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Identification of several non-stationary loads applied to an elastically deformed structure

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Abstract

The technique has been presented for time-dependence identification of several independent beetwen each other loads distributed over a given area of a structure with arbitrary topology by using quantity values more convenient for measurements. In the assumption that the structure's response linearly depends on the loads, the considered problem, which belongs to the class of boundary inverse problems in the mechanics of solids, is reduced to a system of linear algebraic equations for coefficients that approximate the sought-for influences. The system is solved using a regularizing algorithm providing stability of results to random errors in initial data and calculation errors. Concrete calculations, substantiating the efficiency of the presented technique, have been performed as with theoretical data to identify two non-stationary loads applied to a wheel carrier of a race car as with experimental data to restore an impact force applied to a round plate with fixed boundary. To calculate values of a system's elements corresponding to values of measured quantities under unit loads, the finite element method was used. The suggested technique can be used for designing structures with complex geometry based on criterias of their dynamic (fatigue) strength, etc.

Keywords: structure with arbitrary topology, identification of several loads, dependence on time, influence function, FEM, regularization

1. Introduction

The development of modern technology is inseparably linked to the design of new structures and improvement of existed ones, which should satisfy a required set of mechanical properties. To achieve these goals, complete and valid information about applied external loads is important in addition to reliable methods of its calculation. The most rational approach to determine external loads supposes their direct measurement. However, there are many situations when this approach is difficult to implement or requires modification the structure under investigation (for installation of sensors and/or communication means). This substantially reduces the measurement accuracy.

This problem can be solved using the technique of indirect measurements when sought-for loads are restored (identified) by registering more accessible measurement quantities associated with the loads. Restoring of external loads by their indirect manifestations relates to the "boundary inverse problem" in the mechanics of solids. Nowadays, there is an increasing interest in developing effective methods for solving such problems for many applications such as reconstruction/control of stress-strain state of structures, predicting their fatigue life, topology optimization, health monitoring, etc. [8, 20].

Research for this class of problems dwells on problems both in of identifying the spatial distribution of external loads and their time dependence in case of non-stationary processes.

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The methodology for solving such problems often assumes that the response of the structure and associated sensor readings depend linearly on the external loads and that it is oriented at minimizing the discrepancy between the sensor readings and the results of numerical simulations of the structure under the action of the sought-for loads. As noted in several studies, the presence of inevitable measurement errors results in evaluating the values of sought-for functions rather than determining the "true" values. Furthermore, the restoration of an approximate spatial distribution of action based on a finite number of measuring points is possible only by assuming additional a priori information about the nature of the distribution.

Thus, when identifying unsteady forces, it is often assumed that the load is concentrated (i.e. distributed over a small area with an a priori unknown position). At the first stage, the procedure of their identification consists in defining the approximate location of the load application, and at the second stage the approximate time dependence is recovered by localizing the search area. Among many existing methods of identification, the details of which can be found in [12, 13, 20, 31] and literature reviews [21, 25], the most developed methods are based on the frequency domain technique. This technique has been developed to identify loads applied to both elements of canonical form [3, 19, 27] and with an arbitrary geometry [11, 18, 28]. However, for instantaneous loads, more accurate identification can be achieved based on the time domain technique used for example in [2, 5, 10, 12, 14] for structures in canonical form or in [1, 28] for elements with complex geometry. This technique is applicable for real-time identification of external loads. This is important for problems of active control of structure's vibration, health monitoring, etc. Such approach is also widely used in problems of restoring the temporal component of non-stationary distributed loads in the assumption that boundaries of the area of their application and the pattern of distribution inside the area are known [10, 32]. The results obtained so far within the scope of this technique are mainly related to elements of canonical form or are applicable for impulse-type loads, for which numerical and analytical solutions of "direct problems" related to the calculation of structures' strength and vibrations with known loads are available. In doing so, the method of inverse problems' solving is realized by so called deconversion or by the complex procedure of results comparison.

At the same time, for most structures with a complex geometry and various design features, solving of direct problems even in the elastic strain range are only possible with numerical methods such as the widely used the finite element method (FEM). However, numerical methods, including FEM, are not adapted for direct solving of inverse problems. Therefore, the approaches developed so far for identifying external loads, which are based on numerical methods, and FEM in particular, assume the construction of so-called influence functions (coefficients) by solving one or a series of direct auxiliary problems for subsequent identification. This approach is outlined in [8, 23, 24] for an approximate recovering the spatial distribution of quasi-static and time-periodic loads and given in [18] for approximate restoring of external loads represented as a set of concentrated forces.

This paper is devoted to effective and based on the FEM method of inverse boundary problems solving for identification of dynamic external mechanical (quasi-static and non-stationary) loads acting on a structure with arbitrary geometry. The loading is assumed to be either a single load or a system of independent loads whose distribution in spatial coordinates is known. Identification is performed for quantities that are available for measurement and are the indirect response of loads acting on the structure.

Concrete calculations were realized within the scope of the Collaborate Research Centre (CRC) 653 *"Gentelligent Components in their Lifecycle"* for the model problem of identification of loads acting on a wheel carrier of a race car (item 1, Fig. 1). Within the CRC context, new

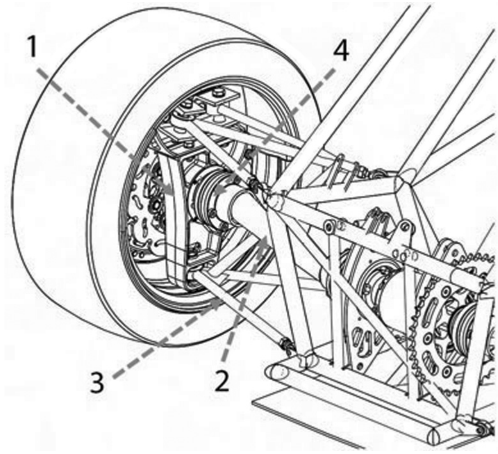


Fig. 1. Wheel suspension with gentelligent technology for measuring loads during the life cycle: 1 – wheel carrier; 2 – driving shaft; 3 – wishbone; 4 – bearing

sensitive materials for collecting data are being developed [7]. Their mechanical properties are applied to identify loads during the life cycle of the structure. This kind of new sensor technology requires new methods to recalculate the measured data into usable information. In this context, a suspension of a race car (Fig. 1) was equipped with different measuring devices to identify loads acting on this structure during its life time. The information obtained allows developing new optimized generations of structures, which are more adapted to actual environmental requirements [16, 17].

This paper is organized as follows. In Section 2 we give the statement of the problem and present accepted designations. Section 3 describes a method of identification of several dynamic loads in the case where these forces can be considered quasi-static. It should be noted that the authors do not claim the scientific novelty of this section, since identical approaches can be found in the literature. At the same time, the illumination of the algorithm for solving the problem in a quasi-static formulation allows us to both generalize the materials outlined in this paper and to touch upon the auxiliary questions that were needed in the development of the method for identifying several non-stationary loads (Section 4).

The choice of the calculation case (quasi-static or non-stationary) is determined by the level of inertial forces — if inertia forces are substantially smaller than external forces, then a quasi-static problem takes place, otherwise the problem is non-stationary. It should be noted that in the present paper we do not consider the case when acting on a structure forces are harmonic, because many researches on this case can be found in the literature (see references above).

In Section 5 we demonstrate the effectiveness of the proposed methods. In particular, in Sections 5.1 and 5.2 problems of identification quasi-static and non-stationary forces acting on a part of race car's suspension are examined. These calculations are based on theoretical data. In Section 5.3 the initial data for identification is experimental data obtained by other authors in the research of vibration of a round plate under a non-stationary force. In Section 6 we give a conclusion.

2. Problem statement

Let there be a structure with a given geometry (topology), a variant of fastening and mechanical characteristics of material/materials. Deformation of the structure is elastic and is the result of the action of R external loads with known distributions over spatial coordinates. These R

loads are designated as $Z^{(r)}(t)$ ($r = \overline{1,R}$) and have to be identified as a function of time. To identify them, S functions $V^{(s)}(t)$ ($s = \overline{1,S}$) are used. These functions show the dynamics of some deformation parameter by action of sought-for loads, and they can be measured with appropriate sensors.

