17), we have

$$\int_0^T \int_{\mathbb{R}^3} \frac{\varrho - 1}{\varepsilon}\mathbf{v}\cdot\nabla_x s \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^T \left\| [\frac{\varrho-1}{\varepsilon}]1_{\{|\varrho-1|<1\}} \right\|_{L^2(\mathbb{R}^3)} \|\mathbf{v}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \|\nabla_x s\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c(M)\varepsilon \left(\log(\varepsilon + T) - \log(\varepsilon)\right) \tag{2.3.43}$$

and

$$\int_0^T \int_{\mathbb{R}^3} \frac{\varrho - 1}{\varepsilon}\mathbf{v}\cdot\nabla_x s \mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^T \left\| [\frac{\varrho-1}{\varepsilon}]1_{\{|\varrho-1|\geq 1\}} \right\|_{L^\gamma(\mathbb{R}^3)} \|\mathbf{v}\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \|\nabla_x s\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c(M)\varepsilon^{\frac{2}{\gamma}} \left(\log(\varepsilon + T) - \log(\varepsilon)\right), \tag{2.3.44}$$

where we used the following interpolation inequality for $\mathbf{v}$

$$\|\mathbf{v}\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \leq \|\mathbf{v}\|^{\frac{\gamma-1}{\gamma}}_{L^1(\mathbb{R}^3;\mathbb{R}^3)} \|\mathbf{v}\|^{1-\frac{\gamma-1}{\gamma}}_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}$$

$$\leq \|\mathbf{v}\|^{\frac{\gamma-1}{\gamma}}_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \|\mathbf{v}\|^{\frac{\gamma-1}{\gamma}}_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \|\mathbf{v}\|^{1/\gamma}_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)} \leq c.$$

For the second term in (2.3.41), performing integration by parts, we have

$$\int_0^T \int_{\mathbb{R}^3} \mathrm{div}_x \mathbf{v}\cdot\partial_t \Psi \mathrm{d}x\mathrm{d}t = 0 \tag{2.3.45}$$

thanks to incompressibility condition, $\mathrm{div}_x\mathbf{v} = 0$. Regarding I_2, we have

$$\int_0^T I_2 \mathrm{d}t \leq \frac{\varepsilon}{2}\int_0^T \int_{\mathbb{R}^3} \left(S(\nabla_x \mathbf{U}) - S(\nabla_x \mathbf{u})\right) : \left(\nabla_x \mathbf{U} - \nabla_x \mathbf{u}\right) \mathrm{d}x\mathrm{d}t$$

$$+ c\varepsilon \int_0^T \int_{\mathbb{R}^3} |S(\nabla_x \mathbf{U})|^2 \mathrm{d}x\mathrm{d}t, \tag{2.3.46}$$

where we used the Young inequality and the following Korn inequality

$$\int_{\mathbb{R}^3} |\nabla_x \mathbf{U} - \nabla_x \mathbf{u}|^2 \mathrm{d}x \leq c\int_{\mathbb{R}^3} \left(S(\nabla_x \mathbf{U}) - S(\nabla_x \mathbf{u})\right) : \left(\nabla_x \mathbf{U} - \nabla_x \mathbf{u}\right) \mathrm{d}x.$$

The first term on the right-hand side of (2.3.46) can be absorbed by the second term on the left-hand side in the relation (2.3.2). For the second term on the right-hand side of (2.3.46), in view of (2.1.8) and (2.2.5), we have

$$c\varepsilon \int_0^T \int_{\mathbb{R}^3} |S(\nabla_x \mathbf{U})|^2 \mathrm{d}x\mathrm{d}t \leq c(M)\varepsilon. \tag{2.3.47}$$

Regarding the terms I_3 and I_4 we deal with the following analysis. First, we have

$$\int_{\mathbb{R}^3} \nabla_x H'(r) \cdot r\mathbf{U} \mathrm{d}x = -\int_{\mathbb{R}^3} p(r) \mathrm{div}_x \mathbf{U} \mathrm{d}x \tag{2.3.48}$$

that it will cancel with its counterpart in I_4. Next,

$$\frac{1}{\varepsilon^2} \int_0^T \int_{\mathbb{R}^3} \nabla_x H'(r) \cdot (\varrho \mathbf{u}) \, \mathrm{d}x \mathrm{d}t = \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^3} H''(r) \nabla_x s \cdot (\varrho \mathbf{u}) \, \mathrm{d}x \mathrm{d}t$$

$$= \int_0^T \int_{\mathbb{R}^3} \frac{H''(1+\varepsilon s) - H''(1)}{\varepsilon} \nabla_x s \cdot (\varrho \mathbf{u}) \, \mathrm{d}x \mathrm{d}t + \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^3} p'(1) \nabla_x s \cdot (\varrho \mathbf{u}) \, \mathrm{d}x \mathrm{d}t. \tag{2.3.49}$$

Observing that

$$\frac{H''(1+\varepsilon s) - H''(1)}{\varepsilon} = H'''(\xi) s, \quad \xi \in (1, 1+\varepsilon s),$$

$$\left| \frac{H''(1+\varepsilon s) - H''(1)}{\varepsilon} \right| \leq cs,$$

the first term on the right-hand side of (2.3.49) can be estimated in the following way

$$\int_0^T \int_{\mathbb{R}^3} \frac{H''(1+\varepsilon s) - H''(1)}{\varepsilon} \nabla_x s \cdot (\varrho \mathbf{u}) \, \mathrm{d}x \mathrm{d}t$$

$$\leq c \int_0^T \|s\|_{L^\infty} \|\nabla_x s\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)} \|\varrho \mathbf{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}t$$

$$\leq c(M) \left(\gamma (\varepsilon + T) \left(\frac{\varepsilon + T}{\varepsilon} \right)^{-1/\gamma} - \varepsilon \gamma \right). \tag{2.3.50}$$

For the second integral on the right-hand side, using the acoustic equation (2.2.1), we get

$$\frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^3} p'(1) \nabla_x s \cdot (\varrho \mathbf{u}) \, \mathrm{d}x \mathrm{d}t$$

$$= -\int_0^T \int_{\mathbb{R}^3} (\varrho \mathbf{u}) \cdot \partial_t \nabla_x \Psi \mathrm{d}x \mathrm{d}t \tag{2.3.51}$$

that it will cancel with its counterpart in (2.3.40). Now, we write

$$\frac{1}{\varepsilon^2} \int_0^T \int_{\mathbb{R}^3} \left[(r - \varrho) \, \partial_t H'(r) - p(\varrho) \mathrm{div}_x \mathbf{U} \right] \mathrm{d}x \mathrm{d}t$$

$$= \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^3} (r - \varrho) \, H''(r) \partial_t s \mathrm{d}x \mathrm{d}t$$

$$-\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^3} p(\varrho)\triangle\Psi \mathrm{d}x\mathrm{d}t$$

$$=\int_0^T\int_{\mathbb{R}^3}\frac{(1-\varrho)}{\varepsilon}H''(r)\partial_t s\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^3} sH''(r)\partial_t s\mathrm{d}x\mathrm{d}t$$

$$-\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^3} p(\varrho)\triangle\Psi \mathrm{d}x\mathrm{d}t. \tag{2.3.52}$$

The last term on the right-hand side can be split as follows

$$-\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^3} p(\varrho)\triangle\Psi \mathrm{d}x\mathrm{d}t$$

$$=-\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^3}\left[p(\varrho)-p(1)\right]\triangle\Psi \mathrm{d}x\mathrm{d}t$$

$$-\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^3} p(1)\triangle\Psi \mathrm{d}x\mathrm{d}t. \tag{2.3.53}$$

Using integration by parts, we have

$$-\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^3}\nabla_x p(1)\nabla_x\Psi \mathrm{d}x\mathrm{d}t=0. \tag{2.3.54}$$

Now, we have

$$-\frac{1}{\varepsilon^2}\int_0^T\int_{\mathbb{R}^3}\left[p(\varrho)-p(1)\right]\triangle\Psi \mathrm{d}x\mathrm{d}t$$

$$=-\int_0^T\int_{\mathbb{R}^3}\frac{\left[p(\varrho)-p'(1)(\varrho-1)-p(1)\right]}{\varepsilon^2}\triangle\Psi \mathrm{d}x\mathrm{d}t$$

$$-\int_0^T\int_{\mathbb{R}^3}\frac{p'(1)(\varrho-1)}{\varepsilon^2}\triangle\Psi \mathrm{d}x\mathrm{d}t. \tag{2.3.55}$$

Then, the following estimates hold

$$\left|\int_0^T\int_{\mathbb{R}^3}\frac{\left[p(\varrho)-p'(1)(\varrho-1)-p(1)\right]}{\varepsilon^2}\triangle\Psi \mathrm{d}x\mathrm{d}t\right|\le c\int_0^T\mathcal{E}\left\|\triangle\Psi\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t. \tag{2.3.56}$$

Now, we have

$$\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}\varrho\partial_t\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t$$

$$=\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}(\varrho-1)\,\partial_t\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t+\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}\partial_t\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t$$

$$=\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}(\varrho-1)\,\partial_t\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t+\left[\frac{1}{2}\int_{\mathbb{R}^3}\left|\nabla_x\Psi\right|^2\mathrm{d}x\right]_{t=0}^{t=T}, \tag{2.3.57}$$

where, using (2.2.1) in the first term on the right-hand side, we have

$$\frac{\varepsilon}{2}\int_0^T\int_{\mathbb{R}^3}\frac{(\varrho-1)}{\varepsilon}\partial_t\left|\nabla_x\Psi\right|^2\mathrm{d}x\mathrm{d}t=a\int_0^T\int_{\mathbb{R}^3}\frac{(\varrho-1)}{\varepsilon}\nabla_x\Psi\cdot\nabla_x s\mathrm{d}x\mathrm{d}t. \quad (2.3.58)$$

Now, using (2.2.6), (2.3.16) and (2.3.17) in (2.3.58), we have

$$\int_0^T\int_{\mathbb{R}^3}\frac{(\varrho-1)}{\varepsilon}\nabla_x\Psi\cdot\nabla_x s\mathrm{d}x\mathrm{d}t$$

$$\leq\int_0^T\left\|[\frac{(\varrho-1)}{\varepsilon}]1_{\{|\varrho-1|<1\}}\right\|_{L^2(\mathbb{R}^3)}\|\nabla_x\Psi\|_{L^2(\mathbb{R}^3)}\|\nabla_x s\|_{L^\infty(\mathbb{R}^3)}\mathrm{d}t$$

$$\leq c(M)\varepsilon\left(\log(\varepsilon+T)-\log(\varepsilon)\right) \quad (2.3.59)$$

and

$$\int_0^T\int_{\mathbb{R}^3}\frac{(\varrho-1)}{\varepsilon}\nabla_x\Psi\cdot\nabla_x s\mathrm{d}x\mathrm{d}t$$

$$\leq\int_0^T\left\|[\frac{(\varrho-1)}{\varepsilon}]1_{\{|\varrho-1|\geq1\}}\right\|_{L^\gamma(\mathbb{R}^3)}\|\nabla_x\Psi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3)}\|\nabla_x s\|_{L^\infty(\mathbb{R}^3)}\mathrm{d}t$$

$$\leq c(M)\varepsilon^{2/\gamma}\left(\gamma\left(\frac{\varepsilon+T}{\varepsilon}\right)^{-1/\gamma}-\gamma\right), \quad (2.3.60)$$

where we have used the following interpolation inequality for $\nabla_x\Psi$

$$\|\nabla_x\Psi\|_{L^{\frac{\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)}\leq\|\nabla_x\Psi\|^{\frac{\gamma-1}{\gamma}}_{L^1(\mathbb{R}^3;\mathbb{R}^3)}\|\nabla_x\Psi\|^{1-\frac{\gamma-1}{\gamma}}_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}$$

$$\leq\|\nabla_x\Psi\|^{\frac{\gamma-1}{\gamma}}_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\|\nabla_x\Psi\|^{\frac{\gamma-1}{\gamma}}_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\|\nabla_x\Psi\|^{1/\gamma}_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\leq c(M)\|\nabla_x\Psi\|^{1/\gamma}_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}.$$

Now, collecting the remained terms, we write

$$\int_0^T\int_{\mathbb{R}^3}\frac{(1-\varrho)}{\varepsilon}H''(r)\partial_t s\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{R}^3}sH''(r)\partial_t s\mathrm{d}x\mathrm{d}t$$

$$-\int_0^T\int_{\mathbb{R}^3}\frac{p'(1)(\varrho-1)}{\varepsilon^2}\triangle\Psi\mathrm{d}x\mathrm{d}t. \quad (2.3.61)$$

For the first integrals in (2.3.61), it is possible to show (see Feireisl, Nečasová and Sun [47]) that

$$\left|\int_0^T\int_{\mathbb{R}^3}\frac{(1-\varrho)}{\varepsilon}H''(r)\partial_t s\mathrm{d}x\mathrm{d}t\right|$$

$$\leq\int_0^T\int_{\mathbb{R}^3}\frac{(\varrho-1)}{\varepsilon^2}p'(1)\triangle\Psi\mathrm{d}x\mathrm{d}t+c(M)\int_0^T\mathcal{E}\|\triangle\Psi\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t, \quad (2.3.62)$$

where the first term on the right hand side of the inequality it will cancel with its counterpart in (2.3.61). While, for the second integral in (2.3.61), we have

$$\left|\int_0^T\int_{\mathbb{R}^3} sH''(r)\partial_t s\mathrm{d}x\mathrm{d}t\right| \leq p'(1)\left[\frac{1}{2}\int_{\mathbb{R}^3} s^2\mathrm{d}x\right]_{t=0}^{t=T} + c(M)\int_0^T \mathcal{E}\left\|\triangle\Psi\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t. \tag{2.3.63}$$

From (2.3.56), (2.3.62), (2.3.63) we need to estimate the following term

$$\int_0^T \mathcal{E}\left\|\triangle\Psi\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t \leq c(M)\int_0^T \mathcal{E}\mathrm{d}t. \tag{2.3.64}$$

Finally, regarding I_5, we have

$$\int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{u}\times\omega)\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{v}\times\omega)\cdot(\mathbf{v}-\mathbf{u})\mathrm{d}x\mathrm{d}t$$

$$= \int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{u}\times\omega)\cdot\mathbf{v}\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{v}\times\omega)\cdot\mathbf{u}\mathrm{d}x\mathrm{d}t$$

$$= \int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{u}\times\omega)\cdot\mathbf{v}\mathrm{d}x\mathrm{d}t - \int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{u}\times\omega)\cdot\mathbf{v}\mathrm{d}x\mathrm{d}t = 0 \tag{2.3.65}$$

and, thanks to (2.1.8), (2.2.6), (2.3.9) and (2.3.13), we have

$$\int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{u}\times\omega)\cdot\nabla_x\Psi\mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^T \left\|\varrho\mathbf{u}\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^3;\mathbb{R}^3)}\left\|\nabla_x\Psi\right\|_{L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t$$

$$\leq c(M)\left(\frac{\gamma\left(\varepsilon+T\right)\left(\frac{\varepsilon+T}{\varepsilon}\right)^{-1/\gamma}}{\gamma-1} - \frac{\gamma\varepsilon}{\gamma-1}\right) \tag{2.3.66}$$

and

$$\int_0^T\int_{\mathbb{R}^3}(\varrho\mathbf{v}\times\omega)\cdot\nabla_x\Psi\mathrm{d}x\mathrm{d}t$$

$$\leq \int_0^T \left\|\varrho\right\|_{L^\gamma(\mathbb{R}^3)}\left\|\mathbf{v}\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}^{\frac{\gamma}{\gamma-1}}\left\|\nabla_x\Psi\right\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^3)}\mathrm{d}t$$

$$\leq c(M)\varepsilon\left(\log(\varepsilon+T)-\log(\varepsilon)\right). \tag{2.3.67}$$

Combining the previous estimates and letting $\varepsilon\to 0$ we can rewrite (2.3.2) as

$$[\mathcal{E}(\varrho,\mathbf{u}\mid r,\mathbf{U})](T) \leq [\mathcal{E}(\varrho,\mathbf{u}\mid r,\mathbf{U})](0) + c(M)\int_0^T \mathcal{E}\mathrm{d}t \tag{2.3.68}$$

In virtue of the integral form of the Gronwall inequality, we have

$$[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](T) \leq ([\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](0)) \left(1 + c(M)Te^{c(M)T}\right) \quad \text{for} \quad t \in [0, T]\,, \tag{2.3.69}$$

where the quantity $\left(1 + c(M)Te^{c(M)T}\right)$ is bounded for fixed $t \in [0, T]$. Theorem 14 is proved and, consequently, Corollary 16.

2.4 Conclusions

The problem we faced above has focused on the inviscid incompressible limit for a compressible barotropic fluid in a "fast" rotating frame. The problem was analyzed in the whole space $\mathbb{R}^3$. However, a possible extension for a fluid in a bounded domain can give light to the interesting analysis of the formation of the boundary layers. Moreover, it is not excluded that the "fast" rotating frame can develop a particular phenomenology in the fluid that can be of some interest, from the mathematical view point, in the analysis of other kind of models in bounded domains or in the whole space.

Chapter 3

Dimension reduction for compressible heat conducting fluids

We consider the scaled compressible Navier-Stokes-Fourier-Poisson system describing the motion of an heat conducting fluid in a rotating frame confined in a straight layer $\Omega_\epsilon = \omega \times (0, \epsilon)$ where ω is a two-dimensional domain and in the presence of the gravity force already mentioned in Introduction. The continuity equation reads

$$\partial_t \varrho + \mathrm{div}_\epsilon (\varrho \mathbf{u}) = 0, \tag{3.0.1}$$

the momentum equation is

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_\epsilon (\varrho \mathbf{u} \otimes \mathbf{u}) + \varrho \mathbf{u} \times \boldsymbol{\chi} + \nabla_\epsilon p(\varrho, \vartheta)$$

$$= \mathrm{div}_\epsilon S (\vartheta, \nabla_\epsilon \mathbf{u}) + \epsilon^{-2\beta} \varrho \nabla_\epsilon \phi + \varrho \nabla_\epsilon |x \times \boldsymbol{\chi}|^2 , \tag{3.0.2}$$

with the stress tensor given by the following relation

$$S (\vartheta, \nabla_\epsilon \mathbf{u}) = \mu (\vartheta) \left(\nabla_\epsilon \mathbf{u} + \nabla_\epsilon^t \mathbf{u} - \frac{2}{3} \mathrm{div}_\epsilon \mathbf{u} \mathbf{I} \right) + \eta (\vartheta) \mathrm{div}_\epsilon \mathbf{u} \mathbf{I}. \tag{3.0.3}$$

The entropy equation is

$$\partial_t (\varrho s (\varrho, \vartheta)) + \mathrm{div}_\epsilon (\varrho s (\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_\epsilon \left(\frac{\mathbf{q} (\vartheta, \nabla_\epsilon \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(S (\vartheta, \nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{u} - \frac{\mathbf{q} (\vartheta, \nabla_\epsilon \vartheta) \cdot \nabla_\epsilon \vartheta}{\vartheta} \right), \tag{3.0.4}$$

with

$$\mathbf{q} = -\kappa (\vartheta) \nabla_\epsilon \vartheta. \tag{3.0.5}$$

The gravitational force is given by the following relation

$$\nabla_\epsilon \phi\left(t,x\right) = \epsilon \int_{\Omega_\epsilon} \alpha \varrho(t,\xi) \frac{\left(x_1 - \xi_1, x_2 - \xi_2, \epsilon\left(x_3 - \xi_3\right)\right)}{\left(\left|x_h - \xi_h\right|^2 + \epsilon^2 \left|x_3 - \xi_3\right|^2\right)^{3/2}} d\xi$$

$$+ \int_{\mathbb{R}^3} \left(1 - \alpha\right) g(y) \frac{\left(x_1 - y_1, x_2 - y_2, \epsilon\left(x_3 - y_3\right)\right)}{\left(\left|x_h - y_h\right|^2 + \epsilon^2 \left|x_3 - y_3\right|^2\right)^{3/2}} dy$$

$$= \epsilon \alpha \mathbf{\Phi}_1 + \left(1 - \alpha\right) \mathbf{\Phi}_2. \tag{3.0.6}$$

The system (3.0.1) - (3.0.4) is completed with the initial conditions

$$\varrho\left(x,0\right) = \varrho_0\left(x\right), \ \mathbf{u}\left(x,0\right) = \mathbf{u}_0\left(x\right), \ \vartheta\left(x,0\right) = \vartheta_0\left(x\right), \ x \in \Omega \tag{3.0.7}$$

and the boundary conditions

$$\mathbf{u}|_{\partial\omega\times(0,1)} = 0, \tag{3.0.8}$$

$$\mathbf{u}\cdot\mathbf{n}|_{\omega\times\{0,1\}} = 0, \ \ [S\cdot\mathbf{n}]\times\mathbf{n}|_{\omega\times\{0,1\}} = \mathbf{0}, \tag{3.0.9}$$

$$\nabla\vartheta\cdot\mathbf{n}|_{\omega\times\{0,1\}} = 0. \tag{3.0.10}$$

$$\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0. \tag{3.0.11}$$

Remark 17. The first condition in (3.0.9) can be written as

$$u_3 = 0 \text{ on } \omega \times \{0,1\}.$$

Remark 18. We consider the no-slip boundary condition holds on the boundary $\omega \times (0,1)$ (on the lateral part of the domain) and the slip boundary condition on the other parts of the boundary $\omega \times \{0,1\}$ (the top and the bottom part of the layer).

Remark 19. We would like to emphasize that we imposed a slip condition on the boundary $\omega \times \{0,\epsilon\}$ in order to avoid difficulties in passing to the dimension reduction limit.

As already mentioned in the Introduction, we will consider two cases: $\beta = 1/2$ and $\beta = 0$. In the first case we will take $\alpha = 1$, assuming only the self-gravitation. In the second case, we will take $\alpha = 0$, assuming only the gravitational force due to external effects.

