Západočeská univerzita v Plzni
Fakulta aplikovaných věd

Regularita zobrazení

Mgr. Tomáš Roubal

rigorózní práce
k získání akademického titulu doktor přírodních věd
v oboru Matematika

Katedra: Katedra matematiky

Plzeň 2018

University of West Bohemia
Faculty of Applied Sciences

Regularity of mappings

Mgr. Tomáš Roubal

Thesis for the award of the degree
of Doctor of Natural Sciences (RNDr.)
in the field: Mathematics

Department: Department of Mathematics

Pilsen 2018

Declaration

I hereby declare that this thesis is my own work, unless clearly stated otherwise.

. . . .
Tomáš Roubal

Acknowledgement

I would like to thank my thesis advisor Radek Cibulka for his scientific leadership, advices, patience, and time.

Abstract

The first aim of this thesis is to discuss metric regularity, metric subregularity, and metric semiregularity of both single-valued and set-valued mappings between metric spaces. Several equivalent properties are formulated and sufficient as well as necessary conditions are presented. Further we discuss stability of these properties with respect to single-valued and set-valued perturbations.

The second aim is to present local convergence theorems, Dennis-Moré theorems, and Kantorovich-type theorems for Newton-type methods for solving generalized equations. The methods are illustrated on non-smooth inequalities.

Keywords: set-valued mapping, generalized equation, metric regularity, metric subregularity, metric semiregularity, generalized equation, openness, Newton-type methods, Kantorovich-type theorem, Dennis-Moré theorem, criteria of regularity, perturbation stability.

Contents

1 Introduction **4**
1.1 Motivation . . . 4
1.2 Regularity of mappings . . . 7

2 Regularity criteria **16**
2.1 Historical background . . . 16
2.2 Regularity . . . 18
2.3 Subregularity . . . 24
2.4 Semiregularity . . . 26

3 Newton-type methods **28**
3.1 Historical background . . . 28
3.2 Local convergence theorems . . . 31
3.3 Dennis-Moré theorems . . . 32
3.4 Semilocal convergence theorems . . . 33
3.5 Numerical experiments . . . 34

Open problems **38**

List of symbols

$\emptyset$	empty set
∞	(positive) infinity
0	zero element
$A \times B$	Cartesian product of sets A and B
$x \in A$	x is an element of the set A
$A \subset B$	A is a subset of B
$A \cup B$	union of sets A and B
$A \cap B$	intersection of sets A and B
$A \setminus B$	set difference between A and B
$A + B$	$\{a + b : a \in A \text{ and } b \in B\}$
$A - B$	$A + (-B)$
$A + x$	$A + \{x\}$
$A - x$	$A + \{-x\}$
λA	$\{\lambda x : x \in A\}, \lambda \in \mathbb{R}$
$\overline{A}$	closure of the set A
$\inf A$	infimum of the set $A \subset \mathbb{R}$
$\min A$	minimum of the set $A \subset \mathbb{R}$
$\sup A$	supremum of the set $A \subset \mathbb{R}$
$\max A$	maximum of the set $A \subset \mathbb{R}$
$\mathbb{N}$	positive integers
$\mathbb{N}_0$	nonnegative integers
$\mathbb{R}$	real numbers
$\mathbb{Q}$	rational numbers
$\mathbb{R}_+$	non-negative real numbers
$\mathbb{R}^n$	Euclidean space of $x = (x_1, ..., x_n)^T$ having n real coordinates
$\mathbb{R}^n_+$	set of $x \in \mathbb{R}^n$ having non-negative coordinates
$a < b$	$b \in \mathbb{R}$ is greater than $a \in \mathbb{R}$
$a \leq b$	$b \in \mathbb{R}$ is greater than or equal to $a \in \mathbb{R}$
$a = b$	a equals to b
$a \neq b$	a is not equal to b
$a := b$	let a be defined by b
$\equiv$	identically equal
(X, d)	metric space X with the metric d
$\mathrm{dist}(x, A)$	$\inf\{d(x, y) : y \in A\}$; the distance between a point $x \in X$ and a set $A \subset X$
$\lvert \cdot \rvert$	absolute value in $\mathbb{R}$
$\lVert \cdot \rVert_X$	norm in X
$(X, \lVert \cdot \rVert_X)$	normed space X with the norm $\lVert \cdot \rVert_X$
$\langle \cdot, \cdot \rangle$	duality pairing or inner product
X^*	set of all bounded linear functionals on X
$\mathcal{L}(X, Y)$	space of all linear continuous mappings between Banach spaces X and Y