3. Problem solution in the quasi-static statement

First, we consider the problem of identification of quasi-static loads when deformation of the structure are so slow that inertial forces can be neglected. To solve the problem, we use the superposition principle that determines the superimposement effect of readings from the s-th sensor under action of each of R applied external loads

$$V^{(s)}(t) \approx \sum_{r=1}^{R} \tilde{V}^{(s,r)} \quad (s = \overline{1,S}).$$

If we introduce a variable $\overline{V}^{(s,r)}$, which represents the response of the s-th sensor to the r-th unit load ($Z^{(r)} = \overline{Z} = 1$), then according to the assumption of linearity of processes, $V^{(s)}(t)$ takes the form

$$V^{(s)}(t) \approx \sum_{r=1}^{R} Z^{(r)}(t) \cdot \overline{V}^{(s,r)} \quad (s = \overline{1,S}). \tag{1}$$

Note that values $\overline{V}^{(s,r)}$ can be determined experimentally or by using mathematical modeling methods based on solving of the corresponding direct problem, for instance, by the FEM, which is a universal and powerful method for engineering analysis.

Since functions $V^{(s)}(t)$ are often presented as arrays of corresponding readout values at fixed points in time t_m (for example, $t_m = m\Delta t$, Δt =const – time step) over the time interval of interest $[0; T_{\text{inv}}]$, the problem of approximate restoration of $Z^{(r)}(t)$ is equivalent to solving equation (1) for each calculation step m in time. Thus, the problem of load identification in quasi-static statement can be reduced to solving the system of linear algebraic equations (SLAE) for every fixed time $t = t_m$ ($m = 0, 1, 2, \ldots$), which in a matrix form can be written as follows

$$\mathbf{A}\mathbf{Z}_m = \mathbf{Y}_m. \tag{2}$$

Here $\mathbf{Y}_m$ and $\mathbf{Z}_m$ are S- and R-column matrices containing known values $V^{(s)}(t_m)$ and sought-for values $Z^{(r)}(t_m)$, respectively; $\mathbf{A}$ is the $(S \times R)$-matrix of constant coefficients which can be called as flexibility influence coefficients [22].

The elements of these matrices are defined by formulas

$$\{\mathbf{Y}_m\}_s = V^{(s)}(t_m) \quad (s = \overline{1,S}); \quad \{\mathbf{Z}_m\}_r = Z^{(r)}(t_m) \quad (r = \overline{1,R}); \quad \{\mathbf{A}\}_{s,r} = \overline{V}^{(s,r)}.$$

Equation (2) is solved using the least-squares method when the required matrix-column $\mathbf{Z}_m$ satisfies a SLAE

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{Z}_m = \mathbf{A}^{\mathrm{T}}\mathbf{Y}_m \tag{3}$$

and represents the normal pseudo-solution of the initial system (2).

In equation (3) and further superscript T denotes the transpose operation.

The obtained result as a set of $(R\times 1)$-column matrices $\mathbf{Z}_m$ ($m = 0, 1, 2, \ldots$) allows restoring in time configurations of quasi-static loads ($Z^{(r)}(t_m) = \{\mathbf{Z}_m\}_r$) applied to the structure. The presented above sequence can be represented in the form of the algorithm given in Fig. 2.

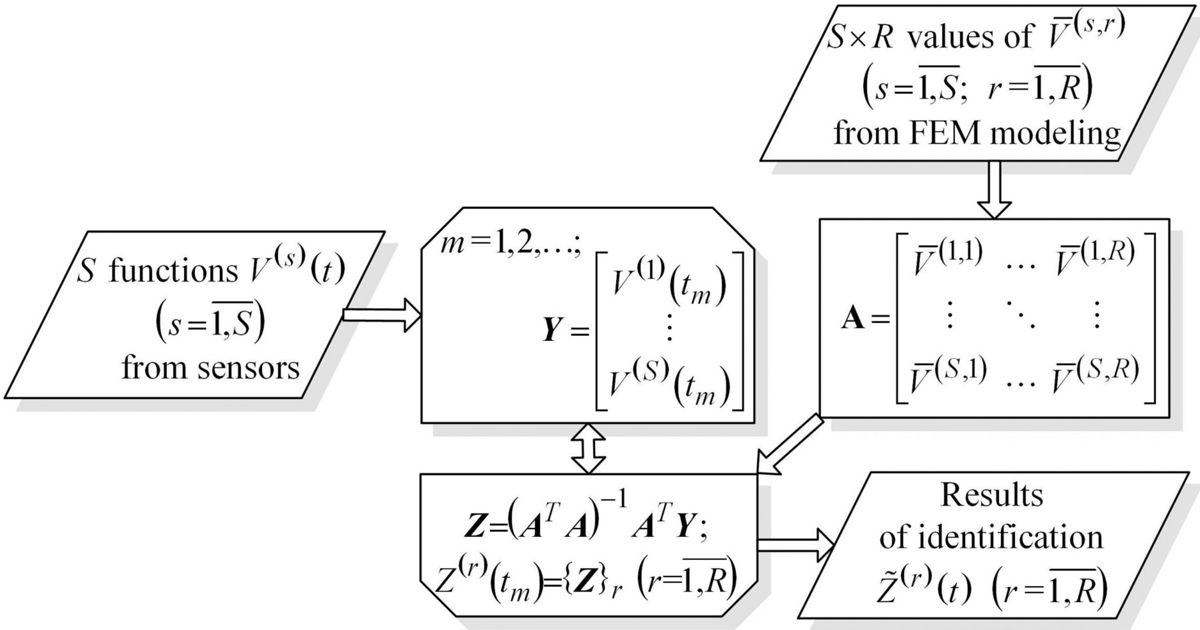


Fig. 2. Algorithm for loads identification for problem in the quasi-static statement

Here and below in block diagrams mirror hexagons, which are connected by a double arrow, designate execution of some cyclic operation (in particular, in Fig. 2 by m).

Obviously, the result of identification as $Z^{(r)}(t)$-vector essentially depends on the matrix $\mathbf{A}$, which, as mentioned earlier, represents response of the sensors to a fixed value of external forces (i.e., contains influence coefficients). Therefore, at the preparatory stage of solving the problem special attention should be paid to the choice of positions and measuring directions of sensors. More detailed information can be found in [20, 32], and an example of filling the matrix $\mathbf{A}$ for some numerical case is demonstrated in Section 5.1. It should be noted that in mathematical context the influence coefficients must provide continuity of a pseudo-inverse operator $\mathbf{A}^+ = (\mathbf{A}^\mathrm{T}\mathbf{A})^{-1}\mathbf{A}^\mathrm{T}$.

4. Problem solution in the non-stationary statement

This section considers the problem of identification of several non-stationary loads (including those of the impulse type). It is well known that identification of external non-stationary loads as functions of time based on their indirect manifestations is a challenging problem due to sensitivity of results to measurement errors. One method to improve solution stability and in some cases the feasibility of solving the problem in general, is to ease requirements to quantity of information obtained from the solution. In this paper, this approach consists in approximation of the sought-for (unknown) dependencies $Z^{(r)}(t)$ using piecewise constant functions (Fig. 3a), which within a certain n-th time interval between points $t = T_{n-1}$ and $t = T_n$ ($T_n - T_{n-1} \geq \Delta t$) are constant and take constant values $\tilde{q}_n^{(r)}$:

$$Z^{(r)}(t) \approx \sum_{n=1}^{N} \tilde{q}_n^{(r)} [H(t - T_{n-1}) - H(t - T_n)]. \tag{4}$$

Here T_n are fixed points in time ($T_0 = 0 < T_1 < T_2 < \ldots < T_N = T_{\mathrm{inv}}$); T_{inv} is the investigation time; $H(t)$ is the Heaviside step function.