We want to show that the weak solution of the Navier-Stokes-Fourier-Poisson system converges to the classical solution of the corresponding two-dimensional system in which the continuity equation reads

$$\partial_t r + \operatorname{div}_h\left(r\mathbf{w}\right) = 0, \tag{3.0.12}$$

the momentum equation is

$$r\partial_t \mathbf{w} + r\mathbf{w}\cdot\nabla_h\mathbf{w} + \nabla_h p(r,\Theta) + r\left(\mathbf{w}\times\boldsymbol{\chi}\right)$$

$$= \mathrm{div}_h S(\Theta, \nabla_h \mathbf{w}) + r\nabla_h \phi_h + r\nabla_h |x \times \boldsymbol{\chi}|^2, \tag{3.0.13}$$

with the stress tensor given by the following relation

$$S_h(\Theta, \nabla_h \mathbf{w}) = \mu \left(\nabla_h \mathbf{w} + \nabla_h^t \mathbf{w} - \mathrm{div}_h \mathbf{w}\right) + \left(\eta + \frac{\mu}{3}\right) \mathrm{div}_h \mathbf{w} \mathrm{I}_h. \tag{3.0.14}$$

where I_h is the unit tensor in $\mathbb{R}^{2\times 2}$ in the domain $(0,T) \times \omega$. The entropy equation is

$$r\partial_t s + r\mathbf{w} \cdot \nabla_h s + \mathrm{div}_h \left(\frac{\mathbf{q}_h(\Theta, \nabla_h \Theta)}{\Theta}\right)$$

$$= \frac{1}{\Theta}\left(S_h(\Theta, \nabla_h \mathbf{w}) : \nabla_h \mathbf{w} - \frac{\mathbf{q}_h(\Theta, \nabla_h \Theta) \cdot \nabla_h \Theta}{\Theta}\right), \tag{3.0.15}$$

with

$$\mathbf{q}_h(\Theta, \nabla_h \Theta) = -\kappa(\Theta) \nabla_h \Theta. \tag{3.0.16}$$

Above,

$$\phi_h(t, x_h) = G \int_\omega \frac{r(t, y_h)}{|x_h - y_h|} dy_h \quad \text{for} \quad \alpha = 1 \tag{3.0.17}$$

and

$$\phi_h(t, x_h) = G \int_{\mathbb{R}^3} \frac{g(y)}{\sqrt{|x_h - y_h|^2 + y_3^2}} dy \quad \text{for} \quad \alpha = 0. \tag{3.0.18}$$

Moreover, $\mathbf{q}_h \cdot \mathbf{n}|_{\partial\omega \times (0,T)} = 0$ and $\mathbf{w}|_{\partial\omega \times (0,T)} = \mathbf{0}$.

3.1 Thermodynamics

The physical properties of heat conduction flows are reflected through various relations which are expressed as typically non-linear functions relating the pressure $p(\varrho, \vartheta)$, the internal energy $e(\varrho, \vartheta)$, the entropy $s(\varrho, \vartheta)$ to the macroscopic variables ϱ, $\mathbf{u}$ and ϑ. The following discussion is based on the general existence theory for the Navier-Stokes-Fourier system developed in [41].

According with the fundamental principles of thermodynamics, the internal energy e is related to the pressure p and the entropy s through Gibbs' relation

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right), \tag{3.1.1}$$

where D denotes the differential with respect to the state variables ϱ, ϑ. We consider the pressure p and the internal energy e in the form

$$p(\varrho, \vartheta) = p_1(\varrho, \vartheta) + \frac{a}{3}\vartheta^4, \tag{3.1.2}$$

$$e(\varrho, \vartheta) = e_1(\varrho, \vartheta) + \frac{a}{3}\frac{\vartheta^4}{\varrho} \tag{3.1.3}$$

where

$$p_1(\varrho,\vartheta) = (\gamma - 1)\,\varrho e(\varrho,\vartheta) \tag{3.1.4}$$

with $\gamma > 1$. The component $\frac{a}{3}\vartheta^4$ represents the effect of "equilibrium" radiation pressure (see [28] for the motivations). Gibbs' equation (3.1.1) can be equivalently written in the form of Maxwell's relation as follows

$$\frac{\partial e(\varrho,\vartheta)}{\partial \varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta\frac{\partial p(\varrho,\vartheta)}{\partial \vartheta}\right). \tag{3.1.5}$$

It follows, under some regularity assumptions on the functions p_1 and e_1, that

$$p_1(\varrho,\vartheta) = \vartheta^{\frac{\gamma}{\gamma-1}} P\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right) \tag{3.1.6}$$

where $P : [0,\infty) \to [0,\infty)$ is a given function with the following properties

$$P \in C^1([0,\infty)) \cap C^2((0,\infty)), \quad P(0) = 0, \quad P'(Z) > 0 \quad \text{for} \quad \text{all} \quad Z \geq 0, \tag{3.1.7}$$

$$0 < \frac{\gamma P(Z) - P'(Z) Z}{Z} \leq c < \infty \quad \text{for} \quad \text{all} \quad Z > 0, \quad \lim_{Z\to\infty} \frac{P(Z)}{Z^\gamma} = p_\infty > 0. \tag{3.1.8}$$

Condition (3.1.8) reflects the fact that the specific heat at constant volume is strictly positive and uniformly bounded. Recalling the Maxwell's relation (3.1.5), for the internal energy we have

$$e_1(\varrho,\vartheta) = \frac{1}{\gamma - 1}\frac{\vartheta^{\frac{\gamma}{\gamma-1}}}{\varrho} P\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right). \tag{3.1.9}$$

Due to the form of the pressure and the internal energy, the entropy is given by

$$s(\varrho,\vartheta) = s_1(\varrho,\vartheta) + \frac{4}{3}a\frac{\vartheta^3}{\varrho}, \tag{3.1.10}$$

with

$$s_1(\varrho,\vartheta) = M\left(\frac{\varrho}{\vartheta^{\frac{1}{\gamma-1}}}\right), \quad M'(Z) = -\frac{1}{\gamma-1}\frac{\gamma P(Z) - P'(Z) Z}{Z^2} < 0,$$

$$\lim_{Z\to\infty} M(Z) = 0. \tag{3.1.11}$$

Note, that it is possible to show that

$$s_1(\varrho,\vartheta) \leq c(1 + |\ln \varrho|) \tag{3.1.12}$$

in the set $\varrho \in (0,\infty)$, $\vartheta \in (0,1)$, and

$$s_1(\varrho,\vartheta) \leq c(1 + |\ln \varrho| + \ln \vartheta) \tag{3.1.13}$$

in the set $\varrho \in (0,\infty)$, $\vartheta \in (1,\infty)$.

The coefficients μ, η and κ are continuously differentiable functions of the temperature, such that

$$0 < c_1 (1 + \vartheta) \leq \mu(\vartheta), \quad \mu'(\vartheta) < c_2, \quad 0 \leq \eta(\vartheta) \leq c_3 (1 + \vartheta), \tag{3.1.14}$$

$$0 < c_4 \left(1 + \vartheta^3\right) \leq \kappa(\vartheta) \leq c_5 \left(1 + \vartheta^3\right) \tag{3.1.15}$$

for any $\vartheta > 0$. For the sake of the simplicity, we consider the particular case

$$\mu(\vartheta) = \mu_0 + \mu_1 \vartheta, \quad \mu_0, \mu_1 > 0, \quad \eta \equiv 0 \tag{3.1.16}$$

and

$$\kappa(\vartheta) = \kappa_0 + \kappa_2 \vartheta^2 + \kappa_3 \vartheta^3, \quad \kappa_i > 0, \quad i = 0, 2, 3. \tag{3.1.17}$$

3.2 Weak and classical solutions

In the following, we introduce the definition of weak solutions for the compressible Navier-Stokes-Fourier-Poisson system (3.0.1) - (3.0.4) and we discuss the global in time existence. Then, we discuss the global existence of the classical solution of the two-dimensional heat conducting system (3.0.12) - (3.0.18).

3.2.1 Weak solutions

To present the weak formulation, we consider the functional space

$$W_{0,n}^{1,2}\left(\Omega;\mathbb{R}^3\right) = \left\{ \mathbf{u} \in W^{1,2}\left(\Omega;\mathbb{R}^3\right); \mathbf{u}\cdot\mathbf{n}|_{\omega\times\{0,1\}} = 0, \, \mathbf{u}|_{\partial\omega\times(0,1)} = \mathbf{0} \right\}.$$

Definition 20. (Weak solution) We say that $[\varrho, \mathbf{u}, \vartheta]$ is a weak solution of the system (3.0.1) - (3.0.4) if

$$\varrho \geq 0, \quad \vartheta > 0, \quad \text{a.e. in} \quad (0,T)\times\Omega,$$

$$\varrho \in C_{weak}\left((0,T), L^{\gamma}(\Omega)\right), \quad \varrho\mathbf{u} \in C_{weak}\left((0,T), L^{\frac{2\gamma}{\gamma+1}}\left(\Omega;\mathbb{R}^3\right)\right),$$

$$\mathbf{u} \in L^2\left((0,T), W_{0,n}^{1,2}\left(\Omega;\mathbb{R}^3\right)\right),$$

$$\vartheta \in L^{\infty}\left((0,T), L^4(\Omega)\right) \cap L^2\left((0,T), W^{1,2}(\Omega)\right),$$

and if $[\varrho, \mathbf{u}, \vartheta]$ satisfy the following integral identities:

$$\int_0^T \int_\Omega (\varrho + b(\varrho))\,\partial_t\varphi + (\varrho + b(\varrho))\,\mathbf{u}\cdot\nabla_\epsilon\varphi + \left(b(\varrho) - b'(\varrho)\,\varrho\right)\mathrm{div}_\epsilon\mathbf{u}\varphi \,\mathrm{d}x\mathrm{d}t$$

$$= -\int_\Omega (\varrho_0 + b(\varrho_0))\,\varphi(0,\cdot)\,\mathrm{d}x \tag{3.2.1}$$

for any $\varphi \in C_c^{\infty}\left([0,T)\times\overline{\Omega}\right)$ and $b \in C^{\infty}([0,\infty))$, $b' \in C_c^{\infty}([0,\infty))$, where (3.2.1) includes as well the initial condition $\varrho(x,0) = \varrho_0(x)$;

$$\int_0^T \int_\Omega \varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_\epsilon\boldsymbol{\varphi} + (\varrho\mathbf{u}\times\boldsymbol{\chi})\cdot\boldsymbol{\varphi} + p(\varrho,\vartheta)\,\mathrm{div}_\epsilon\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t$$

$$-\int_0^T \int_\Omega S(\vartheta,\nabla_\epsilon\mathbf{u}) : \nabla_\epsilon\boldsymbol{\varphi} - \epsilon^{-2\beta}\varrho\nabla_\epsilon\phi\cdot\boldsymbol{\varphi} - \varrho\nabla_\epsilon|x\times\boldsymbol{\chi}|^2\cdot\boldsymbol{\varphi}\,\mathrm{d}x\mathrm{d}t$$

$$= -\int_\Omega \varrho_0\mathbf{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x \tag{3.2.2}$$

for any $\boldsymbol{\varphi}\in C_c^\infty\left([0,T)\times\overline{\Omega};\mathbb{R}^3\right)$, $\boldsymbol{\varphi}|_{[0,T]\times\partial\omega\times(0,1)} = \mathbf{0}$, $\varphi_3|_{[0,T]\times\partial\omega\times\{0,1\}} = 0$, where (3.2.2) includes as well the initial condition $\varrho\mathbf{u}(x,0) = \varrho_0\mathbf{u}_0(x)$;

$$\int_0^T \int_\Omega \varrho s(\varrho,\vartheta)\,\partial_t\varphi + \varrho s(\varrho,\vartheta)\,\mathbf{u}\cdot\nabla_\epsilon\varphi + \frac{\mathbf{q}(\vartheta,\nabla_\epsilon\vartheta)}{\vartheta}\cdot\nabla_\epsilon\varphi\,\mathrm{d}x\mathrm{d}t$$

$$\leq -\int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\,\varphi(0,\cdot)\,\mathrm{d}x$$

$$-\int_0^T \int_\Omega \frac{1}{\vartheta}\left(S(\vartheta,\nabla_\epsilon\mathbf{u}) : \nabla_\epsilon\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_\epsilon\vartheta)\cdot\nabla_\epsilon\vartheta}{\vartheta}\right)\varphi\,\mathrm{d}x\mathrm{d}t \tag{3.2.3}$$

for any $\varphi\in C_c^\infty\left([0,T)\times\overline{\Omega}\right)$, $\varphi\geq 0$; together with the total energy balance

$$\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right)(t,\cdot)\,\mathrm{d}x$$

$$= \int_\Omega \left(\frac{1}{2\varrho_0}|\varrho_0\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0)\right)\mathrm{d}x + \int_0^T \int_\Omega \varrho\boldsymbol{\Phi}_j\cdot\mathbf{u} + \varrho\nabla_\epsilon|x\times\boldsymbol{\chi}|^2\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t \tag{3.2.4}$$

with $j=1,2$, and the integral representation of the gravitational force (3.0.6).

Remark 21. In the weak formulation above, we replace the weak formulation of the continuity equation (3.0.1) with its (weak) renormalized version in the sense of DiPerna and Lions [23].

Remark 22. The concept of weak solution to the Navier-Stokes-Fourier system based on the Second Law of thermodynamic presented above was introduce in [27]. In order to compensate the lack of information resulting from the entropy inequality, the system is supplemented by the total energy balance. Under these circumstances, it can be show (see [41]) that any weak solution of (3.0.1) - (3.0.4) that is sufficiently smooth satisfies the entropy equality (3.0.4).

Remark 23. Concerning the weak formulation introduced above, there are at least two alternative ways by which to replace the entropy balance (3.0.4), namely the total energy balance

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta)\right) + \mathrm{div}_\epsilon\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) + p(\varrho,\vartheta)\right)\mathbf{u}\right] + \mathrm{div}_\epsilon\mathbf{q}(\vartheta,\nabla_\epsilon\vartheta)$$

$$= \mathrm{div}_\epsilon \left[S\left(\vartheta, \nabla_\epsilon \mathbf{u}\right) \mathbf{u} \right] \tag{3.2.5}$$

or by the internal energy balance

$$\partial_t \left(\varrho e\left(\varrho, \vartheta\right)\right) + \mathrm{div}_\epsilon \left(\varrho e\left(\varrho, \vartheta\right) \mathbf{u}\right) + \mathrm{div}_\epsilon \mathbf{q}\left(\vartheta, \nabla_\epsilon \vartheta\right)$$

$$= S\left(\vartheta, \nabla_\epsilon \mathbf{u}\right) : \nabla_\epsilon \mathbf{u} - \mathrm{div}_\epsilon p\left(\varrho, \vartheta\right) \mathbf{u}. \tag{3.2.6}$$

Although relations (3.2.5) and (3.2.6) are equivalent to (3.0.4) for classical solutions, this is, in general, not the case in the framework of weak solutions. Moreover, as mentioned in [43], it is precisely the entropy balance (3.0.4) that gives rise, in combination with the total energy balance, to the relative energy inequality yielding the weak-strong uniqueness property and the convergence we are asking for.

It should also be noted that the term $S\left(\vartheta, \nabla_\epsilon \mathbf{u}\right) \mathbf{u}$ in the total energy balance (3.2.5) is not controlled on the (hypothetical) vacuum zones of vanishing density. Replacing (3.2.5) by the internal energy equation (3.2.6), dividing (3.2.6) on $1/\vartheta$ and using Maxwell's relation (3.1.5), we may rewrite (3.2.6) as the entropy equation (3.0.4) we already introduced in the beginning of the chapter.

The next Theorem concerns with the global-in-time existence of weak solutions for the Navier-Stokes-Fourier-Poisson system (3.0.1) - (3.0.4).

Theorem 24. *Let E_0 and S_0 be non-negative constants. Suppose the thermodynamic functions p, e, s satisfy relations (3.1.2) - (3.1.11), the transport coefficients μ, η, κ comply with (3.1.16) - (3.1.17). Let $\gamma > 3/2$ if $\alpha = 0$ or $\gamma > 12/7$ if $\alpha = 1$. Let g be such that $g \in L^p\left(\mathbb{R}^3\right)$ for $p = 1$ if $\gamma > 6$ and $p = 6\gamma/\left(7\gamma - 6\right)$ for $3/2 < \gamma \leq 6$. Suppose the initial data satisfy*

$$\int_\Omega \left(\frac{1}{2} \varrho \left|\mathbf{u}\right|^2 + \varrho e(\varrho, \vartheta) \right) (0, \cdot)\, dx \equiv \int_\Omega \left(\frac{1}{2\varrho_0} \left|\varrho_0 \mathbf{u}_0\right|^2 + \varrho_0 e(\varrho_0, \vartheta_0) \right) dx \leq E_0,$$

$$\int_\Omega \varrho s(\varrho, \vartheta)\, (0, \cdot)\, dx \equiv \int_\Omega \varrho_0 s(\varrho_0, \vartheta_0) dx \geq S_0. \tag{3.2.7}$$

Then, the system (3.0.1) - (3.0.4) with boundary conditions (3.0.8) - (3.0.10) admits at least one weak solution in the sense of Definition 20.

Proof. The existence of weak solutions to the above problem can be deduced from the works of Feireisl et al. [29], [35], [40] and [45]. In fact, we fix $\epsilon > 0$, we construct a weak solution in Ω_ϵ and then we rescale the solution. □

3.2.2 Classical solutions

The next Theorem concerns with the existence of classical solution for the two-dimensional heat conducting system (3.0.12) - (3.0.18). From classical results of Matsumura and Nishida [76], we know that the target system admits a unique global strong solution provided the initial data are close to a stationary solution. Another possible result is the existence of local-in-time smooth solution (see for example Tani [106]). More precisely

Theorem 25. *Let E be a given positive constant. Suppose that* $p \in C^2\left((0,\infty)^2\right)$, $\mu,\eta,\kappa \in C^1(0,\infty)$ *and that*

$$r_0 \in W^{2,2}(\omega),\ \inf_{\omega} r_0 > 0,\ \mathbf{w}_0 \in W^{3,2}\left(\omega;\mathbb{R}^2\right) \cap W_0^{1,2}\left(\omega;\mathbb{R}^2\right),$$

$$\Theta_0 \in W^{3,2}(\omega),\ \inf_{\omega} \Theta_0 > 0. \tag{3.2.8}$$

Moreover, assume that the following condition holds

$$\left.\frac{1}{r_0}\left(\nabla_h p(r_0,\Theta_0) + r_0(\mathbf{w}_0 \times \boldsymbol{\chi}) - div_h S(\Theta_0, \nabla_h \mathbf{w}_0) - r_0 \nabla_h \phi_h - r_0 \nabla_h |x \times \boldsymbol{\chi}|^2\right)\right|_{\partial\omega} = \mathbf{0}. \tag{3.2.9}$$

Then:

1) (Local solution) There exists a positive parameter T_*, such that $[r,\mathbf{w},\Theta]$ is the unique classical solution to the problem (3.0.12) - (3.0.18) with the boundary conditions

$$\mathbf{w}|_{\partial\omega} = \mathbf{0}, \tag{3.2.10}$$

$$\left.\frac{\partial\Theta}{\partial\mathbf{n}}\right|_{\partial\omega} = 0 \tag{3.2.11}$$

and the initial conditions $[r_0,\mathbf{w}_0,\Theta_0]$ in $(0,T)\times\omega$ for any $T < T_*$ such that

$$r \in C\left([0,T];W^{3,2}(\omega)\right) \cap C^1\left([0,T];W^{2,2}(\omega)\right), \tag{3.2.12}$$

$$\mathbf{w} \in C\left([0,T];W^{3,2}\left(\omega;\mathbb{R}^2\right)\right) \cap C^1\left([0,T];W^{1,2}\left(\omega;\mathbb{R}^2\right)\right), \tag{3.2.13}$$

$$\Theta \in C\left([0,T];W^{3,2}(\omega)\right) \cap C^1\left([0,T];W^{1,2}(\omega)\right). \tag{3.2.14}$$

2) (Global solution) Let $[r_0,\mathbf{w}_0,\Theta_0]$ and $\boldsymbol{\chi}$ be such that for a sufficiently small $\varepsilon > 0$

$$\left\|r_0 - \overline{r}, \mathbf{w}_0, \Theta_0 - \overline{\Theta}\right\|_{3,2} + |\boldsymbol{\chi}| \le \varepsilon, \tag{3.2.15}$$

where $\left[\overline{r},\mathbf{0},\overline{\Theta}\right]$ is a stationary solution to (3.0.12) - (3.0.18) with the boundary condition

$$\left.\frac{\partial\overline{\Theta}}{\partial\mathbf{n}}\right|_{\partial\omega} = 0. \tag{3.2.16}$$

Then, for any $T_* < +\infty$ there exists a global unique strong solution to to (3.0.12) - (3.0.18) with the boundary condition (3.2.10) - (3.2.11) and the initial conditions $[r_0,\mathbf{w}_0,\Theta_0]$ in the class (3.2.12) - (3.2.14).

Proof. It follows from [76, Theorem 1.1] and [106] with slight modifications due to the rotation and the self-gravitation. □

3.3 Convergence analysis

For the purpose of the convergence analysis, we introduce the relative energy functional and the relative energy inequality associated to the system (3.0.1) - (3.0.4) already mentioned in the Introduction.