$[a,b]$	closed interval in $\mathbb{R}$ with $a<b$
(a,b)	open interval in $\mathbb{R}$ with $a<b$
$\mathbb{B}(x,r)$	open ball centered at $x\in X$ with the radius $r>0$
$\mathbb{B}[x,r]$	closed ball centered at $x\in X$ with the radius $r>0$
$\mathbb{S}_X$	unit sphere centered at $0\in X$
$\mathbb{B}_X$	unit ball centered at $0\in X$
$f:X\to Y$	single-valued mapping f from X to Y
$F:X\rightrightarrows Y$	set-valued mapping F from X to Y
(x_k)	sequence of elements x_k
$\lim\limits_{n\to\infty} x_n$	limit of the sequence (x_k)
$\to$	converges to or maps to
$\longmapsto$	maps to
$a_k\uparrow b$	(a_k) converges to b with $a_k<b$
$a_k\downarrow b$	(a_k) converges to b with $b<a_k$
$\nabla f(x)$	derivative of a mapping $f:\mathbb{R}^n\to\mathbb{R}^m$ at $x\in\mathbb{R}^n$

Chapter 1

Introduction

1.1 Motivation

Let us consider a single-valued mapping f from X into Y, where X and Y are metric spaces and let $\bar{x} \in X$ be fixed. The mapping f is called open at $\bar{x}$ if the image of every neighborhood of $\bar{x}$ in X is a neighborhood of $f(\bar{x})$ in Y. The mapping f is said to be open if the image of every open set in X is an open set in Y. Suppose for a moment that f is one-to-one taking X onto Y so that there exists the single-valued inverse mapping f^{-1} defined on whole of Y. Then the openness of f at $\bar{x}$ is equivalent to the continuity of f^{-1} at $f(\bar{x})$ which means that the unique solution $x \in X$ of the equation

$$f(x) = y \tag{1.1}$$

is close to $\bar{x}$ whenever $y \in Y$ is sufficiently close to $f(\bar{x})$. Suppose now that f is not one-to-one. Then the solutions of the equation (1.1) may not be determined uniquely and the openness of f at $\bar{x}$ expresses the fact that whenever $y \in Y$ is sufficiently close to $f(\bar{x})$, then there exists a solution $x \in X$ of the equation (1.1) which is close to $\bar{x}$. In this case the inverse mapping f^{-1} is set-valued and we will see later that openness of f is equivalent to a certain kind of continuity of f^{-1}.

A set-valued mapping G from X into Y, denoted by $G : X \rightrightarrows Y$, is determined by a subset of $X \times Y$ called the graph of G denoted by Graph G. Then G assigns to a point $x \in X$ a (possibly empty) subset $G(x)$ of Y, which contains all $y \in Y$ such that $(x, y) \in$ Graph G and is called the *image* of x under G or the *value* of G at x. The *domain* of G, denoted by dom G, is the set of points $x \in X$ such that the set $G(x)$ is nonempty, and the *range* of G, denoted by rge G, is the union of all sets $G(x)$ for $x \in$ dom G. Such a mapping G has always the *inverse*, denoted by G^{-1}, which is the set-valued mapping from Y to X such that, for each $(x, y) \in X \times Y$, the point $(y, x) \in$ Graph G^{-1} if and only if $(x, y) \in$ Graph G. To emphasize that a mapping from X into Y is single-valued, we use lower-case letters and write $g : X \to Y$.

Let a set-valued mapping $F : X \rightrightarrows Y$ and a point $(\bar{x}, \bar{y}) \in$ Graph F be given. Consider the problem, for a fixed $y \in Y$, to find $x \in X$ such that

$$F(x) \ni y. \tag{1.2}$$

The openness of F at $(\bar{x}, \bar{y})$ means again that, for each neighborhood U of $\bar{x}$ in X, the set $F(U) := \bigcup_{x \in U} F(x)$ is a neighborhood of $\bar{y}$ in Y. In other words, for each $y \in Y$ sufficiently close to $\bar{y}$, there is a solution $x \in X$ of the inclusion (1.2) which is close to $\bar{x}$.

In both the cases the openness gives us the existence of a solution but does not say anything about the distance between the solution x and the reference point $\bar{x}$. In order to get such an estimate, we define openness of F *at* $(\bar{x}, \bar{y})$ with a linear rate which means there is a constant $c > 0$ such that for each $r > 0$ small enough the image of a ball around $\bar{x}$ with the radius r contains a ball around $\bar{y}$ with the radius $c \cdot r$. This property is equivalent to a certain calmness property of the inverse F^{-1}.