Obviously, relation (4) can be rewritten as a set of step functions $\overline{Z}(t) = H(t)$ (Fig. 3b), i.e. as

$$Z^{(r)}(t) \approx \sum_{n=1}^{N} q_n^{(r)} H(t - T_{n-1})\overline{Z}(t - T_{n-1}). \tag{5}$$

Thus, relation $\tilde{q}_n^{(r)} = \sum_{k=1}^{n} q_k^{(r)}$ links coefficients $\tilde{q}_n^{(r)}$ and $q_n^{(r)}$.

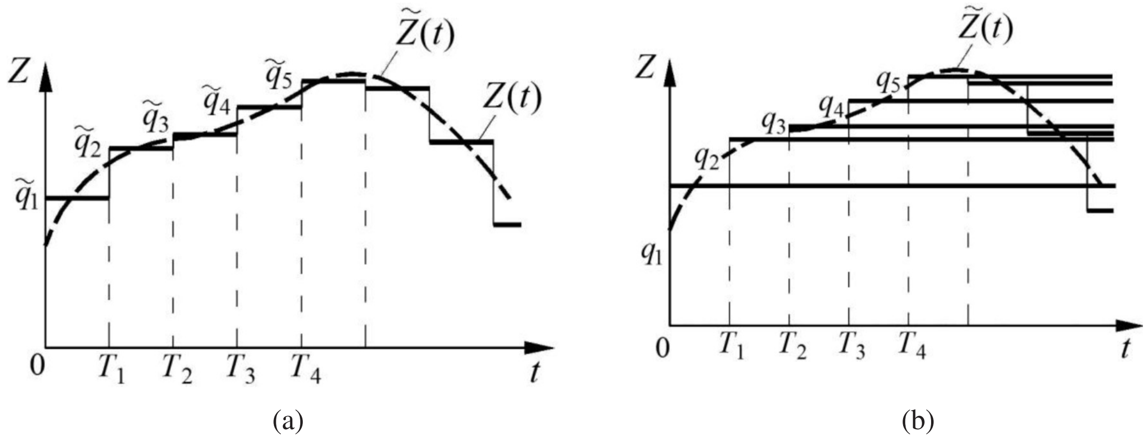


Fig. 3. Approximation of unknown dependencies by step functions: (a) representation as a piecewise constant function; (b) representation as a combination of step functions

If by analogy with the previous case we introduce $S \cdot R$ functions $\overline{V}^{(s,r)}(t)$ ($s = \overline{1,S}$; $r = \overline{1,R}$), which determine changing in time of the s-th registered quantity during the application of exclusively the r-th load in the form of a unit step influence ($Z^{(r)}(t) = \overline{Z}(t)$), then, according to the principle of superposition, the specified values $V^{(s)}(t)$ will be approximately equal to the sum of these functions with account of the time shift T_{n-1} and coefficients $q_n^{(r)}$ (see (5)) with the meaning of weight coefficients

$$V^{(s)}(t) \approx \sum_{r=1}^{R}\sum_{n=1}^{N} q_n^{(r)} H(t - T_{n-1})\overline{V}^{(s,r)}(t - T_{n-1}). \tag{6}$$

As before, functions $\overline{V}^{(s,r)}(t)$, called hereinafter "influence functions", can be defined either experimentally (if it is possible to realize a load similar in configuration to $\overline{Z}(t)$) or by mathematical modeling methods. Given that for the current practice in both experimental investigations and computer simulations continuous quantities $V^{(s)}(t)$ and $\overline{V}^{(s,r)}(t)$ as a rule are replaces with discrete arrays of values representing corresponding functions for equidistant points with step Δt for the investigated time interval, instead of equation (6) the following equation is more appropriate

$$\mathbf{V}_m^{(s)} \approx \sum_{r=1}^{R}\sum_{n=1}^{N} q_n^{(r)} H(m - M_{n-1})\overline{\mathbf{V}}_{m-M_{n-1}}^{(s,r)} \quad (s = \overline{1,S}), \tag{7}$$

where $\mathbf{V}^{(s)}$, $\overline{\mathbf{V}}^{(s,r)}$ are M_N-column matrices with elements $\mathbf{V}_m^{(s)} = V^{(s)}(t_m)$, $\overline{\mathbf{V}}_m^{(s,r)} = \overline{V}^{(s,r)}(t_m)$ ($t_m = m\Delta t$; $m = \overline{1, M_N}$; $M_n = E(T_n \Delta t)$, $E(x)$ is the whole part of the argument).

Thus, the problem of approximately restoring $Z^{(r)}(t)$ (4) is reduced to determination coefficients $q_n^{(r)}$ ($r = \overline{1,R}$; $n = \overline{1,N}$) that would best ensure satisfaction of equation (7), assuming that $\mathbf{V}^{(s)}$, $\overline{\mathbf{V}}^{(s,r)}$, Δt and T_n are known. The last are specified either based on a priori information about the smoothness of solutions or they can be selected based on analysis of obtained results.

In this paper, the technique of coefficients $q_n^{(r)}$ calculating is based on the least-squares method. Coefficients $q_n^{(r)}$ are taken to minimize quadratic functions J_n introduced for each time interval n $(T_{n-1} \dots T_n)$:

$$J_n = \sum_{s=1}^{S} \left(\sum_{m=M_{n-1}+1}^{M_n} \left(\mathbf{V}_m^{(s)} - \sum_{r=1}^{R} \sum_{k=1}^{n} q_k^{(r)} \overline{\mathbf{V}}_{m-M_{k-1}}^{(s,r)} \right)^2 \right) =$$

$$\sum_{s=1}^{S} \left\| \left(\mathbf{V}_{(n)}^{(s)} - \sum_{r=1}^{R} \sum_{k=1}^{n} q_k^{(r)} \overline{\mathbf{V}}_{(n-k+1)}^{(s,r)} \right) \right\|_2^2 \quad (n = \overline{1, N}).$$

Here, values with the lower index in brackets denote sub-matrices of the corresponding columns — $\mathbf{X}_{(n)} = [\,\mathbf{X}_{M_{n-1}+1}\ \mathbf{X}_{M_{n-1}+2}\ \dots\ \mathbf{X}_{M_n}\,]^{\mathrm{T}}$, where $\mathbf{X} = \overline{\mathbf{V}}^{(s,r)}$ or $\mathbf{V}^{(s)}$; $\|\mathbf{X}\|_2$ is the Euclidean norm of $\mathbf{X}$.

The problems of finding $\min\limits_{q_n^{(r)}} J_n$ $(r = \overline{1, R};\ n = \overline{1, N})$ are equivalent to solving equations $\partial J_n / \partial q_n^{(r)} = 0$. Simple mathematical transformations yield a set of $N \cdot R$ linear algebraic equations which can be presented in matrix form as follows:

$$\mathbf{A}\mathbf{q} = \mathbf{Y}.$$

Here, $\mathbf{A}$ is the square $(R \times R)$-matrix with a block structure (these blocks are lower triangular Toeplitz $N \times N$ matrices); $\mathbf{q}$ and $\mathbf{Y}$ are block column-matrices with the height of R blocks equal to N. Thus, notation $\mathbf{A}\mathbf{q} = \mathbf{Y}$ means the following:

$$\begin{bmatrix} \mathbf{A}_{1,1} & \dots & \mathbf{A}_{1,R} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{R,1} & \dots & \mathbf{A}_{R,R} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_R \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_1 \\ \vdots \\ \mathbf{Y}_R \end{bmatrix}. \tag{8}$$

The elements of these matrices are defined by:

$$\{\mathbf{A}_{p,r}\}_{n,k} = \textstyle\sum_{s=1}^{S} \overline{\mathbf{V}}_{(1)}^{(s,p)\,\mathrm{T}} \overline{\mathbf{V}}_{(n-k+1)}^{(s,r)}; \quad \{\mathbf{q}_r\}_n = q_n^{(r)}; \quad \{\mathbf{Y}_r\}_n = \textstyle\sum_{s=1}^{S} \overline{\mathbf{V}}_{(1)}^{(s,r)\,\mathrm{T}} \mathbf{V}_{(n)}^{(s)}$$
$$(s = \overline{1, S}; \quad p, r = \overline{1, R}; \quad n = \overline{1, N}; \quad k = \overline{1, n}).$$

It should be noted that this matrix $\mathbf{A}$ (and its elements $\mathbf{A}_{p,r}$) is not a matrix of influence coefficients/functions in the classical sense. It is also not symmetric, and existence of its inverse operator $\mathbf{A}^{-1}$ depends both on the functions $\overline{V}^{(s,r)}(t)$ and on the width of time intervals $T_n - T_{n-1}$ for averaging unknown functions $Z^{(r)}(t)$. To ensure the continuity of the inverse/pseudo-inverse operator the same recommendations that are used for identification problem in a quasi-static statement are applicable.