3.3.1 Relative energy inequality

The relative energy functional associated to the Navier-Stokes-Fourier-Poisson system (3.0.1) - (3.0.4) is given by the following relation

$$\mathcal{I}(\varrho, \mathbf{u}, \vartheta \mid \widetilde{r}, \widetilde{\mathbf{w}}, \widetilde{\Theta}) = \int_\Omega \left(\frac{1}{2} \varrho \left| \mathbf{u} - \widetilde{\mathbf{w}} \right|^2 + \mathcal{E}(\varrho, \vartheta \mid \widetilde{r}, \widetilde{\Theta}) \right) (t, \cdot) \mathrm{d}x \tag{3.3.1}$$

where for the Helmholtz potential

$$H_{\widetilde{\Theta}}(\varrho, \vartheta) = \varrho e(\varrho, \vartheta) - \widetilde{\Theta} \varrho s(\varrho, \vartheta) \tag{3.3.2}$$

we have

$$\mathcal{E}(\varrho, \vartheta \mid \widetilde{r}, \widetilde{\Theta}) = H_{\widetilde{\Theta}}(\varrho, \vartheta) - \partial_\varrho H_{\widetilde{\Theta}}(\widetilde{r}, \widetilde{\Theta})(\varrho - \widetilde{r}) - H_{\widetilde{\Theta}}(\widetilde{r}, \widetilde{\Theta}). \tag{3.3.3}$$

While, the relative energy inequality reads as follows

$$\begin{aligned} &\left[\mathcal{I}(\varrho, \mathbf{u}, \vartheta \mid \widetilde{r}, \widetilde{\mathbf{w}}, \widetilde{\Theta}) \right]_{t=0}^{t=T} \\ &+ \int_0^T \int_\Omega \frac{\widetilde{\Theta}}{\vartheta} \left(S(\vartheta, \nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_\epsilon \vartheta) \cdot \nabla_\epsilon \vartheta}{\vartheta} \right) \mathrm{d}x \mathrm{d}t \\ &\leq \int_0^T \mathcal{R}(\varrho, \mathbf{u}, \vartheta, \widetilde{r}, \widetilde{\mathbf{w}}, \widetilde{\Theta}) \mathrm{d}t \end{aligned} \tag{3.3.4}$$

where the reminder $\mathcal{R}$ is expressed as follows

$$\begin{aligned} &\mathcal{R}(\varrho, \mathbf{u}, \vartheta, \widetilde{r}, \widetilde{\mathbf{w}}, \widetilde{\Theta}) \\ &= \int_\Omega \varrho \left(\mathbf{u} - \widetilde{\mathbf{w}} \right) \cdot \nabla_\epsilon \widetilde{\mathbf{w}} \cdot \left(\widetilde{\mathbf{w}} - \mathbf{u} \right) \mathrm{d}x \\ &+ \int_\Omega \varrho \left(s(\varrho, \vartheta) - s(\widetilde{r}, \widetilde{\Theta}) \right) \cdot \left(\widetilde{\mathbf{w}} - \mathbf{u} \right) \cdot \nabla_\epsilon \widetilde{\Theta} \mathrm{d}x \\ &+ \int_\Omega \varrho \left(\partial_t \widetilde{\mathbf{w}} + \widetilde{\mathbf{w}} \cdot \nabla_\epsilon \widetilde{\mathbf{w}} \right) \cdot (\widetilde{\mathbf{w}} - \mathbf{u}) \mathrm{d}x \\ &- \int_\Omega \varrho \left(s(\varrho, \vartheta) - s(\widetilde{r}, \widetilde{\Theta}) \right) \partial_t \widetilde{\Theta} \mathrm{d}x \\ &- \int_\Omega \varrho \left(s(\varrho, \vartheta) - s(\widetilde{r}, \widetilde{\Theta}) \right) \widetilde{\mathbf{w}} \cdot \nabla_\epsilon \widetilde{\Theta} \mathrm{d}x \\ &+ \int_\Omega \left(\left(1 - \frac{\varrho}{\widetilde{r}} \right) \partial_t p(\widetilde{r}, \widetilde{\Theta}) - \frac{\varrho}{\widetilde{r}} \mathbf{u} \cdot \nabla_\epsilon p(\widetilde{r}, \widetilde{\Theta}) \right) \mathrm{d}x \end{aligned}$$

$$+\int_{\Omega}\varrho\left(\boldsymbol{\chi}\times\mathbf{u}\right)\cdot\left(\widetilde{\mathbf{w}}-\mathbf{u}\right)-\varrho\nabla_{\epsilon}\left|\boldsymbol{\chi}\times x\right|^{2}\cdot\left(\widetilde{\mathbf{w}}-\mathbf{u}\right)\mathrm{d}x$$

$$-\int_{\Omega}\left(\epsilon^{-2\beta}\varrho\nabla_{\epsilon}\phi\cdot\left(\widetilde{\mathbf{w}}-\mathbf{u}\right)+\frac{\mathbf{q}(\vartheta,\nabla_{\epsilon}\vartheta)\cdot\nabla_{\epsilon}\widetilde{\Theta}}{\vartheta}\right)\mathrm{d}x$$

$$-\int_{\Omega}p(\varrho,\vartheta)\mathrm{div}_{\epsilon}\widetilde{\mathbf{w}}+S(\vartheta,\nabla_{\epsilon}\mathbf{u}):\nabla_{\epsilon}\widetilde{\mathbf{w}}\mathrm{d}x:=I_{1}+...+I_{11}. \tag{3.3.5}$$

Here, $\widetilde{r}$, $\widetilde{\mathbf{w}}$ and $\widetilde{\Theta}$ are sufficiently smooth functions. Moreover, $\widetilde{r}$ and $\widetilde{\Theta}$ are bounded below away from zero in $[0,T]\times\Omega$, $\widetilde{\mathbf{w}}|_{\partial\omega\times(0,1)}=\mathbf{0}$ and $\widetilde{w_3}|_{\omega\times\{0,1\}}=0$. The particular choice of $\widetilde{r}$, $\widetilde{\mathbf{w}}$ and $\widetilde{\Theta}$ will be clarified later.

Remark 26. Any weak solution of the Navier-Stokes-Fourier-Poisson system (3.0.1) - (3.0.4) satisfies the relative energy inequality (3.3.4).

3.3.2 Main results

Our main result reads

Theorem 27. *Suppose that the thermodynamic functions p, e and s satisfy the hypothesis (3.1.2) - (3.1.11), the transport coefficients μ, λ and κ comply with (3.1.15) and (3.1.16) and the stress tensor is given by (1.1.25). Let $[r_0,\mathbf{w}_0,\Theta_0]$ satisfy assumptions of Theorem 25 and let $T_*>0$ be the time interval of existence of the strong solution to problem (3.0.12) - (3.0.14).*

Let

- *either $\mathcal{F}r=1$, $\beta=0$, $\alpha=0$, $\gamma>3/2$ and $g\in L^p\left(\mathbb{R}^3\right)$ with $p=1$ for $\gamma>6$ and $p=6\gamma/\left(7\gamma-6\right)$ for $\gamma\in(3/2,6]$, and*

$$\int_{\mathbb{R}^3}\frac{g(y)y_3}{\left(\sqrt{\left|x_h-y_h\right|^2+y_3^2}\right)^3}dx=0 \tag{3.3.6}$$

for all $x_h\in\omega$.

- *or $\mathcal{F}r=\sqrt{\epsilon}$ $\beta=1/2$, $\alpha=1$ and $\gamma\geq 12/5$.*

Let $[\varrho,\mathbf{u},\vartheta]$ be a sequence of weak solutions to the three-dimensional compressible Navier-Stokes-Fourier-Poisson system (3.0.12) - (3.0.14) with (3.0.6), emanating from initial data $[\varrho_0,\mathbf{u}_0,\vartheta_0]$.

Suppose that

$$\left[\mathcal{I}(\varrho_0,\mathbf{u}_0,\vartheta_0\mid r_0,\mathbf{w}_0,\Theta_0)\right]\to 0. \tag{3.3.7}$$

Then,

$$\left[\mathcal{I}(\varrho,\mathbf{u},\vartheta\mid r,\mathbf{w},\Theta)\right](t)\to 0,\quad when\quad \epsilon\to 0\quad for\quad t\in[0,T], \tag{3.3.8}$$

$$\mathbf{u}\to\mathbf{w}\ \ strongly\ \ in\ \ L^2\left(0,T;W^{1,2}\left(\Omega;\mathbb{R}^3\right)\right), \tag{3.3.9}$$

$$\vartheta\to\Theta\ \ strongly\ \ in\ \ L^2\left(0,T;W^{1,2}\left(\Omega\right)\right), \tag{3.3.10}$$

$$\log\vartheta\to\log\Theta\ \ strongly\ \ in\ \ L^2\left(0,T;W^{1,2}\left(\Omega\right)\right), \tag{3.3.11}$$

where the triple $[r,\mathbf{w},\Theta]$ satisfies the two-dimensional Navier-Stokes-Fourier-Poisson system (3.0.12) - (3.0.14) with the boundary conditions (3.0.8) and (3.0.10) on the time interval $[0,T]$ for any $0<T<T_*$

Remark 28. For $\beta = 0$ we may also include the self-gravitation of the fluid. However, passing with $\epsilon \to 0$, this term tends to zero. Therefore we do not consider here as it would lead to an additional restriction to γ.

Remark 29. Condition (3.3.6) is the necessary condition for the validity of the two-dimensional system, as it means that the gravitational force in the x_3-direction in ω is zero.

Remark 30. From (3.3.8) it follows

$$\varrho \to r \ \text{ in } \ C_{weak}\left([0,T]; L^{\gamma}(\Omega)\right), \quad \varrho \to r \ \text{ a.a. in } \ (0,T)\times\Omega.$$

Remark 31. For $\alpha = 1$ and $\beta = 1/2$, we assume more stronger assumptions then in Theorem 24 since we need a priori estimates independent of ϵ.

As a consequence, we have the following Corollary.

Corollary 32. *Suppose that the thermodynamics functions p, e and s satisfy hypothesis (3.1.2) - (3.1.11), that the coefficients μ, λ and κ comply with (3.1.16) and (3.1.17) and the stress tensor is given by (1.1.25).*

Assume that $[\varrho_0, \mathbf{u}_0, \vartheta_0]$, $\varrho_0 \geq 0$, $\vartheta_0 \geq 0$ satisfy

$$\int_0^1 \varrho_0(x)\, dx_3 \to r_0 \ \textit{ weakly in } \ L^1(\omega),$$

$$\int_0^1 \varrho_0 \mathbf{u}_0(x)\, dx_3 \to \mathbf{w}_0 \ \textit{ weakly in } \ L^1\left(\omega; \mathbb{R}^2\right),$$

$$\int_0^1 \Theta_0(x)\, dx_3 \to \Theta_0 \ \textit{ weakly in } \ L^1(\omega),$$

where $[r_0, \mathbf{w}_0, \Theta_0]$ belong to the regularity class (3.2.8), and

$$\int_\Omega \left(\frac{1}{2}\varrho_0 |\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0, \vartheta_0)\right) dx \to \int_\omega \left(\frac{1}{2} r_0 |\mathbf{w}_0|^2 + r_0 e(r_0, \Theta_0)\right) dx_h.$$

Let $[\varrho, \mathbf{u}, \vartheta]$ be a sequence of weak solution to the three-dimensional compressible Navier-Stokes-Fourier-Poisson system (3.0.1) - (3.0.7) emanating from the initial data $[\varrho_0, \mathbf{u}_0, \vartheta_0]$. Then (3.3.8) - (3.3.11) holds.

3.3.3 Convergence

The following discussion is devoted to the proof of Theorem 27. Here and hereafter, the symbol C will denote a positive generic constant, independent by ϵ, usually found in inequalities, that will not have the same value when used in different parts in the analysis.

We start with the a priori bounds. It is easy to verify that

$$S(\vartheta, \nabla_\epsilon \mathbf{v}) : \nabla_\epsilon \mathbf{v} = \left(\eta(\vartheta) - \frac{2}{3}\mu(\vartheta)\right) |\mathrm{div}_\epsilon \mathbf{v}|^2 + \mu(\vartheta)\left(|\nabla_\epsilon \mathbf{v}|^2 + \nabla_\epsilon \mathbf{v} : \nabla_\epsilon^t \mathbf{v}\right) \tag{3.3.12}$$

for any $\mathbf{v} \in W^{1,2}\left(\Omega; \mathbb{R}^3\right)$. As for any $\mathbf{v} \in W^{1,2}_{0,n}\left(\Omega; \mathbb{R}^3\right)$,

$$\int_\Omega \nabla_\epsilon \mathbf{v} : \nabla_\epsilon^t \mathbf{v} \mathrm{d}x = \int_\Omega (\mathrm{div}_\epsilon \mathbf{v})^2\, \mathrm{d}x$$

we have

$$\int_\Omega S(\vartheta, \nabla_\epsilon \mathbf{v}) : \nabla_\epsilon \mathbf{v} \mathrm{d}x \geq C \|\mathbf{v}\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}, \tag{3.3.13}$$

$$\int_\Omega \frac{1}{\vartheta} S(\vartheta, \nabla_\epsilon \mathbf{v}) : \nabla_\epsilon \mathbf{v} \mathrm{d}x \geq C \|\mathbf{v}\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}, \tag{3.3.14}$$

provided μ fulfills (3.1.16), $\eta \equiv 0$, $\epsilon \leq 1$ and $\vartheta > 0$ in $(0,T) \times \Omega$. Moreover, we have

$$\int_\Omega S_h(\Theta, \nabla_h \mathbf{w}) : \nabla_h \mathbf{w} \mathrm{d}x \geq C \|\mathbf{w}\|^2_{W^{1,2}(\omega;\mathbb{R}^2)}, \tag{3.3.15}$$

$$\int_\Omega \frac{1}{\Theta} S_h(\Theta, \nabla_h \mathbf{w}) : \nabla_h \mathbf{w} \mathrm{d}x \geq C \|\mathbf{w}\|^2_{W^{1,2}(\omega;\mathbb{R}^2)}, \tag{3.3.16}$$

and the Poincaré inequality in the form

$$\|\mathbf{w}\|_{L^2(\omega;\mathbb{R}^2)} \leq C \|\nabla_h \mathbf{w}\|_{L^2(\omega;\mathbb{R}^{2\times 2})} \tag{3.3.17}$$

for any $\mathbf{w} \in W_0^{1,2}(\omega;\mathbb{R}^2)$ and $\Theta > 0$ in $(0,T) \times \omega$.

Due to the energy equality (3.2.4) combined with the entropy inequality (3.2.3) and the inequality (3.3.14), we have the following estimates for $[\varrho, \mathbf{u}, \vartheta]$

$$\|\varrho\|_{L^\infty(0,T;L^\gamma(\Omega))} + \|\sqrt{\varrho}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\mathbf{u}\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}$$

$$+ \|\vartheta\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\vartheta\|_{L^\infty(0,T;L^4(\Omega))} + \|\vartheta\|_{L^3(0,T;L^9(\Omega))} \leq C \tag{3.3.18}$$

with the constant C independent by ϵ. These estimates hold if $\gamma \geq 12/5$ (if $\alpha = 1$) or under the assumptions on g from Theorem 27 (if $\alpha = 0$), for any $\gamma \geq 3/2$. The limit on γ comes from the gravitational potential, as

$$\left\| \int_\Omega \frac{\varrho(y)(x_1 - y_1, x_2 - y_2, \epsilon(x_3 - y_3))}{\left(\sqrt{(x_h - y_h) + \epsilon^2 (x_3 - y_3)}\right)^3} \mathrm{d}y \right\|_{L^p(\Omega;\mathbb{R}^3)} \leq C \|\varrho\|_{L^p(\Omega)}$$

for $1 < p < \infty$, with C independent of ϵ. Thus

$$\left| \int_0^T \int_\Omega \varrho \mathbf{\Phi}_2 \cdot \mathbf{u} \mathrm{d}x\mathrm{d}t \right| \leq \|\varrho\|_{L^\infty(0,T;L^\gamma(\Omega))} \|\mathbf{u}\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))} \|\mathbf{\Phi}_2\|_{L^\infty\left(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega)\right)}$$

$$\leq \|\varrho\|^2_{L^\infty(0,T;L^\gamma(\Omega))} \|\mathbf{u}\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))} \tag{3.3.19}$$

if $\gamma \geq 12/5$. On the other hand,

$$\left| \int_0^T \int_\Omega \varrho \mathbf{\Phi}_1 \cdot \mathbf{u} \mathrm{d}x\mathrm{d}t \right| \leq \|\varrho\|_{L^\infty(0,T;L^\gamma(\Omega))} \|\mathbf{u}\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))} \|\mathbf{\Phi}_1\|_{L^\infty\left(0,T;L^{\frac{6\gamma}{5\gamma-6}}(\Omega)\right)}$$

$$\leq \|\varrho\|_{L^\infty(0,T;L^\gamma(\Omega))} \|\mathbf{u}\|_{L^2(0,T;L^6(\Omega;\mathbb{R}^3))} \|g\|_{L^p(\mathbb{R}^3)} \tag{3.3.20}$$

with p from Theorem 27, as

$$\left\| \int_{\mathbb{R}^3} \frac{g(y)(x_1 - y_1, x_2 - y_2, \epsilon(x_3 - y_3))}{\left(\sqrt{(x_h - y_h) + \epsilon^2 (x_3 - y_3)}\right)^3} \mathrm{d}y \right\|_{L^{\frac{6\gamma}{5\gamma-6}}(\Omega;\mathbb{R}^3)} \leq C \|g\|_{L^p(\Omega)} \tag{3.3.21}$$

where we used the embedding $W^{1,p} \hookrightarrow L^{\frac{6\gamma}{5\gamma-6}}$.

Following [41], it is convenient to introduce the set of essential values $\mathcal{O}_{ess} \subset (0,\infty)^2$,

$$\mathcal{O}_{ess} = \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \,;\, \overline{\varrho}/2 < \varrho < 2\overline{\varrho},\, \overline{\vartheta}/2 < \vartheta < 2\overline{\vartheta} \right\} \tag{3.3.22}$$

and the residual set

$$\mathcal{O}_{res} = (0,\infty)^2 \cap \mathcal{O}_{ess}^c. \tag{3.3.23}$$

We next define the essential and residual set of points as follows

$$\mathcal{M}_{ess} \subset (0,T) \times \Omega, \tag{3.3.24}$$

$$\mathcal{M}_{ess} = \left\{ (x,t) \in (0,T) \times \Omega \,;\, (\varrho\,(x,t)\,, \vartheta\,(x,t)) \in \mathcal{O}_{ess} \right\}, \tag{3.3.25}$$

$$\mathcal{M}_{res} = ((0,T) \times \Omega) \cap (\mathcal{M}_{ess})^c\,. \tag{3.3.26}$$

Finally, each measurable function g can be decomposed as

$$g = [g]_{ess} + [g]_{res} \tag{3.3.27}$$

and we set

$$[g]_{ess} = g1_{\mathcal{M}_{ess}}, \quad [g]_{res} = g1_{\mathcal{M}_{res}} = g - [g]_{ess}\,. \tag{3.3.28}$$

Now, we need to investigate the structural properties of the Helmholtz function. More precisely, we would like to show that the quantity (3.3.3) is non-negative and strictly coercive, attaining its global minimum zero at $\left(\overline{\varrho}, \overline{\vartheta}\right)$. The structural properties of the Helmholtz function follow as

Lemma 1. *Let $H_{\overline{\vartheta}}(\varrho,\vartheta)$ be the Helmholtz function defined in (3.3.2) and $\overline{\varrho} > 0$, $\overline{\vartheta}$ be constants. Let $\mathcal{O}_{ess}$, $\mathcal{O}_{res}$ be the sets of essential and residual values in (3.3.3) and (3.3.3). Then, there exists $c_i = c_i(\overline{\varrho}, \overline{\vartheta})$, $i = 1, ..., 4$, such that*

$$c_1 \left(|\varrho - \overline{\varrho}|^2 + \left|\vartheta - \overline{\vartheta}\right|^2 \right) \leq H_{\overline{\vartheta}}(\varrho,\vartheta) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})(\varrho - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$$

$$\leq c_2 \left(|\varrho - \overline{\varrho}|^2 + \left|\vartheta - \overline{\vartheta}\right|^2 \right) \tag{3.3.29}$$

for all $(\varrho,\vartheta) \in \mathcal{O}_{ess}$

$$H_{\overline{\vartheta}}(\varrho,\vartheta) - \partial_\varrho H_{\overline{\vartheta}}(r,\overline{\vartheta})(\varrho - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$$

$$\geq \inf_{(r,\Theta)\in\partial\mathcal{O}_{ess}} H_{\overline{\vartheta}}(r,\Theta) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})(r - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) = c_3\left(\overline{\varrho},\overline{\vartheta}\right) > 0 \tag{3.3.30}$$

for all $(\varrho,\vartheta) \in \mathcal{O}_{res}$

$$H_{\overline{\vartheta}}(\varrho,\vartheta) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})(\varrho - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) \geq c_4\left(\varrho e\,(\varrho,\vartheta) + \varrho\,|s\,(\varrho,\vartheta)|\right) \tag{3.3.31}$$

for all $(\varrho,\vartheta) \in \mathcal{O}_{res}$

Proof. See [41] Lemma 5.1. ☐

As a consequence we have the following lemma

Lemma 2. *There exists a constant* $C = C\left(\overline{\varrho}, \underline{\varrho}, \overline{\vartheta}, \underline{\vartheta}\right) > 0$ *such that for all* $\varrho \in [0,\infty)$, $r \in \left[\underline{\varrho}/2, 2\overline{\varrho}\right]$, $\vartheta \in (0,\infty)$ *and* $\Theta \in \left[\underline{\vartheta}/2, 2\overline{\vartheta}\right]$

$$\begin{aligned}&\mathcal{E}(\varrho,\vartheta \mid r,\Theta)(t,\cdot)\\ &\geq C\left(1_{\mathcal{O}_{ess}} + \varrho^{\gamma} 1_{\mathcal{O}_{res}} + \vartheta^4 1_{\mathcal{O}_{res}} + (\varrho - r)\, 1_{\mathcal{O}_{ess}} + (\vartheta - \Theta)\, 1_{\mathcal{O}_{ess}}\right)\end{aligned} \tag{3.3.32}$$

The lemma yields the lower bound of the relative energy functional

$$\begin{aligned}&\mathcal{I}(\varrho,\mathbf{u},\vartheta \mid r,\mathbf{w},\Theta)\\ &\geq C\int_{\Omega}\left(\varrho\left|\mathbf{u}-\mathbf{w}\right|^2 + 1_{res} + [\varrho^{\gamma}]_{res} + [\varrho - r]_{ess}^2 + [\vartheta]_{ess}^4 + [\vartheta-\Theta]_{ess}^2\right)\mathrm{d}x\end{aligned} \tag{3.3.33}$$

Now, the basic idea is to apply (3.3.4) to $\left[\widetilde{r}, \widetilde{\mathbf{w}}, \widetilde{\Theta}\right] = [r,\mathbf{w},\Theta]$. We assume that $[r,\mathbf{w},\Theta]$, $\mathbf{w} = (\overline{\mathbf{w}},0)$, is such that $[r,\overline{\mathbf{w}},\Theta]$ solves the two-dimensional Navier-Stokes-Fourier-Poisson system (3.0.12) - (3.0.14) in $(0,T)\times\omega$. In order to integrate over Ω, we assume that the functions defined only on ω are extended being constant in x_3 for $0 \leq x_3 \leq 1$. Moreover, we write $\mathbf{w}$ instead of $\overline{\mathbf{w}}$ when we need to use a vector field with three components. Finally, we denote $\underline{\varrho} = \inf_{(0,T)\times\Omega} r$, $\overline{\varrho} = \sup_{(0,T)\times\Omega} r$, $\underline{\vartheta} = \inf_{(0,T)\times\Omega}\Theta$, $\overline{\vartheta} = \sup_{(0,T)\times\Omega}\Theta$ and use these numbers in order to define the essential and residual sets defined above.