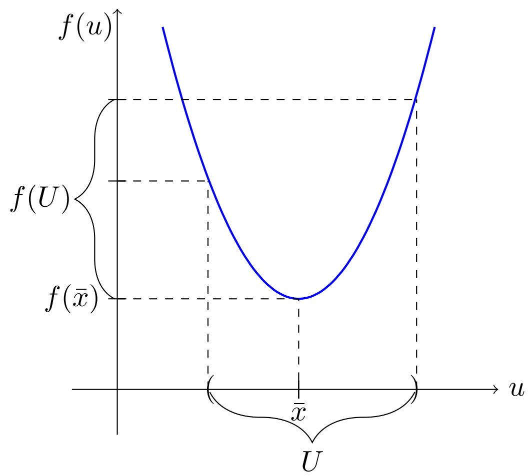

Figure 1.1: The function which is not open at $\bar{x}$.

Even we can request the above property to be satisfied for each point (x, y) close to $(\bar{x}, \bar{y})$, with the same constant c independent of (x, y). This property is called openness *around* $(\bar{x}, \bar{y})$ with a linear rate and is equivalent to a certain kind of Lipschitz property of the inverse mapping F^{-1} called Aubin property. There is the third equivalent property called metric regularity which will be defined later. If X and Y are Banach spaces, then the well-known Banach open mapping principle says that *a continuous linear operator from X to Y is open with a linear rate around any reference point if and only if it maps X onto Y.* A generalization of this principle to nonlinear mappings, proved by L.M. Graves, says that *a continuously differentiable mapping f from X to Y is open around a point $\bar{x} \in X$ with a linear rate if and only if its derivative $f'(\bar{x})$ is surjective.*

Now let $X := \mathbb{R}^n$ and $Y := \mathbb{R}$. Given $f : \mathbb{R}^n \to \mathbb{R}$, consider a problem

$$\text{(1.3)} \qquad \text{minimize} \quad f(u) \quad \text{subject to} \quad u \in \mathbb{R}^n.$$

Let $\bar{x} \in \mathbb{R}^n$ be a solution of (1.3). Then there is a neighborhood U of $\bar{x}$ in $\mathbb{R}^n$ such that $f(U)$ is not a neighborhood of $f(\bar{x})$, hence f is not open at $\bar{x}$, see Figure 1.1. Hence negation of any sufficient condition for openness (or openness with a linear rate) gives us a necessary condition for f to attain its minimum (or maximum) at $\bar{x}$. An example of such a condition is Graves theorem. Suppose that f is a smooth function on $\mathbb{R}^n$. The derivative of f at $\bar{x}$ can be represented by the gradient $\nabla f(\bar{x})$ of f at $\bar{x}$ and the linear function $\mathbb{R}^n \ni u \longmapsto \langle \nabla f(\bar{x}), u \rangle$ is not surjective if and only if $\nabla f(\bar{x}) = 0$. So we have derived Euler-Fermat necessary condition.

This idea can be generalized, for example, to a constrained minimization problem in the form:

$$\text{(1.4)} \qquad \text{minimize } f(u) \quad \text{subject to} \quad g_i(u) = 0 \quad \text{for} \quad i = 1, \dots, m,$$

where functions $g_i : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable. Let $\bar{x} \in \mathbb{R}^n$ be a solution of (1.4) and define a mapping $h : \mathbb{R}^n \to \mathbb{R}^{m+1}$ by

$$h(u) := (f(u), g_1(u), g_2(u), \dots, g_m(u))^T, \quad u \in \mathbb{R}^n.$$

Consequently we have

$$h(\bar{x}) = (f(\bar{x}), 0, 0, \dots, 0)^T \quad \text{and} \quad \nabla h(\bar{x}) = (\nabla f(\bar{x}), \nabla g_1(\bar{x}), \nabla g_2(\bar{x}), \dots, \nabla g_m(\bar{x}))^T.$$