The structure of matrix $\mathbf{A}$ allows to implement a step by step calculation of elements $\{\mathbf{q}_r\}_n$ by a recurrence formula with a constant matrix and a variable right-hand side that includes values $\{\mathbf{q}_r\}_k$ $(k = \overline{1, n-1})$ calculated during the previous steps. However, column Y contains random measurement errors, as well as data processing and transfer errors. Therefore, based on the mentioned recurrence formula results, in spite of a smoothing effect by the piecewise constant approximation (see (4)), will likely by void of any physical meaning. Construction of a solution, resistant to errors in the source data, requires using special regularizing procedures, developed for solving ill-posed problems in computational mathematics. In this paper to determine the approximate solutions of equation (8) a technique is used that combines implementation of the generalized Cramer or Gauss method (to extract from initial system (8) a SLAE of type $\hat{\mathbf{A}}_r\mathbf{q}_r = \hat{\mathbf{Y}}_r$ ($r = \overline{1,R}$) for each sub-matrix $\mathbf{q}_r$) and the Tikhonov regularization method. Note that the first stage takes into account the commutativity property of blocks in matrix $\mathbf{A}$ ($\mathbf{A}_{r,i}\mathbf{A}_{p,j} = \mathbf{A}_{p,j}\mathbf{A}_{r,i}$; $i,j = \overline{1,R}$) and the equality of elements on the diagonals parallel to the main one (to reduce quantity of computational operations). At the second stage, problem $\hat{\mathbf{A}}_r\mathbf{q}_r = \hat{\mathbf{Y}}_r$ is substituted with the solution of the regularized SLAE [26]

$$\left(\hat{\mathbf{A}}_r^{\mathrm{T}}\hat{\mathbf{A}}_r + \alpha\hat{\mathbf{L}}^{\mathrm{T}}\hat{\mathbf{L}}\right)\mathbf{q}_r = \hat{\mathbf{A}}_r^{\mathrm{T}}\hat{\mathbf{Y}}_r, \tag{9}$$

which follows from the condition $\min\limits_{\mathbf{q}_r}\left(\left\|\hat{\mathbf{A}}_r\mathbf{q}_r - \hat{\mathbf{Y}}_r\right\|_2^2 + \alpha\left\|\hat{\mathbf{L}}\mathbf{q}_r\right\|_2^2\right)$ ($\hat{\mathbf{L}}$ is the regularizing matrix).

The presented above sequence can be represented in the form of the following algorithm (Fig. 4).

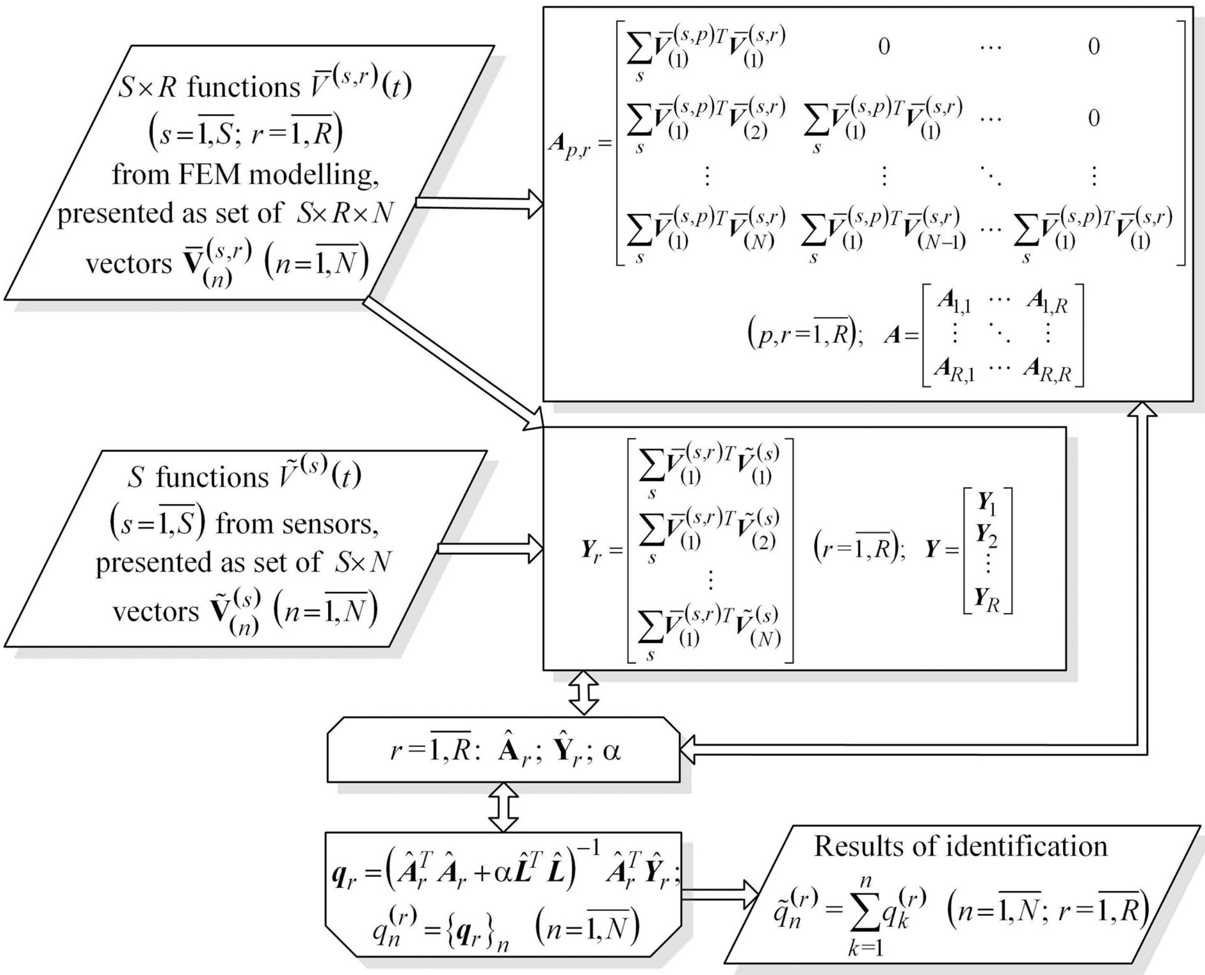


Fig. 4. Algorithm for non-stationary loads identification

An important step of Tikhonov regularization is choosing regularization parameter α ($\alpha > 0$) in the SLAE (9). Selecting of sufficiently big values of α is known to yield a zero solution, whereas sufficiently small ones yield not a normal pseudo-solution of the initial SLAE but an unstable computation process with ill-conditioned matrix $\hat{\mathbf{A}}_r$. Hence, these choices of α-values are excluded. The methods of choosing α, used widely in ill-posed problems of computational mathematics, in particular, residual, asymptotic, quasi-optimal and discrepancy methods [8, 9, 26], demand adjusting α to initial data errors. However, even with modern advanced measurement techniques, the approximate evaluation of these data poses a challenge. Therefore, some authors use the procedure of direct search for values, with a focus on selecting such an α that in their opinion will provide the best identification result. The applicability of such approach can be explained by the fact that, with invariable matrix $\hat{\mathbf{A}}_r$ and the right-hand side of $\hat{\mathbf{Y}}_r$ of the initial SLAE, the mean-square error of solving δ first decreases with an increasing regularisation parameter, and its effect on α diminishes. With α in the neighborhood of some α_δ, the value of δ is practically independent of α. This is the value of the parameter that is often taken as the calculation one because the error of δ starts growing with increasing α. Note that the quantity of iterations and computational operations, respectively, can be substantially decreased to find α_δ if the order of α will be varied by taking as the initial value and calculating for the range of values of $\mathrm{Sp}(\hat{\mathbf{A}}_r^{\mathrm{T}}\hat{\mathbf{A}}_r)/\mathrm{Sp}(\hat{\mathbf{L}}^{\mathrm{T}}\hat{\mathbf{L}})$ and carry out calculations in the range of $\alpha \in (\sigma_r^{2/3}; \sigma_1^{2/3})$, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ are singular values of matrix $\hat{\mathbf{A}}_r$ [8, 26].