Now, we have

$$I_1 = \int_{\Omega}\varrho\left(\mathbf{u}-\mathbf{w}\right)\cdot\nabla_{\epsilon}\mathbf{w}\cdot\left(\mathbf{w}-\mathbf{u}\right)\mathrm{d}x \leq CD(t)\mathcal{I}(\varrho,\mathbf{u},\vartheta \mid r,\mathbf{w},\Theta) \tag{3.3.34}$$

with

$$D(t) = \left\|\nabla_h\overline{\mathbf{w}}\right\|_{L^{\infty}(\Omega;\mathbb{R}^{2\times 2})} \in L^1\left(0,T\right).$$

Next

$$\begin{aligned}I_2 &= \int_{\Omega}\varrho\left(s\left(\varrho,\vartheta\right) - s\left(r,\Theta\right)\right)\left(\mathbf{w}-\mathbf{u}\right)\cdot\nabla_{\epsilon}\Theta\mathrm{d}x\\ &\leq \left\|\nabla_h\Theta\right\|_{L^{\infty}(\Omega;\mathbb{R}^2)}\end{aligned}$$

$$\cdot\left[2\overline{\varrho}\int_{\Omega}\left|\left[s\left(\varrho,\vartheta\right)-s\left(r,\Theta\right)\right]_{ess}\right|\cdot\left|\mathbf{w}-\mathbf{u}\right|\mathrm{d}x + \int_{\Omega}\left|\left[s\left(\varrho,\vartheta\right)-s\left(r,\Theta\right)\right]_{res}\right|\cdot\left|\mathbf{w}-\mathbf{u}\right|\mathrm{d}x\right] \tag{3.3.35}$$

Lemma 2 together with the properties of entropy (3.1.12) and (3.1.13) yields

$$\int_{\Omega}\left|\left[s\left(\varrho,\vartheta\right)-s\left(r,\Theta\right)\right]_{ess}\right|\cdot\left|\mathbf{w}-\mathbf{u}\right|\mathrm{d}x \leq \delta\left\|\mathbf{w}-\mathbf{u}\right\|_{L^2(\Omega;\mathbb{R}^3)}^2 + C(\delta)\int_{\Omega}\mathcal{E}(\varrho,\vartheta \mid r,\Theta)\mathrm{d}x$$

for $\delta > 0$, and

$$\begin{aligned}&\int_{\Omega}\left|\left[s\left(\varrho,\vartheta\right)-s\left(r,\Theta\right)\right]_{res}\right|\cdot\left|\mathbf{w}-\mathbf{u}\right|\mathrm{d}x\\ &\leq \delta\left\|\mathbf{w}-\mathbf{u}\right\|_{L^6(\Omega;\mathbb{R}^3)}^2 + C(\delta)\left\|\left[s\left(\varrho,\vartheta\right)-s\left(r,\Theta\right)\right]_{res}\right\|_{L^{6/5}(\Omega)}^2.\end{aligned}$$

Using again the properties of the entropy (3.1.12) and (3.1.13) together with the fact that the mapping $t \to \int_{\Omega}\mathcal{E}(\varrho,\vartheta \mid r,\Theta)\mathrm{d}x \in L^{\infty}\left(0,T\right)$, we conclude that

$$\left\|\left[s\left(\varrho,\vartheta\right)-s\left(r,\Theta\right)\right]_{res}\right\|_{L^{6/5}(\Omega)}^2 \leq C\int_{\Omega}\mathcal{E}(\varrho,\vartheta \mid r,\Theta)\mathrm{d}x.$$

Finally, we end up with

$$I_2 \le \delta \left\| \mathbf{w} - \mathbf{u} \right\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + C\left(\delta; r, \overline{\mathbf{w}}, \Theta\right) \int_\Omega \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \mathrm{d}x.$$

Next, using the fact that $[r, \mathbf{w}, \Theta]$ solve the two-dimensional Navier-Stokes-Fourier-Poisson system, we have

$$I_3 = \int_\Omega \varrho \left(\partial_t \mathbf{w} + \mathbf{w} \cdot \nabla_h \mathbf{w}\right) \cdot \left(\mathbf{u} - \mathbf{w}\right) \mathrm{d}x = I_{3,1} + I_{3,2},$$

where

$$I_{3,1} = \int_\Omega \frac{\varrho}{r} \left(\mathbf{w} - \mathbf{u}\right) \cdot \left(\mathrm{div}_\epsilon S\left(\Theta, \nabla_\epsilon \mathbf{w}\right) - \nabla_\epsilon p\left(r, \Theta\right)\right) \mathrm{d}x,$$

$$I_{3,2} = \int_\Omega \varrho \left(\mathbf{w} - \mathbf{u}\right) \cdot \Big(- \left(\boldsymbol{\chi} \times \mathbf{w}\right) + \nabla_\epsilon \left|\boldsymbol{\chi} \times x\right|^2 + \nabla_h \phi_h \Big) \mathrm{d}x = \sum_{i=1}^{3} K_i.$$

We write

$$I_{3,1} = \int_\Omega \frac{\varrho - r}{r} \left(\mathbf{w} - \mathbf{u}\right) \cdot \left(\mathrm{div}_\epsilon S\left(\Theta, \nabla_\epsilon \mathbf{w}\right) - \nabla_\epsilon p\left(r, \Theta\right)\right) \mathrm{d}x$$

$$+ I_{3,1} = \int_\Omega \left(\mathbf{w} - \mathbf{u}\right) \cdot \left(\mathrm{div}_\epsilon S\left(\Theta, \nabla_\epsilon \mathbf{w}\right) - \nabla_\epsilon p\left(r, \Theta\right)\right) \mathrm{d}x.$$

Similarly to I_2, we have

$$\left| \int_\Omega \frac{\varrho - r}{r} \left(\mathbf{w} - \mathbf{u}\right) \cdot \left(\mathrm{div}_\epsilon S\left(\Theta, \nabla_\epsilon \mathbf{w}\right) - \nabla_\epsilon p\left(r, \Theta\right)\right) \mathrm{d}x \right|$$

$$\le C\left(\delta; r, \overline{\mathbf{w}}, \Theta\right) \left\| [\varrho - r]_{ess} \right\|^2_{L^2(\Omega)} + \delta \left\| \mathbf{w} - \mathbf{u} \right\|^2_{L^2(\Omega;\mathbb{R}^3)}$$

$$+ C\left(\delta; r, \overline{\mathbf{w}}, \Theta\right) \Big(\left\| [\varrho]_{res} \right\|^2_{L^{6/5}(\Omega)} + \left\| [1]_{res} \right\|^2_{L^{6/5}(\Omega)} \Big) + \delta \left\| \mathbf{w} - \mathbf{u} \right\|^2_{L^6(\Omega;\mathbb{R}^3)}.$$

Integrating by parts the second integral of $I_{3,1}$, we have

$$\int_\Omega \left(\mathbf{w} - \mathbf{u}\right) \cdot \left(\mathrm{div}_\epsilon S\left(\Theta, \nabla_\epsilon \mathbf{w}\right) - \nabla_\epsilon p\left(r, \Theta\right)\right) \mathrm{d}x$$

$$= - \int_\Omega \left(S\left(\Theta, \nabla_\epsilon \mathbf{w}\right) : \nabla_\epsilon \left(\mathbf{w} - \mathbf{u}\right) - p\left(r, \Theta\right) \cdot \mathrm{div}_\epsilon \left(\mathbf{w} - \mathbf{u}\right)\right) \mathrm{d}x.$$

We conclude

$$I_{3,1} \le \int_\Omega \left(p\left(r, \Theta\right) \cdot \mathrm{div}_h \left(\overline{\mathbf{w}} - \mathbf{u}\right) - S\left(\Theta, \nabla_\epsilon \mathbf{w}\right) : \nabla_\epsilon \left(\mathbf{w} - \mathbf{u}\right)\right) \mathrm{d}x + \delta \left\| \mathbf{w} - \mathbf{u} \right\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)}$$

$$+ C\left(\delta; r, \overline{\mathbf{w}}, \Theta\right) \int_\Omega \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \mathrm{d}x$$

for any $\delta > 0$. The terms K_1 - K_3 will be treated below in combination with I_7 and I_9. Now,

$$I_4 = - \int_\Omega \varrho \left(s\left(\varrho, \vartheta\right) - s\left(r, \Theta\right)\right) \partial_t \Theta \mathrm{d}x$$

$$= - \int_{\Omega} (\varrho - r) \left(s(\varrho, \vartheta) - s(r, \Theta) \right) \partial_t \Theta \mathrm{d}x - \int_{\Omega} r \left(s(\varrho, \vartheta) - s(r, \Theta) \right) \partial_t \Theta \mathrm{d}x.$$

For the first term above, we have

$$- \int_{\Omega} (\varrho - r) \left(s(\varrho, \vartheta) - s(r, \Theta) \right) \partial_t \Theta \mathrm{d}x$$

$$= - \int_{\Omega} (\varrho - r) \left[s(\varrho, \vartheta) - s(r, \Theta) \right]_{ess} \partial_t \Theta \mathrm{d}x - \int_{\Omega} (\varrho - r) \left[s(\varrho, \vartheta) - s(r, \Theta) \right]_{res} \partial_t \Theta \mathrm{d}x$$

$$\leq C(\delta; r, \overline{\mathbf{w}}, \Theta) \int_{\Omega} \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \mathrm{d}x.$$

Now,

$$- \int_{\Omega} r \left(s(\varrho, \vartheta) - s(r, \Theta) \right) \partial_t \Theta \mathrm{d}x$$

$$= - \int_{\Omega} r \left(s(\varrho, \vartheta) - s(r, \Theta) - \partial_{\varrho} s(r, \Theta) (\varrho - r) - \partial_{\vartheta} s(r, \Theta) (\vartheta - \Theta) \right) \partial_t \Theta \mathrm{d}x$$

$$- \int_{\Omega} r \left(\partial_{\varrho} s(r, \Theta) (\varrho - r) + \partial_{\vartheta} s(r, \Theta) (\vartheta - \Theta) \right) \partial_t \Theta \mathrm{d}x,$$

and in analogy as before, we end up with

$$I_4 \leq C(\delta; r, \overline{\mathbf{w}}, \Theta) \int_{\Omega} \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \mathrm{d}x$$

$$- \int_{\Omega} r \left(\partial_{\varrho} s(r, \Theta) (\varrho - r) + \partial_{\vartheta} s(r, \Theta) (\vartheta - \Theta) \right) \partial_t \Theta \mathrm{d}x.$$

For I_5 we use the same procedure as for I_4, obtaining

$$I_5 = - \int_{\Omega} \varrho \left(s(\varrho, \vartheta) - s(r, \Theta) \right) \overline{\mathbf{w}} \cdot \nabla_h \Theta \mathrm{d}x$$

$$\leq C(\delta; r, \overline{\mathbf{w}}, \Theta) \int_{\Omega} \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \mathrm{d}x$$

$$- \int_{\Omega} r \left(\partial_{\varrho} s(r, \Theta) (\varrho - r) + \partial_{\vartheta} s(r, \Theta) (\vartheta - \Theta) \right) \overline{\mathbf{w}} \cdot \nabla_h \Theta \mathrm{d}x.$$

Moreover,

$$I_6 = \int_{\Omega} \left(\left(1 - \frac{\varrho}{r} \right) \partial_t p(r, \Theta) - \frac{\varrho}{r} \mathbf{u} \cdot \nabla_{\epsilon} p(r, \Theta) \right) \mathrm{d}x$$

$$= \int_{\Omega} \left(1 - \frac{\varrho}{r} \right) \left(\partial_t p(r, \Theta) + \overline{\mathbf{w}} \cdot \nabla_h p(r, \Theta) \right) dx + \int_{\Omega} p(r, \Theta) \operatorname{div}_{\epsilon} \mathbf{u} \mathrm{d}x$$

$$+ \int_{\Omega} \left(1 - \frac{\varrho}{r} \right) \nabla_{\epsilon} p(r, \Theta) \cdot (\mathbf{u} - \mathbf{w}) \mathrm{d}x.$$

Using the same argument as for I_2, we have

$$\left| \int_{\Omega} \left(1 - \frac{\varrho}{r} \right) \nabla_{\epsilon} p(r, \Theta) \cdot (\mathbf{u} - \mathbf{w}) \mathrm{d}x \right|$$

$$\leq \delta \left\| \mathbf{w} - \mathbf{u} \right\|^2_{W^{1,2}_0(\Omega; \mathbb{R}^3)} + C(\delta; r, \overline{\mathbf{w}}, \Theta) \int_{\Omega} \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \mathrm{d}x$$

for any $\delta > 0$. We end with

$$I_6 \leq \int_\Omega \left(1 - \frac{\varrho}{r}\right)\left(\partial_t p\left(r,\Theta\right) + \overline{\mathbf{w}}\cdot\nabla_h p\left(r,\Theta\right)\right)dx + \int_\Omega p\left(r,\Theta\right)\mathrm{div}_\epsilon \mathbf{u} \mathrm{d}x$$

$$\delta \left\|\mathbf{w}-\mathbf{u}\right\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + C\left(\delta; r, \overline{\mathbf{w}}, \Theta\right)\int_\Omega \mathcal{E}(\varrho, \vartheta \mid r, \Theta)\mathrm{d}x.$$

Finally, we have $I_7 + K_1 = 0$ and $I_8 + K_2 = 0$. We consider now the gravitational potential. We start with the case $\alpha = 0$. We assumed

$$\int_{\mathbb{R}^3} \frac{g(y)y_3}{\left(\sqrt{\left|x_h - y_h\right|^2 + y_3^2}\right)^3}\mathrm{d}y = 0. \tag{3.3.36}$$

Therefore, we have to show that

$$\lim_{\epsilon\to 0^+}\int_\Omega r\left(\mathbf{w}-\mathbf{u}\right)\cdot$$

$$\left[\int_{\mathbb{R}^3} g(y)\left(\frac{\left(x_h - y_h, -y_3\right)}{\left(\sqrt{\left|x_h - y_h\right|^2 + y_3^2}\right)^3} - \frac{\left(x_h - y_h, \epsilon x_3 - y_3\right)}{\left(\sqrt{\left|x_h - y_h\right|^2 + \left(\epsilon x_3 - y_3\right)^2}\right)^3}\right)\mathrm{d}y\right]\mathrm{d}x = 0. \tag{3.3.37}$$

First, due to the estimates above, it is enough to verify

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^3} \overline{g}(y)\left(\frac{\left(x_h - y_h, -y_3\right)}{\left(\sqrt{\left|x_h - y_h\right|^2 + y_3^2}\right)^3} - \frac{\left(x_h - y_h, \epsilon x_3 - y_3\right)}{\left(\sqrt{\left|x_h - y_h\right|^2 + \left(\epsilon x_3 - y_3\right)^2}\right)^3}\right)\mathrm{d}y = \mathbf{0}$$

for all $x_h \in \omega$, $x_3 \in (0,1)$ and $\overline{g} \in C_c^\infty\left(\mathbb{R}^3\right)$. As

$$\lim_{\epsilon\to 0^+}\left(\frac{\left(x_h - y_h, -y_3\right)}{\left(\sqrt{\left|x_h - y_h\right|^2 + y_3^2}\right)^3} - \frac{\left(x_h - y_h, \epsilon x_3 - y_3\right)}{\left(\sqrt{\left|x_h - y_h\right|^2 + \left(\epsilon x_3 - y_3\right)^2}\right)^3}\right)\mathrm{d}y = \mathbf{0}$$

for almost all $(x_h, x_3) \in \Omega$, $(y_h, y_3) \in \mathbb{R}^3$, and

$$\left|\frac{\left(x_h - y_h, \epsilon x_3 - y_3\right)}{\left(\sqrt{\left|x_h - y_h\right|^2 + \left(\epsilon x_3 - y_3\right)^2}\right)^3}\right| \leq \left|\frac{1}{\sqrt{\left|x_h - y_h\right|^2 + \left(\epsilon x_3 - y_3\right)^2}^2}\right| \in L^1_{loc}\left(\mathbb{R}^3\right),$$

for all $\epsilon \in [0,1]$. The Lebesgue dominated converge theorem yields the require identity (3.3.37). For the case $\alpha = 1$, we have to show that

$$\int_\Omega \varrho\left(\mathbf{w}-\mathbf{u}\right)\cdot\left[\int_\Omega \frac{\varrho\left(t,y\right)\left(x_h - y_h, \epsilon\left(x_3 - y_3\right)\right)}{\left(\sqrt{\left|x_h - y_h\right|^2 + \epsilon^2\left(x_3 - y_3\right)^2}\right)^3}\mathrm{d}y + \nabla_\epsilon \int_\omega \frac{r\left(t, y_h\right)}{\left|x_h - y_h\right|}\mathrm{d}y_h\right]\mathrm{d}x$$

$$\leq \delta \|\mathbf{w}-\mathbf{u}\|_{L^6(\Omega;\mathbb{R}^3)} + c(\delta; r, \mathbf{w}, \Theta) \int_\Omega I(\varrho, r; \vartheta, \Theta)\,\mathrm{d}x + H_\epsilon, \tag{3.3.38}$$

where $H_\epsilon = O(\epsilon)$. The derivative of the integral over ω with respect to x_3 is zero. For $\gamma \geq 12/5$, as in (3.3.18), we have to verify

$$\lim_{\epsilon\to 0^+} \int_\Omega r\mathbf{w}\cdot \left[\int_\Omega \frac{r(t,y_h)\left(x_h - y_h, \epsilon^2 (x_3-y_3)\right)}{\left(\sqrt{|x_h-y_h|^2 + \epsilon^2 (x_3-y_3)^2}\right)^3} \mathrm{d}y + \nabla_\epsilon \int_\omega \frac{r(t,y_h)}{|x_h-y_h|} \mathrm{d}y_h \right] \mathrm{d}x = 0.$$

Again, it is enough to show

$$\lim_{\epsilon\to 0^+} \left[\int_\Omega \frac{r(t,y_h)\left(x_h - y_h, \epsilon^2 (x_3-y_3)\right)}{\left(\sqrt{|x_h-y_h|^2 + \epsilon^2 (x_3-y_3)^2}\right)^3} \mathrm{d}y + \nabla_\epsilon \int_\omega \frac{r(t,y_h)}{|x_h-y_h|} \mathrm{d}y_h \right] \mathrm{d}x = \mathbf{0}.$$

First, we note that

$$\nabla_\epsilon \int_\omega \frac{r(t,y_h)}{|x_h-y_h|} \mathrm{d}y_h = -\text{p.v.} \int_\omega \frac{r(t,y_h)(x_h-y_h)}{|x_h-y_h|^{3/2}} \mathrm{d}y_h,$$

where p.v. denotes the integral in the principal value sense. Therefore, we have to verify that

$$\lim_{\epsilon\to 0^+} \int_\Omega \frac{\epsilon r(t,y_h)(x_3-y_3)}{\left(\sqrt{|x_h-y_h|^2 + \epsilon^2 (x_3-y_3)^2}\right)^3} \mathrm{d}y = 0 \tag{3.3.39}$$

and

$$\lim_{\epsilon\to 0^+} \int_\Omega \frac{\epsilon r(t,y_h)(x_h-y_h)}{\left(\sqrt{|x_h-y_h|^2 + \epsilon^2 (x_3-y_3)^2}\right)^3} \mathrm{d}y = \text{p.v.} \int_\omega \frac{r(t,y_h)(x_h-y_h)}{|x_h-y_h|^{3/2}} \mathrm{d}y_h. \tag{3.3.40}$$

Let us fix $x_0 \in \omega$, $\triangle > 0$, sufficiently small, and denote $B_\triangle(x_0) = \{x \in \omega; |x - x_0| < \Delta\}$ and $C_\triangle(x_0) = \{x \in \Omega; |x_h - x_0| < \Delta, 0 < x_3 < 1\}$. We consider (3.3.39). Let us fix $\delta > 0$. Then, there exists $\Delta > 0$ such that for any $0 < \epsilon \leq 1$ and $0 < x_3 < 1$, we have

$$\left| \int_{C_\triangle(x_0)} \frac{\epsilon r(t,y_h)(x_3-y_3)}{\left(\sqrt{|x_0-y_h|^2 + \epsilon^2 (x_3-y_3)^2}\right)^3} \mathrm{d}y \right| < \delta$$

and for this $\triangle > 0$ there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leq \epsilon_0$

$$\left| \int_{\Omega/C_\triangle(x_0)} \frac{\epsilon r(t,y_h)(x_3-y_3)}{\left(\sqrt{|x_0-y_h|^2 + \epsilon^2 (x_3-y_3)^2}\right)^3} \mathrm{d}y \right| < \delta,$$

whence (3.3.39). We consider (3.3.40). Since $(x_h - y_h) / \left(|x_h - y_h|^3\right)$ is a singular integral kernel in the sense of the Calderon-Zygmund theory, for a fixed $x_0 \in \omega$, $0 < x_3 < 1$ and $\delta > 0$, there exists $\Delta > 0$ such that

$$\left| \int_{C_{\triangle(x_0)}} \frac{r(t, y_h)(x_0 - y_h)}{\left(\sqrt{|x_0 - y_h|^2 + \epsilon^2 (x_3 - y_3)^2}\right)^3} \mathrm{d}y \right| < \delta$$

and

$$\left| \text{p.v.} \int_{B_{\triangle(x_0)}} \frac{r(t, y_h)(x_h - y_h)}{|x_h - y_h|^3} \mathrm{d}y_h \right| < \delta.$$

For this $\Delta > 0$, using that

$$\frac{1}{\left(\sqrt{|x_0 - y_h|^2 + \epsilon^2 (x_3 - y_3)^2}\right)^3} - \frac{1}{|x_0 - y_h|^3} \to 0 \quad \text{as } \epsilon \to 0$$

for any $y_h \in \omega$, $0 < x_3, y_3 < 1$, except $x_0 = y_h$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \le \epsilon_0$

$$\left| \int_{\Omega / C_{\triangle(x_0)}} \frac{\epsilon r(t, y_h)(x_h - y_h)}{\left(\sqrt{|x_h - y_h|^2 + \epsilon^2 (x_3 - y_3)^2}\right)^3} \mathrm{d}y - \text{p.v.} \int_{\omega / B_{\triangle(x_0)}} \frac{r(t, y_h)(x_h - y_h)}{|x_h - y_h|^3} \mathrm{d}y_h \right| < \delta,$$

whence (3.3.40). In conclusion, we have

$$I_9 + K_3 \le \delta \|\mathbf{w} - \mathbf{u}\|^2_{L^6(\Omega;\mathbb{R}^3)} + C(\delta; r, \overline{\mathbf{w}}, \Theta) \int_\Omega \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \mathrm{d}x + H_\epsilon.$$

Plugging all the previous estimates in (3.3.4), we obtain

$$\int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{w}|^2 + \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \right)(t, \cdot)\, \mathrm{d}x$$

$$+ \int_0^T \int_\Omega \left(\frac{\Theta}{\vartheta} S(\vartheta, \nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{u} - S(\Theta, \nabla_\epsilon \mathbf{w}) : (\nabla_\epsilon \mathbf{u} - \nabla_\epsilon \mathbf{w}) - S(\vartheta, \nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{w} \right) \mathrm{d}x \mathrm{d}t$$

$$+ \int_0^T \int_\Omega \left(\frac{\mathbf{q}(\vartheta, \nabla_\epsilon \vartheta) \cdot \nabla_\epsilon \Theta}{\vartheta} - \frac{\Theta}{\vartheta} \frac{\mathbf{q}(\vartheta, \nabla_\epsilon \vartheta) \cdot \nabla_\epsilon \vartheta}{\vartheta} \right) \mathrm{d}x \mathrm{d}t$$

$$\le \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{w}(0, \cdot)|^2 + \mathcal{E}(\varrho_0, \vartheta_0 \mid r(0, \cdot), \Theta(0, \cdot)) \right) \mathrm{d}x + H_\epsilon$$

$$+ \int_0^T \left[\delta \|\mathbf{w} - \mathbf{u}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + C(\delta; r, \overline{\mathbf{w}}, \Theta) \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{w}|^2 + \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \right) \mathrm{d}x \right] \mathrm{d}t$$

$$+ \int_0^T \int_\Omega (p(r, \Theta) - p(\varrho, \vartheta)) \operatorname{div}_h \overline{\mathbf{w}} \mathrm{d}x \mathrm{d}t$$

$$+\int_0^T\int_\Omega\left(1-\frac{\varrho}{r}\right)\left(\partial_t p\left(r,\Theta\right)+\overline{\mathbf{w}}\cdot\nabla_h p\left(r,\Theta\right)\right)\mathrm{d}x\mathrm{d}t$$

$$-\int_0^T\int_\Omega r\left(\partial_\varrho s\left(r,\Theta\right)\left(\varrho-r\right)+\partial_\vartheta s\left(r,\Theta\right)\left(\vartheta-\Theta\right)\right)\left(\partial_t\Theta+\overline{\mathbf{w}}\cdot\nabla_h\Theta\right)\mathrm{d}x\mathrm{d}t.$$