However, despite of simplicity of this approach and its efficiency at least for cases when there is some a priori information about the domain of possible values of the sought-for solution taken from the processing of preceding close examples (degree of smoothness, number of extreme points, etc.), this approach should be considered as a subjective one. This explains the recent emergence of automated methods of selecting α that do not require a priori information about the solution, in particular, the L-curve method [23], the U-curve method [4] and the generalized cross-validation method [6]. Detailed information about them can also be found in monograph [8] and paper [15]. However, the practice of inverse problems solving, as it is also shown in [30], has demonstrated many situations where the implementation of these methods is linked to some mathematical challenges arising, for instance, from absence of a clearly defined angle in the L-curve or a local minimum of the U-curve in the interval of definition of the values of regularization parameter α. This requires further theoretical and applied studies in methods for optimal searching of α.

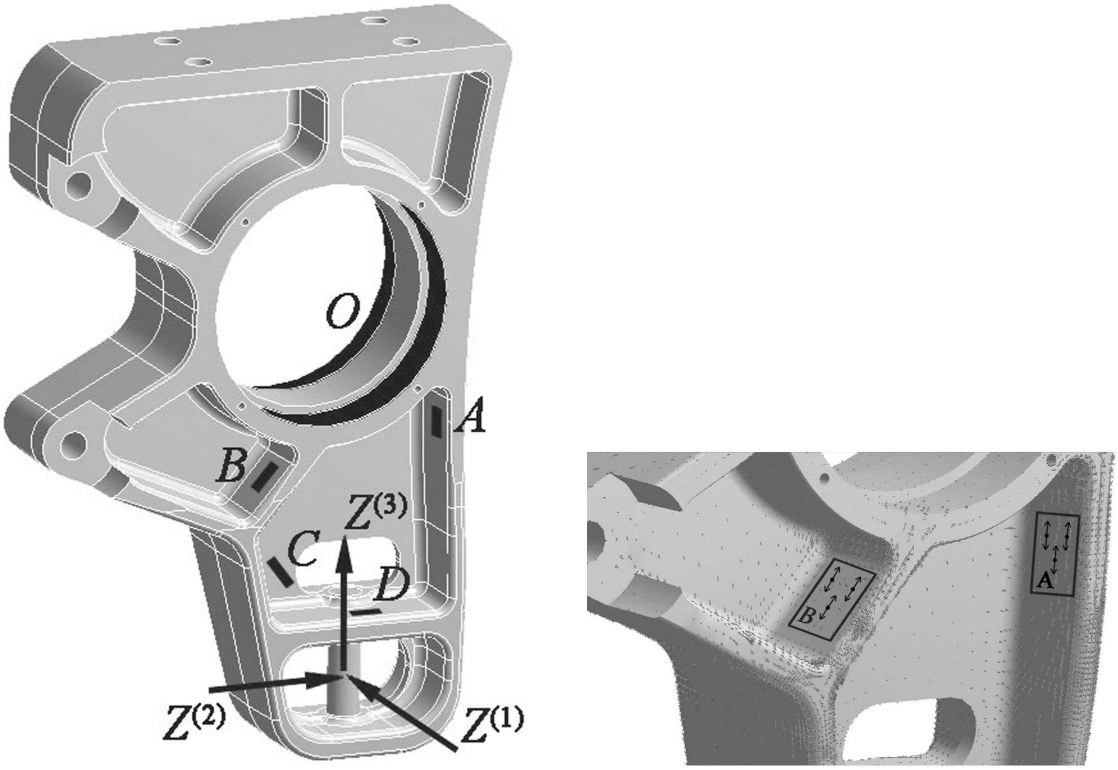
In this paper, α was calculated based on the residual principle [26], in which the relative level of the residual was taken to be 0.001.

Having solved system (9) and with available values $q_n^{(r)}$ ($r = \overline{1, R}$; $n = \overline{1, N}$), the final stage consists in restoring the approximate profile of sought-for functions $Z^{(r)}(t)$ with account of relationship $\tilde{q}_n^{(r)} = \sum_{k=1}^{n} q_k^{(r)}$ and equality (4).

5. Numerical results

5.1. Identification of three quasi-static forces using theoretical data

For numerical experiment, a wheel carrier (item 1 in Fig. 1) was considered to identify a loading presented as a system of three mutually perpendicular forces $Z^{(r)}(t)$ ($r = \overline{1, R}$; $R = 3$) applied to the lower part of the carrier (Fig. 5a). This structure was tightly attached to supporting cylindrical surfaces O (Fig. 5a). Physical properties of the structural material and its geometrical characteristics were known. Identification of $Z^{(r)}(t)$ was assumed to be carried out by the reading



(a) (b)

Fig. 5. Structure under investigation: (a) scheme of fastening and loading; (b) choosing the orientation of measurement axes of sensors and locations of measurement points

of four sensors ($S = 4$) measuring strains $V^{(s)}(t)$ ($s = \overline{1, S}$) at areas indicated in Fig. 5a (black rectangles A, B, C and D).

Locations of these areas, as well as directions of measurement axes of "sensors", were chosen by consecutive solving R "direct static problems" in case of action exclusively the r-th unit force ($Z^{(r)} = \overline{Z} = 1$, $Z^{(p)} = 0$; $r, p = \overline{1, R}$; $p \neq r$) and analysis of the vector strain field in the structure in order to reveal maximum values. For the considered problem these areas are indicated in Fig. 5a as A, B, C and D, and the directions of measurement axes are shown schematically for areas A and B in Fig. 5b.

For the first model example, the quasi-static loads were identified. The initial datum as functions $V^{(s)}$ were taken in the form shown in Fig. 6a (curves 1–4). The investigated time interval T_{inv} was 20 sec; the sampling time interval (step) Δt was constant and equal to 0.02 sec ($t_m = m\Delta t$; $m = 0, 1, \ldots, T_{\text{inv}}/\Delta t$).

The elements of the matrix of influence coefficients $\mathbf{A}$ (see (2)) were determined by averaging the values of strain in nodes of FE-model in measurement areas. So, at the stage of solving the r-th "direct static problems" the r-th column of the matrix $\mathbf{A}$ is calculated. As a result, the following matrix

$$\mathbf{A} = 10^{-8} \begin{bmatrix} -4.4 & 1.24 & 9.63 & 0.76 \\ 29.8 & 17.6 & 1.24 & -7.1 \\ -3.4 & -1.0 & -3.9 & 6.43 \end{bmatrix}^{\mathrm{T}}$$

was obtained. It should be noted that calculation was performed with FEM, the size and order of finite elements were chosen according to the convergence condition of results.

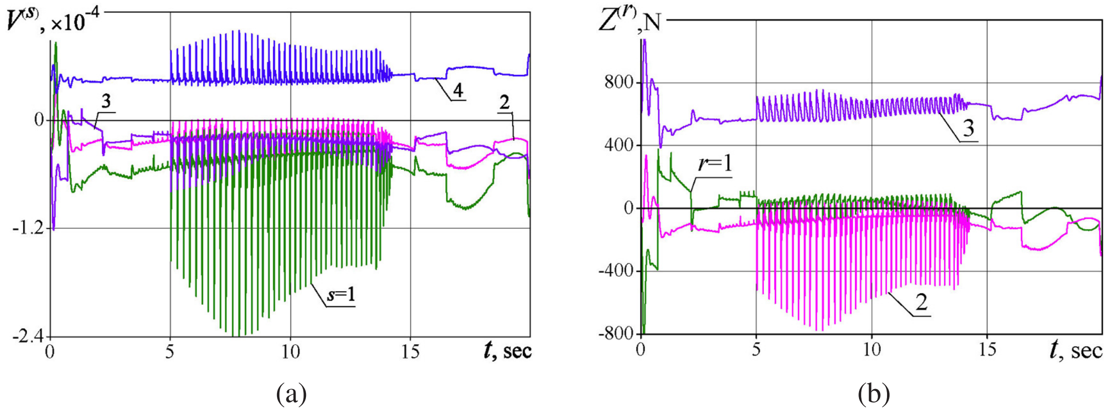


Fig. 6. Result of identification of quasi-static forces: (a) values of strain as input data; (b) values of identified loads

Having values $\{\mathbf{Y}_m\}_s = V^{(s)}(t_m)$ (Fig. 6a) and the matrix $\mathbf{A}$ at each time step t_m the system of equations (3) with unknowns $\{\mathbf{Z}_m\}_r$, that determine the approximated values of the sought-for functions $Z^{(r)}(t)$ in fixed time points $t = t_m$, has been solved (Fig. 2). The result of this calculation is shown in Fig. 6b. The subsequent FEM-solution of the problem for structure's strain computation in areas A, B, C and D, caused by application of loads according to Fig. 6b, yields a result with a close fit to the initial one (Fig. 6a).

Note that analysis of elements of matrix $\mathbf{A}$ allows establishing sensor s that is most sensitive to the action of the load $Z^{(r)}$. For this sensor $\mathbf{A}_{s,r} = \max(\mathbf{A}_{,r})$. Thus, the biggest sensitivity to the action of restoring $Z^{(1)}$ belongs to "sensor" C ($s = 3$): $\mathbf{A}_{3,1} = \max(\mathbf{A}_{,1})$. This helps working out recommendations for choosing an optimal number of registration quantities, as well as areas and directions of measurement. In particular, analysis of values for $\mathbf{A}$-matrix elements allows concluding about possibility of excluding the sensor in area B ($s = 2$), which actually duplicates the sensor in area A ($s = 1$).

5.2. Identification of two non-stationary forces using theoretical data

The second model example is the problem of identification non-stationary loads $Z^{(1)}$ and $Z^{(2)}$ acting on the wheel carrier (Fig. 5a). In doing so, it is assumed that $Z^{(3)} = 0$ ($R = 2$), and identification is carried out by registered strains $V^{(s)}(t)$ ($s = \overline{1, S}$) in areas A, B and C (Fig. 5a; $S = 3$).

At the first stage, to form functions $V^{(s)}(t)$, the corresponding "direct non-stationary problems" were solved by using a FE software system. Loads $Z^{(r)}(t)$ ($r = 1, 2$) were assumed known as time-variable and shown as dash-dotted curves $r = 1$ and $r = 2$ in Fig. 7a. The start point for applying the second load does not coincide with the initial time point $t = 0$, and it is shifted by an arbitrary time interval as shown in Fig. 7a. Investigation time T_{inv} was chosen to be 0.003 sec; time step Δt for the numerical solving of the system of differential equations was constant and equal to $T_{\text{inv}}/150$. Next, values $V^{(s)}(t)$ were superimposed with a "noise" with a zero mean value and an amplitude of 10% of the maximum value of its functions in the range of the investigated interval. This operation served as an imitation of random errors of in-situ measurements and data transmission, processing by measuring instruments. The chosen amplitude value has a gain margin over the order of accuracy corresponding to modern measurement practice. The resulting "noisy" functions $V^{(s)}(t)$ are shown in Fig. 7b and were taken as initial data.

The $S \cdot R = 6$ influence functions $\overline{V}^{(s,r)}(t)$ (see (6)) were determined by using a FE software system also. Some of these functions are shown in Fig. 7c, and designated in the s, r format.

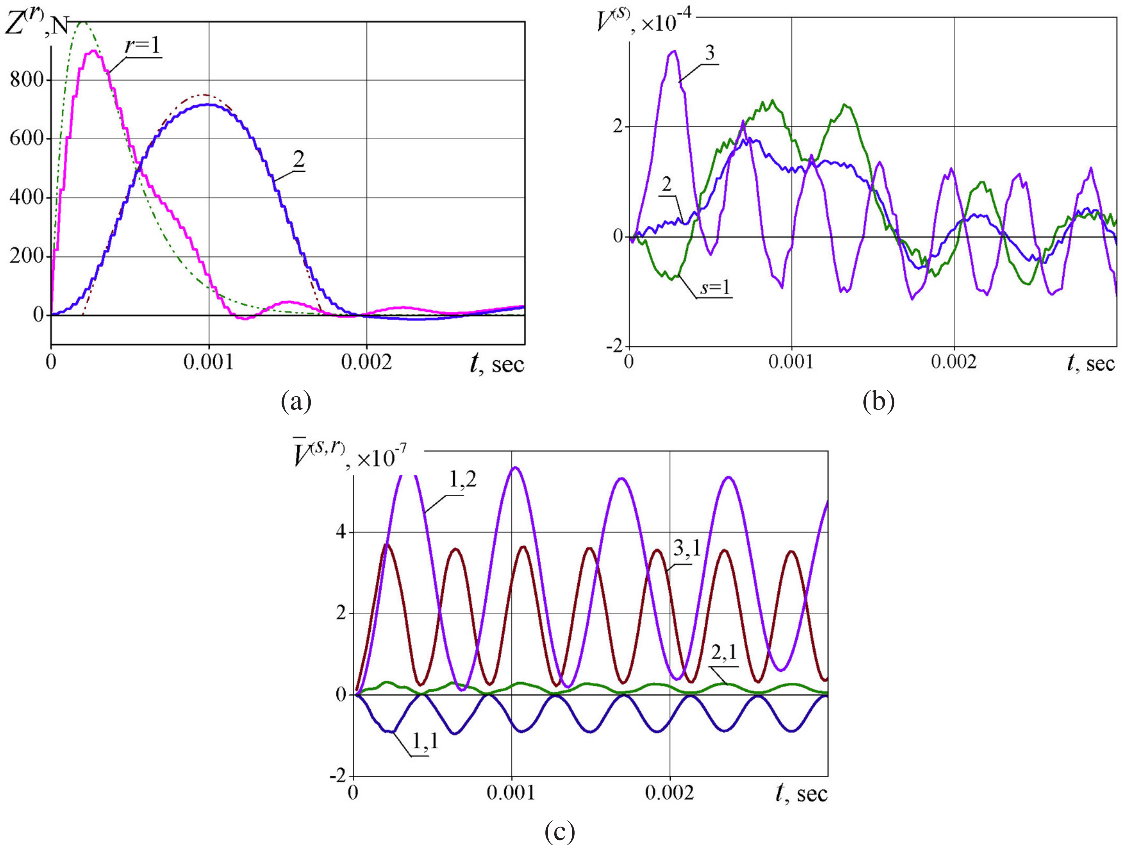


Fig. 7. Result of identification of non-stationary loads: (a) configurations of true and identified loads; (b) strain vs. time; (c) "influence"/"transfer" functions

Identified functions $Z^{(r)}$ $(r = 1, 2)$, presented by solid curves in Fig. 7a, agree satisfactorily with exact values (the dash-dotted curves). The maximum values divergence in the investigated time interval is within 13%.

Note that the transition from (8) to SLAE (9) was done using the generalized Gauss method. Equation (9) was solved using the classic version of the Tikhonov algorithm with a zero-order stabilizer ($\hat{\mathbf{L}}$ is an identity $(N \times N)$-matrix). The time intervals for approximation of $Z^{(r)}$ by piecewise constant functions (see (4)) were assumed to be $2\Delta t$ ($T_n - T_{n-1} = 2\Delta t$; $n = \overline{1, N}$; $N = T_{\text{inv}}/2\Delta t$).