Using the Maxwell (3.1.5), the Gibbs (3.1.1) relations and the continuity equation (3.0.12), we write

$$\int_\Omega\left(p\left(r,\Theta\right)-p\left(\varrho,\vartheta\right)\right)\mathrm{div}_h\overline{\mathbf{w}}\mathrm{d}x+\int_\Omega\left(1-\frac{\varrho}{r}\right)\left(\partial_t p\left(r,\Theta\right)+\overline{\mathbf{w}}\cdot\nabla_h p\left(r,\Theta\right)\right)\mathrm{d}x$$

$$-\int_0^T\int_\Omega r\left(\partial_\varrho s\left(r,\Theta\right)\left(\varrho-r\right)+\partial_\vartheta s\left(r,\Theta\right)\left(\vartheta-\Theta\right)\right)\left(\partial_t\Theta+\overline{\mathbf{w}}\cdot\nabla_h\Theta\right)\mathrm{d}x\mathrm{d}t$$

$$=\int_\Omega\left(p\left(r,\Theta\right)-p\left(\varrho,\vartheta\right)\right)\mathrm{div}_h\overline{\mathbf{w}}\mathrm{d}x+r\left(\Theta-\vartheta\right)\partial_\vartheta s\left(r,\Theta\right)\left(\partial_t\Theta+\overline{\mathbf{w}}\cdot\nabla_h\Theta\right)\mathrm{d}x\mathrm{d}t$$

$$-\int_\Omega\left(r-\varrho\right)\partial_\varrho p\left(r,\Theta\right)\mathrm{div}_h\overline{\mathbf{w}}\mathrm{d}x.$$

Using the same identities as above and the entropy balance (3.0.15), the second term on the right-hand side can be rewritten as follows

$$\int_\Omega r\left(\Theta-\vartheta\right)\partial_\vartheta s\left(r,\Theta\right)\left(\partial_t\Theta+\overline{\mathbf{w}}\cdot\nabla_h\Theta\right)\mathrm{d}x$$

$$=\int_\Omega r\left(\Theta-\vartheta\right)\left(\partial_t s\left(r,\Theta\right)+\overline{\mathbf{w}}\cdot\nabla_h s\left(r,\Theta\right)\right)\mathrm{d}x-\int_\Omega\left(\Theta-\vartheta\right)\partial_\vartheta p\left(r,\Theta\right)\mathrm{div}_h\overline{\mathbf{w}}\mathrm{d}x$$

$$=\int_\Omega\left(\Theta-\vartheta\right)\left[\frac{1}{\Theta}\left(S_h\left(\Theta,\nabla_h\overline{\mathbf{w}}\right):\nabla_h\overline{\mathbf{w}}-\frac{\mathbf{q}_h\left(\Theta,\nabla_h\Theta\right)\cdot\nabla_h\Theta}{\Theta}\right)-\mathrm{div}_h\left(\frac{\mathbf{q}_h\left(\Theta,\nabla_h\Theta\right)}{\Theta}\right)\right]\mathrm{d}x$$

$$-\int_\Omega\left(\Theta-\vartheta\right)\partial_\vartheta p\left(r,\Theta\right)\mathrm{div}_h\overline{\mathbf{w}}\mathrm{d}x.$$

Observing that

$$\left|\int_\Omega\left(p\left(r,\Theta\right)-p\left(\varrho,\vartheta\right)+\partial_\varrho p\left(r,\Theta\right)\left(\varrho-r\right)+\partial_\vartheta p\left(r,\Theta\right)\left(\vartheta-\Theta\right)\right)\mathrm{div}_h\overline{\mathbf{w}}\mathrm{d}x\right|$$

$$\leq\left\|\mathrm{div}_h\overline{\mathbf{w}}\right\|_{L^\infty(\Omega)}\int_\Omega\mathcal{E}(\varrho,\vartheta\mid r,\Theta)\mathrm{d}x,$$

we reduce to

$$\int_\Omega\left(\frac{1}{2}\varrho\left|\mathbf{u}-\mathbf{w}\right|^2+\mathcal{E}(\varrho,\vartheta\mid r,\Theta)\right)(t,\cdot)\,\mathrm{d}x$$

$$+\int_0^T\int_\Omega\left(\frac{\Theta}{\vartheta}S\left(\vartheta,\nabla_\epsilon\mathbf{u}\right):\nabla_\epsilon\mathbf{u}-S\left(\Theta,\nabla_\epsilon\mathbf{w}\right):\left(\nabla_\epsilon\mathbf{u}-\nabla_\epsilon\mathbf{w}\right)-S\left(\vartheta,\nabla_\epsilon\mathbf{u}\right):\nabla_\epsilon\mathbf{w}\right)\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_\Omega\frac{\Theta-\vartheta}{\vartheta}S_h\left(\Theta,\nabla_h\overline{\mathbf{w}}\right):\nabla_h\overline{\mathbf{w}}\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T\int_\Omega\left(\frac{\mathbf{q}\left(\vartheta,\nabla_\epsilon\vartheta\right)\cdot\nabla_\epsilon\Theta}{\vartheta}-\frac{\Theta}{\vartheta}\frac{\mathbf{q}\left(\vartheta,\nabla_\epsilon\vartheta\right)\cdot\nabla_\epsilon\vartheta}{\vartheta}\right)\mathrm{d}x\mathrm{d}t$$

$$+\int_0^T \int_\Omega \left((\Theta - \vartheta) \frac{\mathbf{q}_h (\Theta, \nabla_h \Theta) \cdot \nabla_h \Theta}{\Theta^2} + \frac{\mathbf{q} (\Theta, \nabla_\epsilon \Theta) \cdot \nabla_\epsilon (\vartheta - \Theta)}{\Theta} \right) \mathrm{d}x \mathrm{d}t$$

$$\leq \int_\Omega \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0 - \mathbf{w}(0, \cdot)|^2 + \mathcal{E}(\varrho_0, \vartheta_0 \mid r(0, \cdot), \Theta(0, \cdot)) \right) \mathrm{d}x + H_\epsilon$$

$$+\int_0^T \left[\delta \|\mathbf{w} - \mathbf{u}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + C(\delta; r, \overline{\mathbf{w}}, \Theta) \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{w}|^2 + \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \right) \mathrm{d}x \right] \mathrm{d}t. \tag{3.3.41}$$

Now, following the discussion in [43], we study the terms in the left-hand side in order to show that the terms containing $\nabla_\epsilon \mathbf{u}$ and $\nabla_\epsilon \vartheta$ are strong enough to control the $W^{1,2}$-norm of the velocity. In accordance with hypothesis (3.1.16) we write

$$S(\vartheta, \nabla_\epsilon \mathbf{u}) = S^0(\vartheta, \nabla_\epsilon \mathbf{u}) + S^1(\vartheta, \nabla_\epsilon \mathbf{u})$$

where

$$S^0(\vartheta, \nabla_\epsilon \mathbf{u}) = \mu_0 \left(\nabla_\epsilon \mathbf{u} + (\nabla_\epsilon \mathbf{u})^T - \frac{2}{3} \mathrm{div}_\epsilon \mathbf{u} \mathbf{I} \right),$$

$$S^1(\vartheta, \nabla_\epsilon \mathbf{u}) = \mu_1 \vartheta \left(\nabla_\epsilon \mathbf{u} + (\nabla_\epsilon \mathbf{u})^T - \frac{2}{3} \mathrm{div}_\epsilon \mathbf{u} \mathbf{I} \right).$$

Then

$$\frac{\Theta}{\vartheta} S^1(\vartheta, \nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{u} - S^1(\Theta, \nabla_\epsilon \mathbf{w}) : (\nabla_\epsilon \mathbf{u} - \nabla_\epsilon \mathbf{w}) - S^1(\vartheta, \nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{w}$$

$$+ \left(\frac{\vartheta - \Theta}{\Theta} S^1_h(\Theta, \nabla_h \overline{\mathbf{w}}) : \nabla_h \overline{\mathbf{w}} \right)$$

$$= \Theta \left(\frac{S^1(\vartheta, \nabla_\epsilon \mathbf{u})}{\vartheta} - \frac{S^1(\Theta, \nabla_\epsilon \mathbf{w})}{\Theta} \right) : (\nabla_\epsilon \mathbf{u} - \nabla_\epsilon \mathbf{w})$$

$$+ (\Theta - \vartheta) \left(\frac{S^1(\vartheta, \nabla_\epsilon \mathbf{u})}{\vartheta} - \frac{S^1(\Theta, \nabla_\epsilon \mathbf{w})}{\Theta} \right) : \nabla_\epsilon \mathbf{w}.$$

Using the Korn inequality in the first term and the splitting in essential and residual sets for the second one, we obtain

$$\left| (\Theta - \vartheta) \left(\frac{S^1(\vartheta, \nabla_\epsilon \mathbf{u})}{\vartheta} - \frac{S^1(\Theta, \nabla_\epsilon \mathbf{w})}{\Theta} \right) : \nabla_\epsilon \mathbf{w} \right|$$

$$\leq \delta \|\mathbf{w} - \mathbf{u}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + C(\delta) \int_\Omega \mathcal{E}(\varrho, \vartheta \mid r, \Theta) \mathrm{d}x.$$

Now, for $0 < \Theta \leq \vartheta$, we have

$$\frac{\Theta}{\vartheta} \left(S^0(\nabla_\epsilon \mathbf{u}) - S^0(\nabla_\epsilon \mathbf{w}) \right) : (\nabla_\epsilon \mathbf{u} - \nabla_\epsilon \mathbf{w}) + \Theta \left(\frac{1}{\vartheta} - \frac{1}{\Theta} \right) S^0(\nabla_\epsilon \mathbf{w}) : (\nabla_\epsilon \mathbf{u} - \nabla_\epsilon \mathbf{w})$$

$$+ \frac{\vartheta - \Theta}{\vartheta} \left(S^0(\nabla_\epsilon \mathbf{w}) - S^0(\nabla_\epsilon \mathbf{u}) \right) : \nabla_\epsilon \mathbf{w}$$

$$\leq \frac{\Theta}{\vartheta} S^0(\nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{u} - S^0(\nabla_\epsilon \mathbf{w}) : (\nabla_\epsilon \mathbf{u} - \nabla_\epsilon \mathbf{w}) + S^0(\nabla_\epsilon \mathbf{u}) : \nabla_\epsilon \mathbf{w}$$

$$+\frac{\vartheta-\Theta}{\vartheta}S_h^0\left(\nabla_h\overline{\mathbf{w}}\right):\nabla_h\overline{\mathbf{w}}.$$

As $(1/\vartheta)\leq(1/\Theta)$, the term on the left-hand side of the inequality can be controlled on the right-hand side by

$$\delta\left\|\mathbf{w}-\mathbf{u}\right\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)}+C\left(\delta\right)\int_\Omega\mathcal{E}(\varrho,\vartheta\mid r,\Theta)\mathrm{d}x.$$

Now, for $0<\vartheta\leq\Theta$, we have

$$\begin{aligned}\left(S^0\left(\nabla_\epsilon\mathbf{u}\right)-S^0\left(\nabla_\epsilon\mathbf{w}\right)\right):\left(\nabla_\epsilon\mathbf{u}-\nabla_\epsilon\mathbf{w}\right)+\frac{\Theta-\vartheta}{\vartheta}\left(S^0\left(\nabla_\epsilon\mathbf{u}\right):\nabla_\epsilon\mathbf{u}-S_h^0\left(\nabla_h\overline{\mathbf{w}}\right):\nabla_h\overline{\mathbf{w}}\right)\\ \leq\frac{\Theta}{\vartheta}S^0\left(\nabla_\epsilon\mathbf{u}\right):\nabla_\epsilon\mathbf{u}-S^0\left(\nabla_\epsilon\mathbf{w}\right):\left(\nabla_\epsilon\mathbf{u}-\nabla_\epsilon\mathbf{w}\right)-S^0\left(\nabla_\epsilon\mathbf{u}\right):\nabla_\epsilon\mathbf{w}\\ +\frac{\vartheta-\Theta}{\vartheta}S_h^0\left(\nabla_h\overline{\mathbf{w}}\right):\nabla_h\overline{\mathbf{w}}\end{aligned}$$

As $\nabla_\epsilon\mathbf{u}\to S^0\left(\nabla_\epsilon\mathbf{u}\right):\nabla_\epsilon\mathbf{u}$ is convex, we have

$$\frac{\Theta-\vartheta}{\vartheta}\left(S^0\left(\nabla_\epsilon\mathbf{u}\right):\nabla_\epsilon\mathbf{u}-S_h^0\left(\nabla_h\overline{\mathbf{w}}\right):\nabla_h\overline{\mathbf{w}}\right)\geq\frac{\Theta-\vartheta}{\vartheta}S^0\left(\nabla_\epsilon\mathbf{w}\right):\left(\nabla_\epsilon\mathbf{u}-\nabla_\epsilon\mathbf{w}\right).$$

This term can be controlled on the right-hand side by

$$\delta\left\|\mathbf{w}-\mathbf{u}\right\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)}+C\left(\delta\right)\int_\Omega\mathcal{E}(\varrho,\vartheta\mid r,\Theta)\mathrm{d}x.$$

Summing up, we have

$$\begin{aligned}\int_\Omega\left(\frac{1}{2}\varrho\left|\mathbf{u}-\mathbf{w}\right|^2+\mathcal{E}(\varrho,\vartheta\mid r,\Theta)\right)(t,\cdot)\,\mathrm{d}x++k_1\int_0^T\int_\Omega\left|\nabla_\epsilon\mathbf{u}-\nabla_\epsilon\mathbf{w}\right|^2\mathrm{d}x\mathrm{d}t\\ +\int_0^T\int_\Omega\left(\frac{\mathbf{q}\left(\vartheta,\nabla_\epsilon\vartheta\right)\cdot\nabla_\epsilon\Theta}{\vartheta}-\frac{\Theta}{\vartheta}\frac{\mathbf{q}\left(\vartheta,\nabla_\epsilon\vartheta\right)\cdot\nabla_\epsilon\vartheta}{\vartheta}\right)\mathrm{d}x\mathrm{d}t\\ +\int_0^T\int_\Omega\left(\left(\Theta-\vartheta\right)\frac{\mathbf{q}_h\left(\Theta,\nabla_h\Theta\right)\cdot\nabla_h\Theta}{\Theta^2}+\frac{\mathbf{q}\left(\Theta,\nabla_\epsilon\Theta\right)\cdot\nabla_\epsilon\left(\vartheta-\Theta\right)}{\Theta}\right)\mathrm{d}x\mathrm{d}t\\ \leq\int_\Omega\left(\frac{1}{2}\varrho_0\left|\mathbf{u}_0-\mathbf{w}\left(0,\cdot\right)\right|^2+\mathcal{E}(\varrho_0,\vartheta_0\mid r\left(0,\cdot\right),\Theta\left(0,\cdot\right))\right)\mathrm{d}x+H_\epsilon\\ +\int_0^T\left[\delta\left\|\mathbf{w}-\mathbf{u}\right\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)}+C\left(\delta;r,\overline{\mathbf{w}},\Theta\right)\int_\Omega\left(\frac{1}{2}\varrho\left|\mathbf{u}-\mathbf{w}\right|^2+\mathcal{E}(\varrho,\vartheta\mid r,\Theta)\right)\mathrm{d}x\right]\mathrm{d}t\end{aligned}\tag{3.3.42}$$

For the remaining terms, the procedure is exactly as in [43]. We end up with

$$\begin{aligned}\int_\Omega\left(\frac{1}{2}\varrho\left|\mathbf{u}-\mathbf{w}\right|^2+\mathcal{E}(\varrho,\vartheta\mid r,\Theta)\right)(t,\cdot)\,\mathrm{d}x+k_1\int_0^T\int_\Omega\left|\nabla_\epsilon\mathbf{u}-\nabla_\epsilon\mathbf{w}\right|^2\mathrm{d}x\mathrm{d}t\\ k_2\int_0^T\int_\Omega\left|\nabla_\epsilon\vartheta-\nabla_\epsilon\Theta\right|^2\mathrm{d}x\mathrm{d}t+k_3\int_0^T\int_\Omega\left|\nabla_\epsilon\log\vartheta-\nabla_\epsilon\log\Theta\right|^2\mathrm{d}x\mathrm{d}t\\ \leq\int_\Omega\left(\frac{1}{2}\varrho_0\left|\mathbf{u}_0-\mathbf{w}\left(0,\cdot\right)\right|^2+\mathcal{E}(\varrho_0,\vartheta_0\mid r\left(0,\cdot\right),\Theta\left(0,\cdot\right))\right)\mathrm{d}x+H_\epsilon\\ k_4\int_0^T\int_\Omega\left(\frac{1}{2}\varrho\left|\mathbf{u}-\mathbf{w}\right|^2+\mathcal{E}(\varrho,\vartheta\mid r,\Theta)\right)\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.3.43}$$

The positive constants k_j depends on $(r,\overline{\mathbf{w}},\Theta)$ through the norms involved in Theorem 27 and $H_\epsilon\to0$ as $\epsilon\to0$. The Gronwall lemma finishes the proof.

3.4 Conclusions

The problem we faced above has focused on the dimension reduction limit for a compressible heat conducting fluid in which the analysis on the gravity force has played the main role. We believe that the strategy used, or its analogue, could be applied for other kind of models describing systems in which the dynamics is essentially two-dimensional due to the predominance of gravitational effects. Moreover, further extensions of the above problem are not excluded. For example, fluids where the magnetic field is taken into account.

Chapter 4

Global regularity for incompressible fluids

We consider the incompressible Navier-Stokes equations in whole space $\mathbb{R}^3$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} - \mu \Delta_x \mathbf{u} + \nabla_x p = \mathbf{f}, \quad \mathrm{div}_x \mathbf{u} = 0. \tag{4.0.1}$$

The shear viscosity coefficient μ is assumed to be constant and without loss of generality we put $\mu = 1$. Moreover, we put $\mathbf{f} \equiv 0$ for simplicity.

In the following we will discuss some preliminary results necessary for our analysis. In particular, we will introduce the the anisotropic Lebesgue spaces as key tool of our analysis and the so-called Troisi inequality, proving several Lemmas.

4.1 Preliminary results

First, we define the anisotropic Lebesgue spaces.

Definition 33. Let $\bar{p} = (p_1, p_2, p_3)$, $p_i \in [1, \infty], i = 1, 3$. We say that a function f belongs to $L^{\bar{p}}$ if f is measurable on $\mathbb{R}^3$ and the following norm is finite:

$$||f||_{L^{\bar{p}}} \equiv \left\| \left\| \|f\|_{L_1^{p_1}} \right\|_{L_2^{p_2}} \right\|_{L_3^{p_3}}$$

$$:= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, x_3)|^{p_1} \, dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} dx_3 \right)^{\frac{1}{p_3}}.$$

Second, we introduce the Troisi inequality, which has been proved in [110].

Lemma 3. *(Troisi inequality) Suppose that* $r, p_1, p_2, p_3 \in (1, \infty)$ *and*

$$1 + \frac{3}{r} = \sum_{i=1}^{3} \frac{1}{p_i}.$$

Then there exists a constant $c > 0$ such that for every $f \in L^2 \cap C^\infty$

$$\|f\|_r \le c \prod_{i=1}^{3} \|\partial_i f\|_{p_i}^{1/3}. \tag{4.1.1}$$

Now, the following inequality generalizes the Troisi inequality.

Lemma 4. *(Generalized Troisi inequality) Let $r \in (1, \infty)$. Suppose that $\alpha \in (1, \infty)$, $\gamma_1, \gamma_2, \gamma_3 \in (0, 1)$ and $\gamma_1 + \gamma_2 + \gamma_3 = 1$. Let the following conditions be satisfied:*

$$\frac{(\alpha - 1)\, r}{\alpha\gamma_1 r - \alpha + 1} > 1, \tag{4.1.2}$$

$$\frac{r}{\alpha\gamma_2 r - 1} > 1, \tag{4.1.3}$$

$$\frac{r}{\alpha\gamma_3 r - 1} > 1, \tag{4.1.4}$$

$$\frac{(\alpha - 1)\, r}{\alpha\gamma_3 r - 1} > 1. \tag{4.1.5}$$

Then there exists a constant $c > 0$ such that for every $f \in L^2 \cap C^\infty$

$$\|\mathbf{u}\|_r \le \|\partial_1 \mathbf{u}\|_{\frac{r}{r - \frac{\alpha\gamma_1 r}{\alpha - 1} + 1}}^{\frac{\alpha-1}{\alpha+1}} \|\partial_2 \mathbf{u}\|_{\frac{r}{r - \alpha\gamma_2 r + 1}}^{\frac{1}{\alpha+1}} \left\| \|\partial_3 \mathbf{u}\|_{L_{23}^{\frac{r}{r - \alpha\gamma_3 r + 1}}} \right\|_{L_1^{\frac{r}{\alpha r - r - \alpha\gamma_3 r + 1}}}^{\frac{1}{\alpha+1}}. \tag{4.1.6}$$

Remark 34. Let $r \in (3/2, \infty)$, $p_1, p_2, p_3 \in (1, \infty)$, $1 + 3/r = \sum_{i=1}^{3} 1/p_i$. Then, putting in the previous lemma $\alpha = 2$, $\gamma_i = (p_i r + p_i - r)/(2p_i r)$, the conditions (4.1.2) - (4.1.5) are satisfied and (4.1.6) yields (4.1.1). So, for $r \in (3/2, \infty)$ the Troisi inequality can be viewed as a special case of Lemma 4.