5.3. Identification of non-stationary load using experimental data

The presented in this paper algorithm was also tested for identification of a concentrated non-stationary load applied axisymmetrically to a round plate with a rigidly fastened boundary. An experimental investigation of the vibration of plate with geometrical parameters $R = 60$ mm (radius) and $h = 2$ mm (thickness) and material properties $E = 70$ GPa, $\rho = 2\,700\,\text{kg/m}^3$ and $\nu = 0.33$ (the Young's modulus, density and Poisson's ratio, respectively) is presented in [29]. Schematic of experimental setup (Fig. 8) is also taken from this manuscript. Its authors used an instrumented hammer as to realize concentrated loading, as to record the real time dependence of the contact force (this result is shown in Fig. 9a as dashed curve 0). They also used a instrumentation complex to measure strains $\varepsilon_\theta(t)$ in the circumferential direction of the plate under the contact force in points positioned at 10 mm, 20 mm and 30 mm from the plate centre (curves $V^{(1)}(t)$, $V^{(2)}(t)$ and $V^{(3)}(t)$ in Fig. 9b, respectively). More detailed information about this experiment and used equipment can be found in [29].

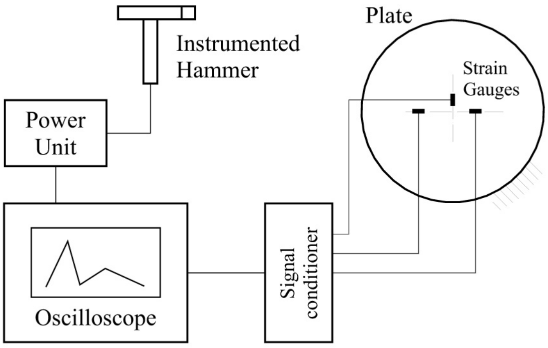


Fig. 8. Schematic of experimental setup

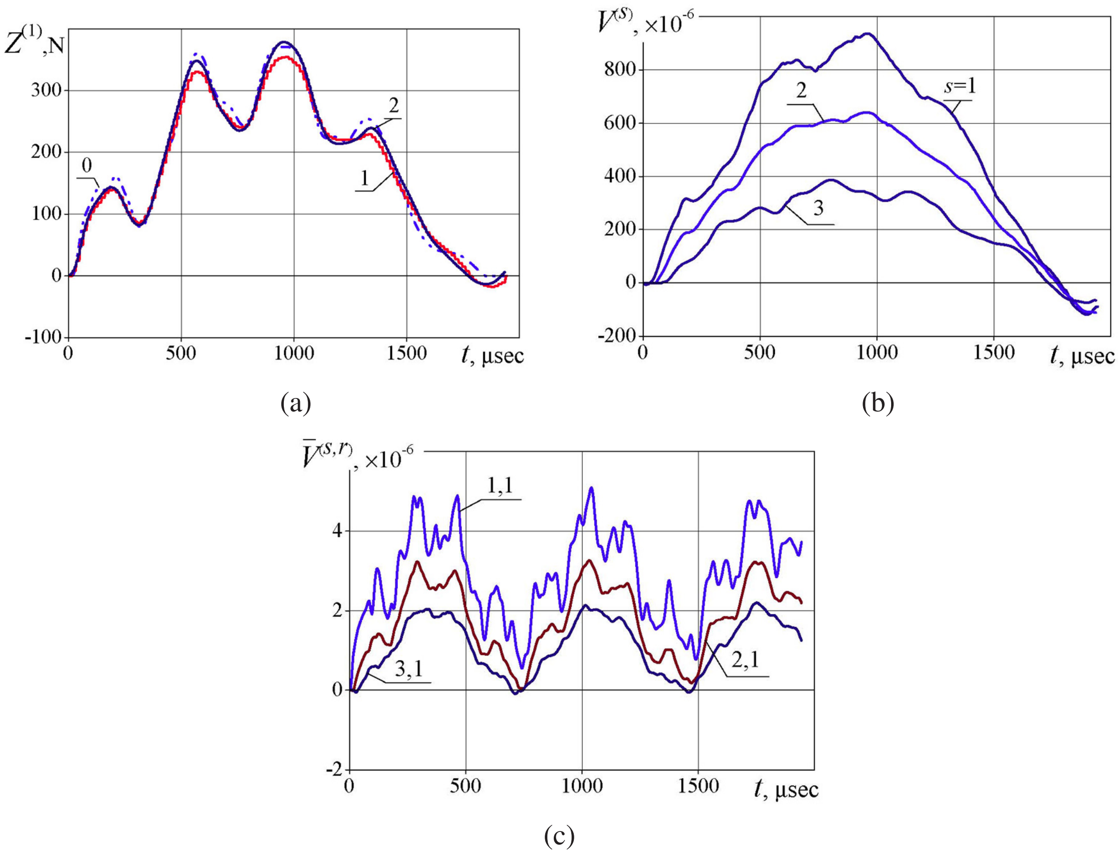


Fig. 9. Contact force identification based on experimental data: (a) configurations of true and identified loads; (b) strain vs. time; (c) "influence"/"transfer" functions (see Fig. 7c)

In our case, the inverse problem consists in restoring the contact force as a function of time $Z(t) = Z^{(R)}(t)$ $(R = 1)$ by the values of strains $V^{(s)}(t)$ $(s = \overline{1, S};\ S = 3)$. To solve it according to the algorithm suggested in this paper, the first stage consisted in using FEM-software to solve the "direct non-stationary problem". The purpose of this stage was to build transfer functions $\overline{V}^{(s,R)}(t)$ that define strains $\varepsilon_\theta(t)$ in points 10 mm $(s = 1)$, 20 mm $(s = 2)$ and 30 mm $(s = 3)$ for the plate under concentrated load in the form of Heaviside function $(Z^{(R)}(t) = \overline{Z} = 1)$. FE-analysis results are shown in Fig. 9c. During $\overline{V}^{(s,R)}(t)$ calculation, as well as previously, convergence of results was controlled. The time step was taken to be 2 μsec, the same as in [29]. Having both functions $V^{(s)}(t)$ (obtained by digitizing of graphs) and functions $\overline{V}^{(s,R)}(t)$ (calculated by FEM), at the second stage the SLAE (8) was solved for weight numbers $q_n^{(R)}$. These coefficients define the configuration of the required load $Z^{(R)}(t)$ (see (4)). The result of identification is presented as curve 2 in Fig. 9a. Curve 1 in this figure shows the result of contact load restoration based on data exclusively from the first strain gauge ($S = 1$). By comparing curves 1 and 2 with curve 0, one can conclude that both in the first case ($S = 3$) and in the second one ($S = 1$) the identified loads are in good agreement with the real load (dotted curve). However, for the given concrete example, the result of identification with an acceptable accuracy can be obtained based on data from one measuring point only ($S = 1$). This significantly reduces number of computational operations, and accordingly, the computational runtime.

6. Conclusion

The paper presents method of identification as a function of time of several non-stationary loads (including impulse ones) that acting on a structure with arbitrary geometry (topology). The initial data are quantities considered to be available for in-situ measurements including their variation with time due to the influence of sought-for loads. The method is based on the principle of superposition and the assumption that registered data are linearly dependent on loads acting on the structure.

Obviously that linearity of the problem statement contributes some limitations on application area of the presented results. At the same time advantages of the method are simplicity of numerical implementation and sustainability of results to random errors in initial data. This is ensured not only by involving a regularizing procedure but also by replacing a continuous function with a piecewise constant analog to significantly reduce the size of an computational system of linear algebraic equations (SLAE).

Identification quality depends not so much on the system solution accuracy as on the validity and accuracy of identification/simulation of dynamic processes in the investigated structure to build a matrix of influence functions (or coefficients for problem in quasi-static statement). This issue completely depends on modern achievements in solving direct problems in the mechanics of deformable solids. In this paper, these matrix were computed using the finite element method.

Identification quality also depends on the number of registered values S, which, in general, is not recommended to be smaller than the number R of sought-for functions. In the paper, when numerical experiments were performed, cases when $S > R$ were considered and the results obtained proved the efficiency of the suggested technique. At the same time, analysis of the obtained expressions shows that calculations for $S < R$ also can be done. In doing so, the procedure of identification of non-stationary loads remains invariable; however, in the case of quasi-static loads the Moore-Penrose method should be used for solving the computational SLAE.

Generally, the presented algorithms can be used both to identify the space-time dependence of forces applied to a structure with a complex geometry and for quantities of another physical nature (kinematic, thermal, electrical, etc.). This is indicative of the possibility of application the obtained results for solving of a wide class of inverse boundary problems in the mechanics of solids. Moreover, the demonstrated algorithms can be readily integrated in intergenerational development processes [18]. This process model is aimed at inheriting life cycle information to support the development of the next generations of components. Therefore, the shown algorithms can be used during product development phases.

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References

[1] Allen, M. S., Carne, T. G., Comparison of inverse structural filter (ISF) and sum of weighted accelerations technique (SWAT) time domain force identification methods, Mechanical Systems and Signal Processing 22 (2008) 1036–1054. https://doi.org/10.1016/j.ymssp.2007.11.022

[2] Atobe, S., Fukunaga, H., Hu, N., Impact force identification of CFRP structures using experimental transfer matrices, Computers Materials and Continua 26 (1) (2011) 67–90.

[3] Boukria, Z., Perrotin, P., Bennani, A., Experimental impact force location and identification using inverse problems: Application for a circular plate, International Journal of Mechanics 5 (1) (2011) 48–55.

[4] Chamorro-Servent, J., Aguirre, J., Ripoll, J., Vaquero, J. J., Desco, M., Feasibility of U-curve method to select the regularization parameter for fluorescence diffuse optical tomography in phantom and small animal studies, Optics Express 19 (12) (2011) 11490–11506. https://doi.org/10.1364/OE.19.011490

[5] Chen, C., Li, Y., Yuan, F.-G., Impact source identification in finite isotropic plates using a time-reversal method: Experimental study, Smart Materials and Structures 21 (2012) 105–129. https://doi.org/10.1088/0964-1726/21/10/105025

[6] Choi, H. G., Thite, A. N., Thompson, D. J., Comparison of methods for parameter selection in Tikhonov regularization with application to inverse force determination, Journal of Sound and Vibration 304 (2007) 894–917. https://doi.org/10.1016/j.jsv.2007.03.040

[7] Denkena, B. (Ed.), Gentelligent components in the life cycle: Use of inheritable, component-inherent information in production technology, TEWISS, 2010. (in German)

[8] Doyle, J. F., Modern experimental stress analysis, John Wiley & Sons Ltd., Chichester, 2004. https://doi.org/10.1002/0470861584

[9] Engl, H. W., Hanke, M., Neubauer, A., Regularization of inverse problems, Kluwer Academic Publ., The Netherlands, 1996. https://doi.org/10.1007/978-94-009-1740-8

[10] Fadoua, E. K., Abderrahman, H., Abdellatif, K., Abdelali, E., Rachid, D., Eric, J., Ali, L., Reconstruction of a distributed force impacting an elastic rectangular plate, Journal of Basic and Applied Scientific Research 1 (1) (2010) 20–30.

[11] Gombi, S. L., Ramakrishna, D. S., A solution to the inverse problem of impact force determination from structural responses, International Journal of Engineering and Innovative Technology 1 (3) (2012) 192–196.

[12] Hu, N., Fukunaga, H., Matsumoto, S., Yan, B., Peng, X. H., An efficient approach for identifying impact force using embedded piezoelectric sensors, International Journal of Impact Engineering 34 (2007) 1258–1271. https://doi.org/10.1016/j.ijimpeng.2006.05.004

[13] Inoue, H., Harrigan, J. J., Reid, S. R., Review of inverse analysis for indirect measurement of impact force, Applied Mechanics Reviews 56 (2001) 503–524. https://doi.org/10.1115/1.1420194

[14] Jacquelin, E., Bennani, A., Hamelin, P., Force reconstruction: Analysis and regularization of a deconvolution problem, Journal of Sound and Vibration 265 (2003) 81–107. https://doi.org/10.1016/S0022-460X(02)01441-4

[15] Krawczyk-Stado, D., Rudnicki, M., The use of L-curve and U-curve in inverse electromagnetic modelling, Intelligent Computer Techniques in Applied Electromagnetics 119 (2008) 73–82. https://doi.org/10.1007/978-3-540-78490-6_9

[16] Lachmayer, R., Gottwald, P., Integrated development by the consideration of product experiences, Proceedings of the 10th Int. Workshop on Integrated Design Engineering, Magdeburg, Inst. of Machine Design. Univ., Germany, 2015, pp. 73–82.

[17] Lachmayer, R., Mozgova, I., Sauthoff, B., Gottwald, P., Product evolution and optimization based on gentelligent components and product life cycle data, Smart Product Engineering (2013) 685–694. https://doi.org/10.1007/978-3-642-30817-8_67

[18] Lage, Y. E., Maia, N. M. M., Neves, M. M., Ribeiro, A. M. R., Force identification using the concept of displacement transmissibility, Journal of Sound and Vibration 332 (2013) 1674–1686. https://doi.org/10.1016/j.jsv.2012.10.034

[19] Lee, S.-K., Banerjee, S., Mal, A., Identification of impact force on a thick plate based on the elastodynamic and higher-order time-frequency analysis, Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science 221 (2007) 1249–1263. https://doi.org/10.1243/09544062JMES674

[20] Maia, N. M. M., Lage, Y. E., Neves, M. M., Recent advances on force identification in structural dynamics, In book "Advances in vibration engineering and structural dynamics", edited by F. Beltran-Carbajal, InTech, 2012, pp. 103–132. https://doi.org/10.5772/51650

[21] Martin, M. T., Doyle, J. F., Impact force identification from wave propagation responses, International Journal of Impact Engineering 18 (1996) 65–77. https://doi.org/10.1016/0734-743X(95)00022-4

[22] Rao, S. S., Mechanical vibrations, Prentice Hall, 2011.

[23] Romppanen, A.-J., Inverse load sensing method for line load determination of beam-like structures, Ph.D. thesis, Tampere University of Technology, 2008.

[24] Soloviev, A. N., Direct and inverse problems for finite elastic and electroelastic solids, Ph.D. thesis, Rostov/Don Russian Fed., 2005. (in Russian)

[25] Stevens, K. K., Force identification problem — An overview, Proceedings of Spring Conference on Experimental Mechanics, Florida, USA, 1987, pp. 838–844.

[26] Tikhonov, A. N., Goncharsky, A. V., Stepanov, V. V., Yagola, A.G., Numerical methods for the solution of ill-posed problems, Kluwer Academic Publ., Dordrecht, 1995. https://doi.org/10.1007/978-94-015-8480-7

[27] Wang, B.-T., Chiu, C.-H., Determination of unknown impact force acting on a simply supported beam, Mechanical Systems and Signal Processing 17 (3) (2003) 683–704. https://doi.org/10.1006/mssp.2001.1463

[28] Wang, B.-T., Chiu, C.-H., Determination of unknown impact force acting on arbitrary structures, Proceedings — Spie the International Society for Optical Engineering 2 (3727) (1999) 1653–1659.

[29] Wu, E., Tsai, T.-D., Yen, C.-S., Two methods for determining impact-force history on elastic plates, Experimental Mechanics 35 (1) (1995) 11–18. https://doi.org/10.1007/BF02325828

[30] Yagola, A. G., Ill-posed problems with a priori information, Siberian Electronic Mathematical Reports (2010) C.343–C.361. (in Russian)

[31] Yan, G., Zhou, L., Impact load identification of composite structure using genetic algorithms, Journal of Sound and Vibration 319 (2009) 869–884. https://doi.org/10.1016/j.jsv.2008.06.051

[32] Yanyutin, E. G., Yanchevsky, I. V., et al., Problems of impulse deformation of elements of construction, Publ. of KhNAHU, Kharkiv, 2004. (in Russian)