Proof. By the use of the density argument we can suppose that $f \in C_0^\infty(\mathbb{R}^3)$. Define

$$f(x_1, x_2) = \sup_{x_3} |\mathbf{u}(x_1, x_2, x_3)|^{\gamma_3},$$
$$g(x_1, x_3) = \sup_{x_2} |\mathbf{u}(x_1, x_2, x_3)|^{\gamma_2},$$
$$h(x_2, x_3) = \sup_{x_1} |\mathbf{u}(x_1, x_2, x_3)|^{\gamma_1}.$$

Then

$$\left(\int_{\mathbb{R}} |\mathbf{u}(x_1, x_2, x_3)|^r \, dx_3 \right)^{\frac{1}{r}} \le \left(\int_{\mathbb{R}} f^r g^r h^r \, dx_3 \right)^{\frac{1}{r}}$$
$$\le f(x_1, x_2) \left(\int_{\mathbb{R}} g^r(x_1, x_3) h^r(x_2, x_3) dx_3 \right)^{\frac{1}{r}}$$
$$\le f(x_1, x_2) \left(\int_{\mathbb{R}} g^{\alpha r}(x_1, x_3) dx_3 \right)^{\frac{1}{\alpha r}} \left(\int_{\mathbb{R}} h^{\frac{\alpha r}{\alpha - 1}}(x_2, x_3) dx_3 \right)^{\frac{\alpha-1}{\alpha r}}.$$

It follows that

$$\left(\int_{\mathbb{R}^2} |\mathbf{u}(x_1, x_2, x_3)|^r \, dx_2 dx_3 \right)^{\frac{1}{r}}$$

$$\le \left(\int_{\mathbb{R}} g^{\alpha r}(x_1,x_3)dx_3\right)^{\frac{1}{\alpha r}} \left(\int_{\mathbb{R}} f^r(x_1,x_2)\left(\int_{\mathbb{R}} h^{\frac{\alpha r}{\alpha-1}}(x_2,x_3)dx_3\right)^{\frac{\alpha-1}{\alpha}} dx_2\right)^{\frac{1}{r}}$$

$$\le \left(\int_{\mathbb{R}} g^{\alpha r}(x_1,x_3)dx_3\right)^{\frac{1}{\alpha r}} \left(\int_{\mathbb{R}} f^{\alpha r}(x_1,x_2)dx_2\right)^{\frac{1}{\alpha r}} \left(\int_{\mathbb{R}^2} h^{\frac{\alpha r}{\alpha-1}}(x_2,x_3)dx_2dx_3\right)^{\frac{\alpha-1}{\alpha r}}$$

and

$$\left(\int |\mathbf{u}(x)|^r\, dx\right)^{\frac{1}{r}}$$

$$\le \left(\int_{\mathbb{R}^2} h^{\frac{\alpha r}{\alpha-1}}(x_2,x_3)dx_2dx_3\right)^{\frac{\alpha-1}{\alpha r}}$$

$$\cdot \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} g^{\alpha r}(x_1,x_3)dx_3\right)^{\frac{1}{\alpha}} \left(\int_{\mathbb{R}} f^{\alpha r}(x_1,x_2)dx_2\right)^{\frac{1}{\alpha}} dx_1\right)^{\frac{1}{r}}$$

$$\le \left(\int_{\mathbb{R}^2} h^{\frac{\alpha r}{\alpha-1}}(x_2,x_3)dx_2dx_3\right)^{\frac{\alpha-1}{\alpha r}} \left(\int_{\mathbb{R}^2} g^{\alpha r}(x_1,x_3)dx_1dx_3\right)^{\frac{1}{\alpha r}}$$

$$\cdot \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f^{\alpha r}(x_1,x_2)dx_2\right)^{\frac{1}{\alpha-1}} dx_1\right)^{\frac{\alpha-1}{\alpha r}}. \tag{4.1.7}$$

Now, we will estimate all three terms on the right hand side of (4.1.7). We have

$$\left(\int_{\mathbb{R}^2} g^{\alpha r}(x_1,x_3)dx_1dx_3\right)^{\frac{1}{\alpha r}} \le \left(\int_{\mathbb{R}^2} \sup_{x_2} |\mathbf{u}(x_1,x_2,x_3)|^{\alpha\gamma_2 r}\, dx_1dx_3\right)^{\frac{1}{\alpha r}}$$

$$\le C\left(\int |\mathbf{u}(x_1,x_2,x_3)|^{\alpha\gamma_2 r-1} |\partial_2\mathbf{u}(x_1,x_2,x_3)|\, dx\right)^{\frac{1}{\alpha r}}$$

$$\le C\,\|\mathbf{u}\|_r^{\frac{\alpha\gamma_2 r-1}{\alpha r}} \|\partial_2\mathbf{u}\|_{\frac{r}{r-\alpha\gamma_2 r+1}}^{\frac{1}{\alpha r}}. \tag{4.1.8}$$

Above we used the condition (4.1.3). Analogically, using (4.1.2), we obtain

$$\left(\int_{\mathbb{R}^2} h^{\frac{\alpha r}{\alpha-1}}(x_2,x_3)dx_2dx_3\right)^{\frac{\alpha-1}{\alpha r}} \le C\,\|\mathbf{u}\|_r^{\frac{\alpha\gamma_1 r-\alpha+1}{\alpha r}} \|\partial_1\mathbf{u}\|_{\frac{r}{r-\frac{\alpha\gamma_1 r}{\alpha-1}+1}}^{\frac{\alpha-1}{\alpha r}}. \tag{4.1.9}$$

At last, using (4.1.4) and (4.1.5) we get

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f^{\alpha r}(x_1,x_2)dx_2\right)^{\frac{1}{\alpha-1}} dx_1\right)^{\frac{\alpha-1}{\alpha r}}$$

$$\le \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \sup_{x_3} |\mathbf{u}(x_1,x_2,x_3)|^{\alpha\gamma_3 r}\, dx_2\right)^{\frac{1}{\alpha-1}} dx_1\right)^{\frac{\alpha-1}{\alpha r}}$$

$$\le C\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |\mathbf{u}(x_1,x_2,x_3)|^{\alpha\gamma_3 r-1} |\partial_3\mathbf{u}(x_1,x_2,x_3)|\, dx_2dx_3\right)^{\frac{1}{\alpha-1}} dx_1\right)^{\frac{\alpha-1}{\alpha r}}$$

$$\leq C\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}|\mathbf{u}(x_1,x_2,x_3)|^r\,dx_2dx_3\right)^{\frac{\alpha\gamma_3 r-1}{(\alpha-1)r}}\left(\int_{\mathbb{R}^2}|\partial_3\mathbf{u}(x_1,x_2,x_3)|^{\frac{r}{r-\alpha\gamma_3 r+1}}\,dx_2dx_3\right)^{\frac{r-\alpha\gamma_3 r+1}{(\alpha-1)r}}dx_1\right)^{\frac{\alpha-1}{\alpha r}}$$

$$\leq C\,\|\mathbf{u}\|_r^{\frac{\alpha\gamma_3 r-1}{\alpha r}}\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2}|\partial_3\mathbf{u}(x_1,x_2,x_3)|^{\frac{r}{r-\alpha\gamma_3 r+1}}\,dx_2dx_3\right)^{\frac{r-\alpha\gamma_3 r+1}{(\alpha-1)r-\alpha\gamma_3 r+1}}dx_1\right)^{\frac{(\alpha-1)r-\alpha\gamma_3 r+1}{\alpha r^2}}$$

and

$$\left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}}f^{\alpha r}(x_1,x_2)dx_2\right)^{\frac{1}{\alpha-1}}dx_1\right)^{\frac{\alpha-1}{\alpha r}}$$
$$\leq C\,\|\mathbf{u}\|_r^{\frac{\alpha\gamma_3 r-1}{\alpha r}}\left\|\|\partial_3\mathbf{u}\|_{L_{23}^{\frac{r}{r-\alpha\gamma_3 r+1}}}\right\|_{L_1^{\frac{r}{(\alpha-1)r-\alpha\gamma_3 r+1}}}^{\frac{1}{\alpha r}}. \qquad (4.1.10)$$

It follows from (4.1.7) - (4.1.10) that

$$\|\mathbf{u}\|_r\leq\|\mathbf{u}\|_r^{\frac{\alpha\gamma_2 r-1}{\alpha r}+\frac{\alpha\gamma_1 r-\alpha+1}{\alpha r}+\frac{\alpha\gamma_3 r-1}{\alpha r}}$$
$$\times\|\partial_1\mathbf{u}\|_{\frac{r}{r-\frac{\alpha}{\alpha-1}\gamma_1 r+1}}^{\frac{\alpha-1}{\alpha r}}\|\partial_2\mathbf{u}\|_{\frac{r}{r-\alpha\gamma_2 r+1}}^{\frac{1}{\alpha r}}\left\|\|\partial_3\mathbf{u}\|_{L_{23}^{\frac{r}{r-\alpha\gamma_3 r+1}}}\right\|_{L_1^{\frac{r}{(\alpha-1)r-\alpha\gamma_3 r+1}}}^{\frac{1}{\alpha r}}$$

and (4.1.6) follows immediately. □

The following key lemma is a slight generalization of Lemma 2.2 from [114]. We use here the Fourier transform, which is defined in a standard way, namely $\hat{f}(\xi)=\int_{\mathbb{R}^d}e^{-ix\cdot\xi}f(x)dx$, $x,\xi\in\mathbb{R}^d$, $d\in\mathbb{N}$.

Lemma 5. *Let $p,q,r\in[2,\infty)$ and $1/p+1/q+1/r-1/2\geq 0$. Then there exists a constant c such that for every $f\in L^2\cap C^\infty$*

$$\left\|\left\|\|f\|_{L_1^p}\right\|_{L_2^q}\right\|_{L_3^r}\leq c\,\|\partial_3 f\|_2^{\frac{r-2}{2r}}\|\partial_2 f\|_2^{\frac{q-2}{2q}}\|\partial_1 f\|_2^{\frac{p-2}{2p}}\|f\|_2^{\frac{1}{r}+\frac{1}{q}+\frac{1}{p}-\frac{1}{2}}.$$

Proof. By the use of the density argument we can suppose that $f\in C_0^\infty(\mathbb{R}^3)$. At first, let us remind a well known definition of the homogeneous Sobolev spaces. Let $s\in\mathbb{R}$, $d\in\mathbf{N}$. Then

$$\dot{H}^s(\mathbb{R}^d)\equiv\dot{H}^s:=\left\{f\in S';\hat{f}\in L^1_{loc}\text{ and }\|f\|_{\dot{H}^s}:=\left(\int|\xi|^{2s}\left|\widehat{f}(\xi)\right|^2d\xi\right)^{\frac{1}{2}}<\infty\right\},$$

where S' denotes the space of the tempered distributions on $\mathbb{R}^d$. It is well known that

$$\dot{H}^s\hookrightarrow L^{\frac{2d}{d-2s}};\quad s\in\left[0,\frac{d}{2}\right);\quad d\in\mathbf{N}. \qquad (4.1.11)$$

Define

$$\mathfrak{F}_1 f(\xi_1,x_2,x_3):=\int e^{-i\xi_1 x_1}f(x_1,x_2,x_3)dx_1$$

and analogically $\mathfrak{F}_j$ for $j=2,3$. Define further the operator $\Lambda_1^s, s\in\mathbb{R}$ in the following way

$$\mathfrak{F}_1(\Lambda_1^s f)(\xi_1,x_2,x_3):=|\xi_1|^s\,\mathfrak{F}_1 f(\xi_1,x_2,x_3)$$

and again analogically we can define Λ_j^s for $j = 2, 3$. Clearly, using (4.1.11) for $d = 1$ and the Plancherel theorem we have

$$\|f\|_{L_1^p} \leq \left\| \Lambda_1^{\frac{p-2}{2p}} f \right\|_{L_1^2}. \tag{4.1.12}$$

So combining (4.1.12) and the Minkowski inequality

$$\left\| \left\| \|f\|_{L_1^p} \right\|_{L_2^q} \right\|_{L_3^r} \leq \left\| \left\| \left\| \Lambda_1^{\frac{p-2}{2p}} f \right\|_{L_1^2} \right\|_{L_2^q} \right\|_{L_3^r} \leq \left\| \left\| \left\| \Lambda_1^{\frac{p-2}{2p}} f \right\|_{L_2^q} \right\|_{L_1^2} \right\|_{L_3^r}$$

$$\leq \left\| \left\| \Lambda_2^{\frac{q-2}{2q}} \left(\Lambda_1^{\frac{p-2}{2p}} f \right) \right\|_{L_{12}^2} \right\|_{L_3^r} \leq \left\| \left\| \Lambda_2^{\frac{q-2}{2q}} \left(\Lambda_1^{\frac{p-2}{2p}} f \right) \right\|_{L_3^r} \right\|_{L_{12}^2} \leq \left\| \Lambda_3^{\frac{r-2}{2r}} \left(\Lambda_2^{\frac{q-2}{2q}} \left(\Lambda_1^{\frac{p-2}{2p}} f \right) \right) \right\|_2. \tag{4.1.13}$$

Let $\mathfrak{F}$ denotes the Fourier transform $\mathfrak{F}f(\xi) = \int e^{-ix\cdot\xi} f(x)dx$. Using the Fubini theorem and the definition of the operators $\mathfrak{F}_j$ and Λ_j^s, $j = 1, 2, 3$, we have

$$\mathfrak{F}\Big(\Lambda_3^{\frac{r-2}{2r}} \Big(\Lambda_2^{\frac{q-2}{2q}} \Big(\Lambda_1^{\frac{p-2}{2p}} f \Big) \Big) \Big)(\xi)$$

$$= \int e^{-ix_1\xi_1} \int e^{-ix_2\xi_2} \int e^{-ix_3\xi_3} \Lambda_3^{\frac{r-2}{2r}} \Big(\Lambda_2^{\frac{q-2}{2q}} \Big(\Lambda_1^{\frac{p-2}{2p}} f \Big) \Big)(x_1, x_2, x_3) dx_3 dx_2 dx_1$$

$$= \int e^{-ix_1\xi_1} \int e^{-ix_2\xi_2} \mathfrak{F}_3 \Big(\Lambda_3^{\frac{r-2}{2r}} \Big(\Lambda_2^{\frac{q-2}{2q}} \Big(\Lambda_1^{\frac{p-2}{2p}} f \Big) \Big) \Big)(x_1, x_2, \xi_3) dx_2 dx_1$$

$$= |\xi_3|^{\frac{r-2}{2r}} \int e^{-ix_1\xi_1} \int e^{-ix_2\xi_2} \mathfrak{F}_3 \Big(\Lambda_2^{\frac{q-2}{2q}} \Big(\Lambda_1^{\frac{p-2}{2p}} f \Big) \Big)(x_1, x_2, \xi_3) dx_2 dx_1$$

$$= |\xi_3|^{\frac{r-2}{2r}} \int e^{-ix_1\xi_1} \int e^{-ix_2\xi_2} \int e^{-ix_3\xi_3} \Lambda_2^{\frac{q-2}{2q}} \Big(\Lambda_1^{\frac{p-2}{2p}} f \Big)(x_1, x_2, x_3) dx_3 dx_2 dx_1$$

$$= |\xi_3|^{\frac{r-2}{2r}} \int e^{-ix_3\xi_3} \int e^{-ix_1\xi_1} \int e^{-ix_2\xi_2} \Lambda_2^{\frac{q-2}{2q}} \Big(\Lambda_1^{\frac{p-2}{2p}} f \Big)(x_1, x_2, x_3) dx_2 dx_1 dx_3$$

$$= |\xi_3|^{\frac{r-2}{2r}} \int e^{-ix_3\xi_3} \int e^{-ix_1\xi_1} \mathfrak{F}_2 \Big(\Lambda_2^{\frac{q-2}{2q}} \Big(\Lambda_1^{\frac{p-2}{2p}} f \Big) \Big)(x_1, \xi_2, x_3) dx_1 dx_3$$

$$= |\xi_3|^{\frac{r-2}{2r}} |\xi_2|^{\frac{q-2}{2q}} \int e^{-ix_3\xi_3} \int e^{-ix_1\xi_1} \mathfrak{F}_2 \Big(\Lambda_1^{\frac{p-2}{2p}} f \Big)(x_1, \xi_2, x_3) dx_1 dx_3$$

$$= |\xi_3|^{\frac{r-2}{2r}} |\xi_2|^{\frac{q-2}{2q}} \int e^{-ix_3\xi_3} \int e^{-ix_1\xi_1} \int e^{-ix_2\xi_2} \Lambda_1^{\frac{p-2}{2p}} f(x_1, x_2, x_3) dx_2 dx_1 dx_3$$

$$= |\xi_3|^{\frac{r-2}{2r}} |\xi_2|^{\frac{q-2}{2q}} \int e^{-ix_3\xi_3} \int e^{-ix_2\xi_2} \int e^{-ix_1\xi_1} \Lambda_1^{\frac{p-2}{2p}} f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

$$= |\xi_3|^{\frac{r-2}{2r}} |\xi_2|^{\frac{q-2}{2q}} \int e^{-ix_3\xi_3} \int e^{-ix_2\xi_2} \mathfrak{F}_1 \Big(\Lambda_1^{\frac{p-2}{2p}} f \Big)(\xi_1, x_2, x_3) dx_2 dx_3$$

$$= |\xi_3|^{\frac{r-2}{2r}} |\xi_2|^{\frac{q-2}{2q}} |\xi_1|^{\frac{p-2}{2p}} \int e^{-ix_3\xi_3} \int e^{-ix_2\xi_2} \mathfrak{F}_1 f(\xi_1, x_2, x_3) dx_2 dx_3$$

$$= |\xi_3|^{\frac{r-2}{2r}} |\xi_2|^{\frac{q-2}{2q}} |\xi_1|^{\frac{p-2}{2p}} \int e^{-ix_3\xi_3} \int e^{-ix_2\xi_2} \int e^{-ix_1\xi_1} f(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

$$= |\xi_3|^{\frac{r-2}{2r}} |\xi_2|^{\frac{q-2}{2q}} |\xi_1|^{\frac{p-2}{2p}} \mathfrak{F}f(\xi).$$

So using the last equality together with the Plancherel theorem we can continue with (4.1.13) and complete the proof

$$\left\| \left\| \|f\|_{L^p_1} \right\|_{L^q_2} \right\|_{L^r_3} \leq \Big(\int |\xi_3|^{\frac{r-2}{r}} |\xi_2|^{\frac{q-2}{q}} |\xi_1|^{\frac{p-2}{p}} |\mathfrak{F}f(\xi)|^2 d\xi \Big)^{\frac{1}{2}}$$

$$= \Big(\int |\xi_3|^{\frac{r-2}{r}} |\mathfrak{F}f(\xi)|^{\frac{r-2}{r}} |\xi_2|^{\frac{q-2}{q}} |\mathfrak{F}f(\xi)|^{\frac{q-2}{q}} |\xi_1|^{\frac{p-2}{p}} |\mathfrak{F}f(\xi)|^{\frac{p-2}{p}} |\mathfrak{F}f(\xi)|^{2(\frac{1}{r}+\frac{1}{q}+\frac{1}{p})-1} d\xi \Big)^{\frac{1}{2}}$$

$$\leq \|\partial_3 f\|_2^{\frac{r-2}{2r}} \|\partial_2 f\|_2^{\frac{q-2}{2q}} \|\partial_1 f\|_2^{\frac{p-2}{2p}} \|f\|_2^{\frac{1}{r}+\frac{1}{q}+\frac{1}{p}-\frac{1}{2}}.$$

□

4.2 State of art and main results

In the following we will sum up the present state of art concerning our analysis. Then, we will present our main results.

4.2.1 State of art

Let us sum up the present state of the art. The best result concerning u_3 has been proved in [117], Theorem 1. The regularity of a solution on $(0, T]$ is ensured if $u_3 \in L^\beta(0, T; L^p)$, where

$$\frac{2}{\beta} + \frac{3}{p} \leq \frac{3}{4} + \frac{1}{2p}, \quad p \in \Big(\frac{10}{3}, \infty\Big]. \tag{4.2.1}$$

The condition (4.2.1) is not optimal for any p.

The results for ∇u_3 are optimal for $p \in (3/2, 2]$. The solution is regular on $(0, T]$ if $\nabla u_3 \in L^\beta(0, T; L^p)$, where

$$\frac{2}{\beta} + \frac{3}{p} \leq 2, \quad p \in \Big(\frac{3}{2}, \frac{9}{5}\Big], \quad \text{see [15]} \tag{4.2.2}$$

$$\frac{2}{\beta} + \frac{3}{p} \leq 2, \quad p \in \Big(\frac{9}{5}, 2\Big), \quad \text{see [14]} \tag{4.2.3}$$

$$\frac{2}{\beta} + \frac{3}{p} \leq 2, \quad p = 2, \quad \text{see [111]} \tag{4.2.4}$$

$$\frac{2}{\beta} + \frac{3}{p} \leq \frac{59}{30}, \quad p \in \Big(2, \frac{30}{13}\Big], \quad \text{see [101]} \tag{4.2.5}$$

$$\frac{2}{\beta} + \frac{3}{p} \leq \frac{7}{4} + \frac{1}{2p}, \quad p \in \Big(\frac{30}{13}, 3\Big), \quad \text{see [101]} \tag{4.2.6}$$

$$\frac{2}{\beta} + \frac{3}{p} \leq \frac{7}{4} + \frac{1}{2p}, \quad p \in \Big[3, \frac{10}{3}\Big), \quad \text{see [116]} \tag{4.2.7}$$

$$\frac{2}{\beta} + \frac{3}{p} \leq \frac{7}{4} + \frac{1}{2p}, \quad p \in \Big[\frac{10}{3}, \infty\Big), \quad \text{see [100].} \tag{4.2.8}$$

Remark 35. In fact in [14] the authors proved the following result: if moreover the vorticity $\nabla \times u_0 \in L^{3/2}$ then Leray solutions satisfying $u_3 \in L^q(0, T; \dot{H}^{1/2+2/q})$,

$q \in (4,6)$, are regular on $(0,T]$. It is obvious that (4.2.3) follows as a direct consequence, namely if $\nabla u_3 \in L^q(0,T;L^p)$, where $2/q + 3/p = 2$ and $p \in (9/5, 2)$ and $q \in (4,6)$, then $\nabla u_3 \in L^q(0,T;\dot{H}^{2/q-1/2})$ and $u_3 \in L^q(0,T;\dot{H}^{2/q+1/2})$. Applying the criterion from [14] gives the regularity of u. The criterion from [15] is the extension of the result from [14] for $q \in (4,\infty)$ and it implies immediately (4.2.2).

Concerning $\nabla^2 u_3$ the following result has been proved in [115]. The regularity of u is ensured on $(0,T]$ provided

$$\partial_1\partial_3 u_3, \partial_2\partial_3 u_3 \in L^\beta(0,T;L^p),\ \frac{2}{\beta} + \frac{3}{p} \leq 2 + \frac{1}{p},\ p \in (1,\infty).$$

An almost regular result is so achieved for $p \to 1_+$. It is also noteworthy that the condition is imposed here only on two items of the Hessian tensor.

4.2.2 Main results

We now present the main results. The following Theorem 36 is a slight generalization of a result from [114]. It is interesting that for $p_1 \to 2_+$, $p_2 \to 2_+$, the criterion is almost optimal.

Theorem 36. *Let* $\mathbf{u} = (u_1, u_2, u_3)$ *be a weak solution to (4.0.1) corresponding to the initial condition* $\mathbf{u}_0 \in W^{1,2}_\sigma$ *which satisfies the energy inequality. Suppose that* $p_1, p_2, p_3 \in (2,\infty]$, $3/(4p_1) + 3/(4p_2) + 1/p_3 \leq 3/4$, $\beta \in (2,\infty]$, $\overline{p} = (p_1, p_2, p_3)$ *and*

$$u_3 \in L^\beta\left(0,T;L^{\overline{p}}\right).$$

Then the condition

$$\frac{2}{\beta} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{3}{4} + \frac{1}{4p_1} + \frac{1}{4p_2} \tag{4.2.9}$$

ensures the regularity of $\mathbf{u}$ *on* $(0,T]$.

Putting $p_1 = p_2 = p_3 = p$ in Theorem 36, (4.2.9) reduces to (4.2.1) with one slight improvement, the value $p = 10/3$ is now allowed. So Theorem 36 can also be understood as a generalization of the above mentioned result from [117].

Remark 37. The result from Theorem 36 formulated in the framework of the anisotropic Lebesgue spaces is almost optimal which is not the case for the corresponding result formulated in the framework of the standard Lebesgue spaces (see the result from [117], Theorem 1).

The following Theorem 38 improves the above mentioned result from [101] (see (4.2.5)). It is due to the fact that while the proof from [101] has been based on the Troisi inequality, the proof of Theorem 38 uses a generalized version of the Troisi inequality using the anisotropic Lebesgue spaces (see Lemma 4).

Theorem 38. *Let* $\mathbf{u} = (u_1, u_2, u_3)$ *be a weak solution to (4.0.1) corresponding to the initial condition* $\mathbf{u}_0 \in W^{1,2}_\sigma$ *which satisfies the energy inequality. Suppose that* $\beta \in (2,\infty)$ *and*

$$\nabla u_3 \in L^\beta\left(0,T;L^p\right),$$

where

$$\frac{2}{\beta} + \frac{3}{p} < \frac{75}{38}, \quad p \in \left(2, \frac{38}{17}\right) \tag{4.2.10}$$

and

$$\frac{2}{\beta}+\frac{3}{p}<\frac{7}{4}+\frac{1}{2p}, \quad p\in\Big[\frac{38}{17},\infty\Big). \tag{4.2.11}$$

Then $\mathbf{u}$ *is regular on* $(0,T]$.

Moreover, we have the following Theorem

Theorem 39. *Let* $\mathbf{u}=(u_1,u_2,u_3)$ *be a weak solution to (4.0.1) corresponding to the initial condition* $\mathbf{u}_0\in W^{1,2}_\sigma$ *which satisfies the energy inequality. Suppose that*

$$\nabla u_3 \in L^\beta\left(0,T;L^{\overline{p}}\right) \tag{4.2.12}$$

where

$$\overline{p}=(p_1,p_2,p_3), \quad p_i\in(1,\infty], \quad i=1,2,3, \quad \beta\in(1,\infty].$$

Suppose that there exist numbers $q_i, r_i\in[2,\infty), i=1,2,3$ *such that*

$$\frac{1}{p_i}+\frac{1}{q_i}+\frac{1}{r_i}=1, i=1,2,3, \tag{4.2.13}$$

$$\frac{3}{4q_1}+\frac{3}{4q_2}+\frac{1}{q_3}\geq\frac{1}{2}, \tag{4.2.14}$$

$$\sum_{i=1}^{3}\frac{1}{r_i}\geq\frac{1}{2}. \tag{4.2.15}$$

Then the condition

$$\frac{2}{\beta}+\sum_{i=1}^{3}\frac{1}{p_i}=2-\frac{1}{4q_1}-\frac{1}{4q_2} \tag{4.2.16}$$

ensures the regularity of $\mathbf{u}$ *on* $(0,T]$.

The following Theorem 40 is a consequence of Theorem 39.

Theorem 40. *Let* $\mathbf{u}=(u_1,u_2,u_3)$ *be a weak solution to (4.0.1) corresponding to the initial condition* $\mathbf{u}_0\in W^{1,2}_\sigma$ *which satisfies the energy inequality. Suppose that*

$$\nabla u_3\in L^\beta\left(0,T;L^{\overline{p}}\right),$$

where

$$\overline{p}=(p_1,p_2,p_3), \quad p_1,p_2\in(1,\infty], p_3\in[2,\infty], \quad \beta\in(1,\infty].$$

Suppose further that if $p_1,p_2\in(2,\infty]$ *then*

$$\frac{2}{\beta}+\sum_{i=1}^{3}\frac{1}{p_i}\leq\frac{7}{4}+\frac{1}{4}\Big(\frac{1}{p_1}+\frac{1}{p_2}\Big) \tag{4.2.17}$$

and if at least one of the numbers p_1 *and* p_2 *is not in* $(2,\infty]$ *then*

$$\frac{2}{\beta}+\sum_{i=1}^{3}\frac{1}{p_i}<\frac{7}{4}+\frac{1}{4}\Big(\frac{1}{\max(p_1,2)}+\frac{1}{\max(p_2,2)}\Big).$$

Then $\mathbf{u}$ *is regular on* $(0,T]$.

Putting in Theorem 40 $p_1 = p_2 = p_3 \in (2, \infty]$, we obtain the following Corollary 41. It further improves the above mentioned result from [101] (see (4.2.5)). This improvement is better than the one from Theorem 38 due to the use of the term $\int |\nabla u_3| \, |\nabla \mathbf{u}| \, |\nabla_h \mathbf{u}| \, dx$ instead of $\int |\nabla u_3| \, |\mathbf{u}| \, |\nabla \nabla_h \mathbf{u}| \, dx$ (see the proofs of Theorems 38 and 39), which enables us to use more fully the potential of the anisotropic Lebesgue spaces.

Corollary 41. *Let $\mathbf{u} = (u_1, u_2, u_3)$ be a weak solution to (4.0.1) corresponding to the initial condition $\mathbf{u}_0 \in W^{1,2}_\sigma$ which satisfies the energy inequality. Suppose that*

$$\nabla u_3 \in L^\beta\left(0, T; L^p\right),$$

where

$$\frac{2}{\beta} + \frac{3}{p} \le \frac{7}{4} + \frac{1}{2p}, \;\; p \in (2, \infty).$$

Then $\mathbf{u}$ is regular on $(0, T]$.

The following theorem deals with criteria where conditions are imposed on $\nabla^2 u_3$. Unlike the result from [115], we impose conditions on all items of the Hessian tensor, but unlike [115] we get almost optimal result for a wide range of p.

Theorem 42. *Let $u = (u_1, u_2, u_3)$ be a weak solution to (4.0.1) corresponding to the initial condition $u_0 \in W^{1,2}_\sigma$ which satisfies the energy inequality. Suppose that $\beta \in (1, \infty)$, $p \in (1, 3)$ and*

$$\nabla^2 u_3 \in L^\beta\left(0, T; L^p\right).$$

If, moreover,

$$\frac{2}{\beta} + \frac{3}{p} < 3, \quad p \in (1, 3/2] \tag{4.2.18}$$

or

$$\frac{2}{\beta} + \frac{3}{p} = \frac{5}{2} + \frac{3}{4p}, \quad p \in (3/2, 3), \tag{4.2.19}$$

then $\mathbf{u}$ is regular on $(0, T]$.

4.3 Proofs of main results

In the following, we prove the main results.

4.3.1 Proof of Theorem 36

Proof. Let $T^* = \sup\{\tau > 0; \mathbf{u} \text{ is regular on } (0, \tau)\}$. Since $\mathbf{u}_0 \in W^{1,2}_\sigma$, $\mathbf{u}$ is regular on some positive time interval and T^* is either equal to infinity (in which case the proof is finished) or it is a positive number and $\mathbf{u}$ is regular on $(0, T^*)$, that is $\nabla \mathbf{u} \in L^\infty_{loc}([0, T^*); L^2)$. It is sufficient to prove that $T^* > T$. We proceed by contradiction and suppose that $T^* \le T$. We take $\varepsilon > 0$ sufficiently small (it will be specified later) and fix $T_1 \in (0, T^*)$ such that $||\nabla \mathbf{u}||_{L^2(T_1, T^*; L^2)} < \varepsilon$. Taking arbitrarily $T_2 \in (T_1, T^*)$ the proof will be finished if we show that $||\nabla \mathbf{u}(T_2)||_2 \le C < \infty$, where C is independent of T_2. Actually, the standard extension argument then shows that the regularity of $\mathbf{u}$ can be extended beyond T^* and it contradicts the definition of T^*.

As in [116] we define

$$J(T_2)^2 = \sup_{\tau\in(T_1,T_2)} ||\nabla_h \mathbf{u}(\tau)||_2^2 + \int_{T_1}^{T_2} ||\nabla\nabla_h \mathbf{u}(t)||_2^2 \, dt$$

and

$$L(T_2)^2 = \sup_{\tau\in(T_1,T_2)} ||\partial_3 \mathbf{u}(\tau)||_2^2 + \int_{T_1}^{T_2} ||\nabla\partial_3 \mathbf{u}(t)||_2^2 \, dt,$$

where $\nabla_h \mathbf{u} = (\partial_1 \mathbf{u}, \partial_2 \mathbf{u})$. As was discussed in the first paragraph of this proof, it now suffices to show that $J(T_2)^2 + L(T_2)^2 \le C < \infty$ uniformly in T_2.

To estimate $L(T_2)$ let us fix an arbitrary $\tau \in (T_1, T^*)$, multiply (4.0.1) by $-\partial_{33}\mathbf{u}$ and integrate over $\mathbb{R}^3$ and (T_1,τ). We obtain

$$\frac{1}{2}||\partial_3 \mathbf{u}(\tau)||_2^2 + \int_{T_1}^{\tau} ||\nabla\partial_3 \mathbf{u}(t)||_2^2 \, dt = \frac{1}{2}||\partial_3 \mathbf{u}(T_1)||_2^2 + \int_{T_1}^{\tau}\int u_j\partial_j u_k \partial_{33}^2 u_k \, dx \, dt. \tag{4.3.1}$$

Using integration by parts and the continuity equation, we get

$$\int u_j\partial_j u_k \partial_{33}^2 u_k dx$$

$$= -\int \partial_3 u_j \partial_j u_k \partial_3 u_k dx - \int u_j \partial_{j3}^2 u_k \partial_3 u_k dx = -\int \partial_3 u_j \partial_j u_k \partial_3 u_k dx$$

$$= \sum_{j=1}^{2}\sum_{k=1}^{3}\int u_k \left(\partial_{3j}^2 u_j \partial_3 u_k + \partial_{j3}^2 u_k \partial_3 u_j\right) dx + \sum_{k=1}^{3}\int \left(\partial_1 u_1 + \partial_2 u_2\right)\partial_3 u_k \partial_3 u_k dx$$

$$= \sum_{j=1}^{2}\sum_{k=1}^{3}\int u_k \left(\partial_{3j}^2 u_j \partial_3 u_k + \partial_{j3}^2 u_k \partial_3 u_j\right) dx$$

$$-\sum_{k=1}^{3} 2\int \left(u_1 \partial_3 u_k \partial_{31}^2 u_k + u_2 \partial_3 u_k \partial_{32}^2 u_k\right) dx$$

$$\le c\int |\mathbf{u}|\,|\partial_3 \mathbf{u}|\,|\nabla\nabla_h \mathbf{u}|\, dx$$

$$\le c\,\|\partial_1 \mathbf{u}\|_2^{1/3}\,\|\partial_2 \mathbf{u}\|_2^{1/3}\,\|\partial_3 \mathbf{u}\|_2^{1/3}\,\|\partial_3 \mathbf{u}\|_2^{1/2}\,\|\partial_1\partial_3 \mathbf{u}\|_2^{1/6}\,\|\partial_2\partial_3 \mathbf{u}\|_2^{1/6}\,\|\partial_3\partial_3 \mathbf{u}\|_2^{1/6}\,\|\nabla\nabla_h \mathbf{u}\|_2$$

$$\le c\|\nabla_h \mathbf{u}\|_2^{\frac{2}{3}}\,\|\partial_3 \mathbf{u}\|_2^{1/3}\,\|\partial_3 \mathbf{u}\|_2^{1/2}\,\|\nabla\nabla_h \mathbf{u}\|_2^{\frac{4}{3}}\,\|\partial_3 \nabla \mathbf{u}\|_2^{\frac{1}{6}},$$

where we have also used the Hölder inequality, the interpolation inequality and the Troisi inequality (see Lemma 3). So

$$\int_{T_1}^{\tau}\int u_j\partial_j u_k \partial_{33}^2 u_k \, dx \, dt$$

$$\le c\int_{T_1}^{\tau} \|\nabla_h \mathbf{u}\|_2^{\frac{2}{3}}\,\|\partial_3 \mathbf{u}\|_2^{1/3}\,\|\partial_3 \mathbf{u}\|_2^{1/2}\,\|\nabla\nabla_h \mathbf{u}\|_2^{\frac{4}{3}}\,\|\partial_3 \nabla \mathbf{u}\|_2^{\frac{1}{6}} \, dt$$

$$\leq c\|\nabla_h \mathbf{u}\|^{\frac{2}{3}}_{L^\infty(T_1,\tau;L^2)}\|\partial_3\mathbf{u}\|^{\frac{1}{3}}_{L^\infty(T_1,\tau;L^2)}\|\partial_3\mathbf{u}\|^{\frac{1}{2}}_{L^2(T_1,\tau;L^2)}\|\nabla\nabla_h\mathbf{u}\|^{\frac{4}{3}}_{L^2(T_1,\tau;L^2)}\|\partial_3\nabla\mathbf{u}\|^{\frac{1}{6}}_{L^2(T_1,\tau;L^2)}$$

$$\leq cJ(\tau)^2 L(\tau)^{\frac{1}{2}}.$$

Consequently, the last inequality and (4.3.1) yield

$$\frac{1}{2}||\partial_3\mathbf{u}(\tau)||_2^2+\int_{T_1}^{\tau}||\nabla\partial_3\mathbf{u}(t)||_2^2\,dt \leq \frac{1}{2}||\partial_3\mathbf{u}(T_1)||_2^2+cJ(\tau)^2L(\tau)^{\frac{1}{2}}, \quad \tau\in(T_1,T^*).$$

So specially,

$$\int_{T_1}^{T_2}||\nabla\partial_3\mathbf{u}(t)||_2^2\,dt \leq c + cJ(T_2)^2L(T_2)^{\frac{1}{2}}$$

and due to the fact that J and L are increasing in T_2

$$\sup_{\tau\in(T_1,T_2)}\frac{1}{2}||\partial_3\mathbf{u}(\tau)||_2^2 \leq c + cJ(T_2)^2L(T_2)^{\frac{1}{2}}.$$

So it follows from the definition of $J(T_2)$ and $L(T_2)$ that

$$L(T_2)^2 \leq c + cJ(T_2)^2L(T_2)^{1/2}$$

and consequently

$$L(T_2) \leq c + cJ(T_2)^{4/3}. \tag{4.3.2}$$

The constant c is independent of T_2. It is worthwhile to notice that the estimate of $L(T_2)$ is general and it does not require any additional conditions on $\mathbf{u}$.

To estimate $J(T_2)$ we multiply (4.0.1) by $-\Delta_h\mathbf{u} = -\sum_{j=1}^2 \partial_{jj}^2\mathbf{u}$. We get

$$\frac{1}{2}||\nabla_h\mathbf{u}(T_2)||_2^2+\int_{T_1}^{T_2}||\nabla\nabla_h\mathbf{u}(t)||_2^2\,dt = \frac{1}{2}||\nabla_h\mathbf{u}(T_1)||_2^2+\int_{T_1}^{T_2}\int u_j\partial_ju_k\Delta_hu_k\,dx\,dt. \tag{4.3.3}$$

It is possible to show in a standard way (see, for example [117], proof of Theorem 1 and [61], Lemma 2.2) that

$$\int u_j\partial_ju_k\Delta_hu_kdx \leq c\int |u_3|\,|\nabla\mathbf{u}|\,|\nabla\nabla_h\mathbf{u}|\,dx.$$

So it follows from (4.3.3) that

$$J(T_2)^2 \leq c + c\int_{T_1}^{T_2}\int |u_3|\,|\nabla\mathbf{u}|\,|\nabla\nabla_h\mathbf{u}|\,dxdt.$$

Lemma 5 now yields the following estimate

$$\int |u_3|\,|\nabla\mathbf{u}|\,|\nabla\nabla_h\mathbf{u}|\,dx$$

$$\leq \left\|\left\|\|u_3\|_{L_3^{p_3}}\right\|_{L_2^{p_2}}\right\|_{L_1^{p_1}}\left\|\left\|\|\nabla\mathbf{u}\|_{L_3^{2p_3/(p_3-2)}}\right\|_{L_2^{2p_2/(p_2-2)}}\right\|_{L_1^{2p_1/(p_1-2)}}\|\nabla\nabla_h\mathbf{u}\|_2$$

$$\leq \left\|\left\|\|u_3\|_{L_3^{p_3}}\right\|_{L_2^{p_2}}\right\|_{L_1^{p_1}}\|\partial_1\nabla\mathbf{u}\|_2^{\frac{1}{p_1}}\|\partial_2\nabla\mathbf{u}\|_2^{\frac{1}{p_2}}\|\partial_3\nabla\mathbf{u}\|_2^{\frac{1}{p_3}}\|\nabla\mathbf{u}\|_2^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)}\|\nabla\nabla_h\mathbf{u}\|_2$$

$$\leq \left\| \left\| \|u_3\|_{L_3^{p_3}} \right\|_{L_2^{p_2}} \right\|_{L_1^{p_1}} \|\partial_3 \nabla \mathbf{u}\|_2^{\frac{1}{p_3}} \|\partial_3 \mathbf{u}\|_2^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)} \|\nabla \nabla_h \mathbf{u}\|_2^{1+\frac{1}{p_1}+\frac{1}{p_2}}$$

$$+ \left\| \left\| \|u_3\|_{L_3^{p_3}} \right\|_{L_2^{p_2}} \right\|_{L_1^{p_1}} \|\partial_3 \nabla \mathbf{u}\|_2^{\frac{1}{p_3}} \|\nabla_h \mathbf{u}\|_2^{1-\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)} \|\nabla \nabla_h \mathbf{u}\|_2^{1+\frac{1}{p_1}+\frac{1}{p_2}} = A+B.$$

We now use (4.2.9) and the Hölder inequality gives

$$\int_{T_1}^{T_2} A dt$$

$$\leq \int_{T_1}^{T_2} \left\| \left\| \|u_3\|_{L_3^{p_3}} \right\|_{L_2^{p_2}} \right\|_{L_1^{p_1}} \|\partial_3 \mathbf{u}\|_2^{\frac{3}{4}-\frac{3}{4p_1}-\frac{3}{4p_2}-\frac{1}{p_3}} \|\nabla \mathbf{u}\|_2^{\frac{1}{4}-\frac{1}{4p_1}-\frac{1}{4p_2}} \|\nabla \nabla_h \mathbf{u}\|_2^{1+\frac{1}{p_1}+\frac{1}{p_2}} \|\partial_3 \nabla \mathbf{u}\|_2^{\frac{1}{p_3}} dt$$

$$\leq ||u_3||_{L^{\beta}(T_1,T_2;L^{\bar{p}})} ||\partial_3 \mathbf{u}||_{L^\infty(T_1,T_2;L^2)}^{\frac{3}{4}-\frac{3}{4p_1}-\frac{3}{4p_2}-\frac{1}{p_3}} ||\nabla \mathbf{u}||_{L^2(T_1,T_2;L^2)}^{\frac{1}{4}-\frac{1}{4p_1}-\frac{1}{4p_2}} ||\nabla \nabla_h \mathbf{u}||_{L^2(T_1,T_2;L^2)}^{1+\frac{1}{p_1}+\frac{1}{p_2}} ||\partial_3 \nabla \mathbf{u}||_{L^2(T_1,T_2;L^2)}^{\frac{1}{p_3}}$$

$$\leq c\epsilon^{\frac{1}{4}-\frac{1}{4p_1}-\frac{1}{4p_2}} L(T_2)^{\frac{3}{4}-\frac{3}{4p_1}-\frac{3}{4p_2}} J(T_2)^{1+\frac{1}{p_1}+\frac{1}{p_2}}$$

$$\leq c + c\epsilon^{\frac{1}{4}-\frac{1}{4p_1}-\frac{1}{4p_2}} J(T_2)^2.$$

For the last inequality we used (4.3.2). In the same way

$$\int_{T_1}^{T_2} B dt \leq c\epsilon^{\frac{1}{4}-\frac{1}{4p_1}-\frac{1}{4p_2}} L(T_2)^{\frac{1}{p_3}} J(T_2)^{\frac{7}{4}+\frac{1}{4p_1}+\frac{1}{4p_2}-\frac{1}{p_3}}$$

$$\leq c\epsilon^{\frac{1}{4}-\frac{1}{4p_1}-\frac{1}{4p_2}} J(T_2)^{\frac{7}{4}+\frac{1}{4p_1}+\frac{1}{4p_2}+\frac{1}{3p_3}} \leq c + c\epsilon^{\frac{1}{4}-\frac{1}{4p_1}-\frac{1}{4p_2}} J(T_2)^2.$$

We can conclude that

$$J(T_2)^2 \leq c + c\epsilon^{\frac{1}{4}-\frac{1}{4p_1}-\frac{1}{4p_2}} J(T_2)^2. \tag{4.3.4}$$

Choosing now ε sufficiently small, we can derive from (4.3.2) and (4.3.4) that $J(T_2) + L(T_2)$ is bounded independently of $T_2 \in (T_1, T^*)$ and the proof follows immediately. □

4.3.2 Proof of Theorem 38

Proof. We proceed exactly in the same way as in the proof of Theorem 36 up to the condition (4.3.3). It has been proved in [116] that

$$\int u_j \partial_j u_k \Delta_h u_k dx \leq c \int |\nabla u_3| |\nabla_h \mathbf{u}|^2 dx + c \int |\nabla u_3| |\mathbf{u}| |\nabla \nabla_h \mathbf{u}| dx.$$

So it follows from (4.3.3) that

$$J(T_2)^2 \leq c + c \int_{T_1}^{T_2} \int |\nabla u_3| |\nabla_h \mathbf{u}|^2 dxdt + c \int_{T_1}^{T_2} \int |\nabla u_3| |\mathbf{u}| |\nabla \nabla_h \mathbf{u}| dxdt. \tag{4.3.5}$$

It is possible to prove easily (see also [116]) that

$$\int_{T_1}^{T_2} \int |\nabla u_3| |\nabla_h \mathbf{u}|^2 dxdt \leq c\varepsilon J(T_2)^2. \tag{4.3.6}$$

Further,

$$\int_{T_1}^{T_2}\int |\nabla u_3|\,|\mathbf{u}|\,|\nabla\nabla_h\mathbf{u}|\,dxdt \le \int_{T_1}^{T_2} \|\nabla u_3\|_p\,\|\mathbf{u}\|_r\,\|\nabla\nabla_h\mathbf{u}\|_2\,dt, \tag{4.3.7}$$

where $r = 2p/(p-2)$.

We will now estimate the right hand side of (4.3.7). Suppose that the numbers $\alpha, \gamma_1, \gamma_2, \gamma_3$ satisfy all conditions from Lemma 4. Suppose further that the following conditions are satisfied:

$$\frac{r}{r - \frac{\alpha\gamma_1 r}{\alpha-1} + 1} \in [2,6]\,, \tag{4.3.8}$$

$$\frac{r}{r - \alpha\gamma_2 r + 1} \in [2,6]\,, \tag{4.3.9}$$

$$\frac{r}{r - \alpha\gamma_3 r + 1} \in [2,\infty)\,, \tag{4.3.10}$$

$$\frac{r}{\alpha r - r - \alpha\gamma_3 r + 1} \in [2,\infty)\,, \tag{4.3.11}$$

$$\frac{r + \alpha r - 3\alpha\gamma_3 r + 3}{r} \ge \frac{1}{2}, \tag{4.3.12}$$

$$\gamma_3 \le \frac{3\alpha r + 2r + 10}{10\alpha r}, \tag{4.3.13}$$

$$\gamma_3 < \frac{\alpha r + 2}{2\alpha r}. \tag{4.3.14}$$

By the use of Lemma 5 and (4.3.8) - (4.3.11) we have immediately the following three inequalities:

$$\|\partial_2\mathbf{u}\|_{\frac{r}{r-\alpha\gamma_2 r+1}} \le \|\nabla\partial_2\mathbf{u}\|_2^{\frac{3(2\gamma_2\alpha r - r - 2)}{2r}}\,\|\partial_2\mathbf{u}\|_2^{\frac{5r - 6\alpha\gamma_2 r + 6}{2r}}\,, \tag{4.3.15}$$

$$\|\partial_1\mathbf{u}\|_{\frac{r}{r-\frac{\alpha}{\alpha-1}\gamma_1 r+1}} \le \|\nabla\partial_1\mathbf{u}\|_2^{\frac{3(2\alpha\gamma_1 r - \alpha r - 2\alpha + r + 2)}{2r(\alpha-1)}}\,\|\partial_1\mathbf{u}\|_2^{\frac{5\alpha r - 5r - 6\alpha\gamma_1 r + 6\alpha - 6}{2(\alpha-1)r}} \tag{4.3.16}$$

and

$$\left\|\,\|\partial_3\mathbf{u}\|_{L_{23}^{\frac{r}{r-\alpha\gamma_3 r+1}}}\right\|_{L_1^{\frac{r}{\alpha r - r - \alpha\gamma_3 r+1}}}$$

$$\le \|\partial_2\partial_3\mathbf{u}\|_2^{\frac{2\gamma_3\alpha r - r - 2}{2r}}\,\|\partial_3\partial_3\mathbf{u}\|_2^{\frac{2\gamma_3\alpha r - r - 2}{2r}}\,\|\partial_1\partial_3\mathbf{u}\|_2^{\frac{3r - 2\alpha r + 2\alpha\gamma_3 r - 2}{2r}}\,\|\partial_3\mathbf{u}\|_2^{\frac{2\alpha r + r - 6\alpha\gamma_3 r + 6}{2r}}. \tag{4.3.17}$$

Consequently, assuming that

$$\frac{1}{\beta} + \frac{\alpha r - 2\alpha\gamma_3 r + 2}{8r\,(\alpha+1)} + \frac{3\alpha r - 2\alpha\gamma_3 r - 6\alpha + 4r - 4}{4r(\alpha+1)} + \frac{2\alpha\gamma_3 r - r - 2}{4r(\alpha+1)} = 1, \tag{4.3.18}$$

it follows from Lemma 4, the inequalities (4.3.15) - (4.3.17) and by the use of the Hölder inequality that

$$\int_{T_1}^{T_2} \|\nabla u_3\|_p\,\|\mathbf{u}\|_r\,\|\nabla\nabla_h\mathbf{u}\|_2\,dt$$

$$\begin{aligned}
&\leq \int_{T_1}^{T_2} \|\nabla u_3\|_p \|\nabla_h \mathbf{u}\|_2^{\frac{-\alpha r+6\alpha+6\alpha\gamma_3 r}{2r(\alpha+1)}} \|\partial_3 \mathbf{u}\|_2^{\frac{3\alpha r+2r-10\alpha\gamma_3 r+10}{4r(\alpha+1)}} \\
&\cdot \|\partial_3 \mathbf{u}\|_2^{\frac{\alpha r+2-2\alpha\gamma_3 r}{4r(\alpha+1)}} \|\nabla\nabla_h \mathbf{u}\|_2^{\frac{3\alpha r-2\alpha\gamma_3 r-6\alpha+4r-4}{2r(\alpha+1)}} \|\partial_3 \nabla \mathbf{u}\|_2^{\frac{2\alpha\gamma_3 r-r-2}{2r(\alpha+1)}} \, dt \\
&\leq \|\nabla u_3\|_{L^\beta(T_1,T_2;L^p)} \|\nabla_h \mathbf{u}\|_{L^\infty(T_1,T_2;L^2)}^{\frac{-\alpha r+6\alpha+6\alpha\gamma_3 r}{2r(\alpha+1)}} \|\partial_3 \mathbf{u}\|_{L^\infty(T_1,T_2;L^2)}^{\frac{3\alpha r+2r-10\alpha\gamma_3 r+10}{4r(\alpha+1)}} \\
&\cdot \|\partial_3 \mathbf{u}\|_{L^2(T_1,T_2;L^2)}^{\frac{\alpha r+2-2\alpha\gamma_3 r}{4r(\alpha+1)}} \|\nabla\nabla_h \mathbf{u}\|_{L^2(T_1,T_2;L^2)}^{\frac{3\alpha r-2\alpha\gamma_3 r-6\alpha+4r-4}{2r(\alpha+1)}} \|\partial_3 \nabla \mathbf{u}\|_{L^2(T_1,T_2;L^2)}^{\frac{2\alpha\gamma_3 r-r-2}{2r(\alpha+1)}} \\
&\leq C\epsilon^{\frac{\alpha r+2-2\alpha\gamma_3 r}{4r(\alpha+1)}} J(T_2)^{\frac{2\alpha r+4\alpha\gamma_3 r+4r-4}{2r(\alpha+1)}} L(T_2)^{\frac{3\alpha r-6\alpha\gamma_3 r+6}{4r(\alpha+1)}} \\
&\leq C\epsilon^{\frac{\alpha r+2-2\alpha\gamma_3 r}{4r(\alpha+1)}} J(T_2)^2.
\end{aligned}$$

So it follows from the last inequality and (4.3.5), (4.3.6) and (4.3.7) that

$$J(T_2)^2 \leq c + c\epsilon J(T_2)^2 + c\epsilon^{\frac{\alpha r+2-2\alpha\gamma_3 r}{4r(\alpha+1)}} J(T_2)^2.$$

We can conclude in the same way as in the proof of Theorem 36 that $\mathbf{u}$ is regular on $(0,T]$.

Notice that the condition (4.3.18) is equivalent to the following condition:

$$\frac{2}{\beta} + \frac{3}{p} = \frac{7}{4} + \frac{\alpha\gamma_3}{2(\alpha+1)} + \frac{1}{2p(\alpha+1)}.$$

Thus, to complete the proof we will now discuss the following problem. Denote $f(\alpha,\gamma_3) = \frac{\alpha\gamma_3}{2(\alpha+1)} + \frac{1}{2p(\alpha+1)}$. We want to find maximum (respectively supremum) of f on the set of all $\alpha, \gamma_1, \gamma_2, \gamma_3$ such that $\alpha \in (1,\infty)$; $\gamma_1, \gamma_2, \gamma_3 \in (0,1)$; $\gamma_1 + \gamma_2 + \gamma_3 = 1$ which satisfy conditions (4.1.2) - (4.1.5) and (4.3.8) - (4.3.14). The analysis of this problem leads, for example, to the following choice of $\alpha, \gamma_1, \gamma_2, \gamma_3$. Let $\epsilon > 0$ be sufficiently small. If, firstly, $r \in (19,\infty)$ (which means that $s \in (2, 38/17)$), we take

$$\begin{aligned}
\alpha &= \frac{12}{7} - \epsilon, \\
\gamma_1 &= \frac{5}{24} + \frac{5}{12r} + \frac{5\epsilon}{12-7\epsilon}, \\
\gamma_2 &= \frac{3}{8} - \frac{1}{r}, \\
\gamma_3 &= \frac{5}{12} + \frac{7}{12r} - \frac{5\epsilon}{12-7\epsilon}.
\end{aligned}$$

It is possible to verify that $\alpha \in (1,\infty)$, $\gamma_1, \gamma_2, \gamma_3 \in (0,1)$, $\gamma_1 + \gamma_2 + \gamma_3 = 1$ and all conditions (4.1.2) - (4.1.5) and (4.3.8) - (4.3.14) are satisfied. Moreover, $f(\alpha,\gamma_3) = \frac{17}{76} - \epsilon\,(3113p - 1862)\,/(912p(19-7\varepsilon))$. So, the solution is regular if (4.2.10) is satisfied.

Secondly, let $r \in [10,19]$ (which means that $p \in [38/17, 5/2]$). We put

$$\alpha = \frac{2r-2}{r+2},$$

$$\gamma_1 = \frac{r^2-2r-8}{4r(r-1)} + \frac{\epsilon\,(r+2)}{r-1},$$

$$\gamma_2 = \frac{(r+2)^2}{4r(r-1)},$$
$$\gamma_3 = \frac{r-2}{2r} - \frac{\epsilon(r+2)}{r-1}.$$

Again, the conditions (4.1.2) - (4.1.5) and (4.3.8) - (4.3.14) are satisfied, $f(\alpha, \gamma_3) = 1/(2p) - 2\varepsilon(p-1)/(3p)$ and so the regularity of solution under the condition (4.2.11) for $p \in [38/17, 5/2]$ is proved. For $p > 5/2$ this fact has been proved in [100]. The proof of Theorem 38 is complete. □

4.3.3 Proof of Theorem 39

Proof. We proceed exactly in the same way as in the proof of Theorem 36 up to the condition (4.3.3). It is possible to show (see [117], proof of Theorem 2) that

$$\int u_j \partial_j u_k \Delta_h u_k dx \le c \int |\nabla u_3| \, |\nabla \mathbf{u}| \, |\nabla_h \mathbf{u}| \, dx.$$

So it follows from (4.3.3) that

$$J(T_2)^2 \le c + c \int_{T_1}^{T_2} \int |\nabla u_3| \, |\nabla \mathbf{u}| \, |\nabla_h \mathbf{u}| \, dxdt. \qquad (4.3.19)$$

Using (4.2.13) and the Hölder inequality and then (4.2.14) and (4.2.15) and Lemma 5 we can estimate the right hand side of (4.3.19):

$$\int |\nabla u_3| \, |\nabla \mathbf{u}| \, |\nabla_h \mathbf{u}|$$

$$\le \left\| \left\| \|\nabla u_3\|_{L_3^{p_3}} \right\|_{L_2^{p_2}} \right\|_{L_1^{p_1}} \left\| \left\| \|\nabla \mathbf{u}\|_{L_3^{q_3}} \right\|_{L_2^{q_2}} \right\|_{L_1^{q_1}} \left\| \left\| \|\nabla_h \mathbf{u}\|_{L_3^{r_3}} \right\|_{L_2^{r_2}} \right\|_{L_1^{r_1}}$$
$$\le \left\| \left\| \|\nabla u_3\|_{L_3^{p_3}} \right\|_{L_2^{p_2}} \right\|_{L_1^{p_1}} \|\partial_1 \nabla \mathbf{u}\|_2^{(q_1-2)/(2q_1)} \|\partial_2 \nabla \mathbf{u}\|_2^{(q_2-2)/(2q_2)}$$
$$\cdot \|\partial_3 \nabla \mathbf{u}\|_2^{(q_3-2)/(2q_3)} \|\nabla \mathbf{u}\|_2^{\frac{1}{q_1}+\frac{1}{q_2}+\frac{1}{q_3}-\frac{1}{2}} \|\partial_1 \nabla_h \mathbf{u}\|_2^{(r_1-2)/(2r_1)}$$
$$\cdot \|\partial_2 \nabla_h \mathbf{u}\|_2^{(r_2-2)/(2r_2)} \|\partial_3 \nabla_h \mathbf{u}\|_2^{(r_3-2)/(2r_3)} \|\nabla_h \mathbf{u}\|_2^{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{2}}.$$

So we get using (4.2.14)

$$J(T_2)^2 \le \int_{T_1}^{T3} \left\| \left\| \|\nabla u_3\|_{L_3^{p_3}} \right\|_{L_2^{p_2}} \right\|_{L_1^{p_1}} \|\nabla_h \mathbf{u}\|_2^{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{2}}$$
$$\cdot \|\nabla \mathbf{u}\|_2^{\left(\frac{3}{4q_1}+\frac{3}{4q_2}+\frac{1}{q_3}-\frac{1}{2}\right)} \|\nabla \mathbf{u}\|_2^{\frac{1}{4q_1}+\frac{1}{4q_2}} \|\nabla\nabla_h \mathbf{u}\|_2^{\frac{5}{2}-\frac{1}{q_1}-\frac{1}{q_2}-\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{r_3}} \|\partial_3 \nabla \mathbf{u}\|_2^{\frac{1}{2}-\frac{1}{q_3}} dt$$

and by the use of the Hölder inequality and (4.2.13) and (4.2.16) we have

$$J(T_2)^2 \le \|\nabla u_3\|_{L^\beta(T_1,T_2;L^{\overline{p}})} \|\nabla_h \mathbf{u}\|_{L^\infty(T_1,T_2;L^2)}^{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{2}} \|\nabla \mathbf{u}\|_{L^\infty(T_1,T_2;L^2)}^{\frac{3}{4q_1}+\frac{3}{4q_2}+\frac{1}{q_3}-\frac{1}{2}}$$
$$\cdot \|\nabla \mathbf{u}\|_{L^2(T_1,T_2;L^2)}^{\frac{1}{q_1}+\frac{1}{q_2}} \|\nabla\nabla_h \mathbf{u}\|_{L^2(T_1,T_2;L^2)}^{\frac{5}{2}-\frac{1}{q_1}-\frac{1}{q_2}-\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{r_3}} \|\partial_3 \nabla \mathbf{u}\|_{L^2(T_1,T_2;L^2)}^{\frac{1}{2}-\frac{1}{q_3}}.$$

Using now the choice of T_1, the definition of $J(T_2)$ and $L(T_2)$ and the fact that $L(T_2) \leq J(T_2)^{4/3}$ we finally obtain

$$J(T_2)^2 \leq c\varepsilon^{\frac{1}{q_1}+\frac{1}{q_2}} J(T_2)^{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{2}+\frac{5}{2}-\frac{1}{q_1}-\frac{1}{q_2}-\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{r_3}} L(T_2)^{\frac{3}{4q_1}+\frac{3}{4q_2}+\frac{1}{q_3}-\frac{1}{2}+\frac{1}{2}-\frac{1}{q_3}}$$

$$\leq c\varepsilon^{\frac{1}{q_1}+\frac{1}{q_2}} J(T_2)^{2-\frac{1}{q_1}-\frac{1}{q_2}+\frac{4}{3}\left(\frac{3}{4q_1}+\frac{3}{4q_2}\right)} = c\varepsilon^{\frac{1}{q_1}+\frac{1}{q_2}} J(T_2)^2.$$

Choosing ε sufficiently small we get that $J(T_2)$ and consequently $L(T_2)$ are bounded independently of T_2 and the proof is complete. □

4.3.4 Proof of Theorem 40

Proof. Theorem 40 follows immediately from Theorem 39. Supposing that assumptions in Theorem 40 are satisfied and moreover $p_1, p_2 \in (2, \infty]$ then we proceed in the following way: if moreover $p_3 \in (2, \infty]$, we put

$$q_i = \frac{2p_i}{p_i - 2}, \ i = 1, 2, \ q_3 = 2$$

and

$$r_1 = r_2 = 2, \ r_3 = \frac{2p_3}{p_3 - 2}.$$

If $p_3 = 2$, then we choose $\varepsilon \in (0, 1/4)$ such that

$$\frac{3}{4}\Big(\frac{1}{p_1} + \frac{1}{p_2}\Big) - \frac{1}{4} \leq \frac{1}{2 + \varepsilon}$$

and put

$$q_i = \frac{2p_i}{p_i - 2}, \ i = 1, 2, \ q_3 = 2 + \varepsilon,$$

$$r_1 = r_2 = 2, \ r_3 = \frac{4 + 2\varepsilon}{\varepsilon}.$$

It is possible to verify that in both cases all the conditions (4.2.12)-(4.2.15) are satisfied. Moreover, the veracity of (4.2.16) follows immediately from (4.2.17) and the choice of q_1 and q_2. So using Theorem 39 we get the regularity of u.

If we suppose that $p_3 \in (2, \infty]$ and $p_1, p_2 \in (1, 2]$ then by a possible decrease of β we can suppose without loss of generality that

$$0 < 2 - \Big(\frac{2}{\beta} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_2}\Big) < \min\Big(\frac{2}{3\beta}, \frac{p_1 - 1}{2p_1}, \frac{p_2 - 1}{2p_2}, \frac{1}{4}\Big).$$

Putting

$$\varepsilon = 2 - \Big(\frac{2}{\beta} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_2}\Big)$$

and

$$q_1 = q_2 = \frac{1}{2\varepsilon}, \ q_3 = 2,$$

$$r_i = \frac{p_i}{p_i - 2\varepsilon p_i - 1}, \ i = 1, 2, \ r_3 = \frac{2p_3}{p_3 - 2},$$

we can again verify all the conditions (4.2.12)-(4.2.16) and complete the proof by the use of Theorem 39. We proceed analogically in the remaining cases. □

4.3.5 Proof of Theorem 42

Proof. We proceed in the same way as in the the proof of Theorem 39 up to the condition (4.3.19). Let $q_1, q_2 \in [2, \infty)$ and

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{3p-3}{2p}.$$

Then by the Hölder inequality

$$\int \nabla u_3 \nabla \mathbf{u} \nabla_h \mathbf{u} dx \le \left\| \|\nabla u_3\|_{L^\infty_3} \right\|_{L^{2p/(3-p)}_{12}} \left\| \|\nabla \mathbf{u}\|_{L^2_3} \right\|_{L^{q_1}_{12}} \left\| \|\nabla_h \mathbf{u}\|_{L^2_3} \right\|_{L^{q_2}_{12}}.$$

Further,

$$\left\| \|\nabla u_3\|_{L^\infty_3} \right\|_{L^{2p/(3-p)}_{12}} \le \Big(\int |\nabla u_3|^{\frac{3p-3}{3-p}} |\partial_3 \nabla u_3| dx \Big)^{\frac{3-p}{2p}}$$

$$\le ||\nabla u_3||^{\frac{3p-3}{2p}}_{\frac{3p}{3-p}} ||\partial_3 \nabla u_3||_p^{\frac{3-p}{2p}} \le c||\nabla^2 u_3||_p$$

and using also Lemma 5, we have

$$\int \nabla u_3 \nabla \mathbf{u} \nabla_h \mathbf{u} dx \le c||\nabla^2 u_3||_p ||\nabla \mathbf{u}||_2^{\frac{2}{q_1}} ||\partial_1 \nabla \mathbf{u}||_2^{\frac{q_1-2}{2q_1}}$$

$$\cdot ||\partial_2 \nabla \mathbf{u}||_2^{\frac{q_1-2}{2q_1}} ||\nabla_h \mathbf{u}||_2^{\frac{2}{q_2}} ||\partial_1 \nabla_h \mathbf{u}||_2^{\frac{q_2-2}{2q_2}} ||\partial_2 \nabla_h \mathbf{u}||_2^{\frac{q_2-2}{2q_2}}$$

and

$$\int_{T_1}^{T_2} \int \nabla u_3 \nabla \mathbf{u} \nabla_h \mathbf{u} dx dt$$

$$\le c \int_{T_1}^{T_2} ||\nabla^2 u_3||_p ||\nabla \mathbf{u}||_2^{\frac{3}{2q_1}} ||\nabla \mathbf{u}||_2^{\frac{1}{2q_1}} ||\nabla_h \mathbf{u}||_2^{\frac{2}{q_2}} ||\nabla \nabla_h \mathbf{u}||_2^{\frac{3-p}{p}} dt.$$

Firstly, assuming that (4.2.18) holds, we can choose q_1 and q_2 in such a way that $1/q_1 = 1 - 2/(3\beta) - 1/p$ and $1/q_2 = 1/2 + 2/(3\beta) - 1/(2p)$. Let $1/y = 5(3 - 2/\beta - 3/p)/12$. Then we can estimate by the use of the Hölder inequality

$$\int_{T_1}^{T_2} \int \nabla u_3 \nabla \mathbf{u} \nabla_h \mathbf{u} dx dt \le c(T_2 - T_1)^y ||\nabla^2 u_3||_{L^\beta(0,T;L^p)} ||\partial_3 \mathbf{u}||^{\frac{3}{2q_1}}_{L^\infty(0,T;L^2)}$$

$$\cdot ||\nabla \mathbf{u}||^{\frac{1}{2q_1}}_{L^2(0,T;L^2)} ||\nabla_h \mathbf{u}||^{\frac{2}{q_2}}_{L^\infty(0,T;L^2)} ||\nabla \nabla_h \mathbf{u}||^{\frac{3-p}{p}}_{L^2(0,T;L^2)}$$

$$\le c\varepsilon^{\frac{1}{2q_1}} J(T_2)^{\frac{2}{q_2} + \frac{3-p}{p}} L(T_2)^{\frac{3}{2q_1}} = c\varepsilon^{\frac{1}{2q_1}} J(T_2)^2.$$

Secondly, let (4.2.19) hold. Then we simply put $q_2 = 2$ and $q_1 = 2p/(2p-3)$ and estimate

$$\int_{T_1}^{T_2} \int \nabla u_3 \nabla \mathbf{u} \nabla_h \mathbf{u} dx dt \le ||\nabla^2 u_3||_{L^\beta(0,T;L^p)} ||\partial_3 \mathbf{u}||^{\frac{3}{2q_1}}_{L^\infty(0,T;L^2)}$$

$$\cdot ||\nabla \mathbf{u}||^{\frac{1}{2q_1}}_{L^2(0,T;L^2)} ||\nabla_h \mathbf{u}||^{\frac{2}{q_2}}_{L^\infty(0,T;L^2)} ||\nabla \nabla_h \mathbf{u}||^{\frac{3-p}{p}}_{L^2(0,T;L^2)} \le c\varepsilon^{\frac{1}{2q_1}} J(T_2)^2.$$

As in the proof of Theorem 36 we can now conclude that $J(T_2) + L(T_2)$ is estimated from above independently of T_2 and the proof is complete. □

4.4 Conclusions

The global regularity problem we faced above has focused on the use of the anisotropic Lebesgue space framework, thanks to which results in literature have been improved (see Theorems 38 - 42). We believe that the tool could be useful to improve other results in the literature concerning, for example, other kind of models. Moreover, we would like to mention that since different generalizations of the Troisi inequality can also be derived, it is not excluded that some of these generalizations could lead to an even stronger criteria.

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