Fix any sufficiently small $\varepsilon > 0$ and let

$$y := (f(\bar{x}) - \varepsilon, 0, 0, \dots, 0).$$

Then there is no $x \in \mathbb{R}^n$ with $h(x) = y$. Indeed, for any such a point x, we would have $f(x) = f(\bar{x}) - \varepsilon < f(\bar{x})$ and $g_i(x) = 0$ for each $i = 1, \ldots, m$, which contradicts the assumption that $\bar{x}$ solves (1.4). Consequently, h is not open at $\bar{x}$ and, by Graves theorem, the mapping $\mathbb{R}^n \ni u \longmapsto \nabla h(\bar{x})u$ is not surjective. This means that the rows of the Jacobian matrix $\nabla h(\bar{x})$ are linearly dependent. In other words, there are numbers $\lambda_i \in \mathbb{R}$, for $i = 0, 1, \ldots, m$, such that

$$\lambda_0 \nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) + \cdots + \lambda_m \nabla g_m(\bar{x}) = 0.$$

If all the vectors $\nabla g_i(\bar{x})$, for $i = 1, 2, \ldots, m$, are linearly independent, then the previous equality can be rewritten as

$$\nabla f(\bar{x}) + \lambda_1 \nabla g_1(\bar{x}) + \lambda_2 \nabla g_2(\bar{x}) + \cdots + \lambda_m \nabla g_m(\bar{x}) = 0.$$

The numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are called the Lagrange multipliers. We have derived Karush-Kuhn–Tucker necessary conditions for the problem (1.4). An interesting fact is that W. Karush, who derived these conditions in his master thesis in 1939, was a student of Graves.

1.2 Regularity of mappings

In this section, we will present various regularity properties of a set-valued mapping $F : X \rightrightarrows Y$, that maps from a metric space (X, d) into subsets of a metric space (Y, ρ), around/at the reference point. We will focus on three properties of set-valued mappings called regularity, subregularity, and semiregularity. At the end of this section we will present "stronger" versions of these properties. All of them play a fundamental role in modern variational analysis, non-smooth analysis, and optimization. We will illustrate this on the problems (1.1) and (1.2).

By the term *semiregularity* at the reference point we mean the group of three equivalent properties called metric semiregularity, openness with a linear rate at the reference point, and recession with a linear rate of the inverse. Metric semiregularity was introduced by A.Y. Kruger in [30] in 2009 and can be found under the name *hemiregularity*, in [2, 19].

Definition 1.2.1 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* metrically semiregular at $(\bar{x}, \bar{y})$ *when $\bar{y} \in F(\bar{x})$ and there is a constant $\kappa > 0$ along with a neighborhood V of $\bar{y}$ in Y such that*

$$\text{dist}\,\big(\bar{x}, F^{-1}(y)\big) \leq \kappa\rho(y, \bar{y}) \quad \textit{for every} \quad y \in V. \tag{1.5}$$

The infimum of $\kappa > 0$ for which there exists a neighborhood V of $\bar{y}$ in Y such that (1.5) *holds is called the* semiregularity modulus *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\text{semireg}\, F(\bar{x}, \bar{y})$.

We use the convention that $\inf \emptyset = \infty$, that is, semireg $F(\bar{x}, \bar{y}) < \infty$ if and only if F is metrically semiregular at $(\bar{x}, \bar{y})$. For a single valued mapping $f : X \to Y$ we omit the point $\bar{y} = f(\bar{x})$, that is, we write semireg $f(\bar{x})$ (and the same applies in all the definitions below) and for a linear mapping $A : X \to Y$ we omit even the point $\bar{x}$, that is, we write semireg A (and the same applies for the other properties). Now suppose for a moment that F is metrically semiregular at $(\bar{x}, \bar{y})$. Let $\kappa >$ semireg $F(\bar{x}, \bar{y})$ be arbitrary. From (1.5), for a fixed $y \in V$, we have

$$\text{dist}\,\big(\bar{x}, F^{-1}(y)\big) < \infty,$$

that is, the set $F^{-1}(y)$ is nonempty. Moreover, there is a point $x \in X$ with $y \in F(x)$ such that

$$d(\bar{x}, x) \leq \kappa\rho(y, \bar{y}).$$

Metric semiregularity guarantees the solvability of (1.2) for $y \in V$ and also the estimate of the distance between the reference point $\bar{x}$ and the solution x. In other words it guarantees the stability of a solution with respect to small perturbations of the right-hand side.

Metric semiregularity is equivalent to openness with a linear rate at the reference point which can be found under the name *controllability*, in [19, 18] and [23, 25, 22].

Definition 1.2.2 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* open with a linear rate at $(\bar{x}, \bar{y})$ *when $\bar{y} \in F(\bar{x})$ and there are positive constants c and ε such that*

$$\mathbb{B}[\bar{y}, ct] \subset F(\mathbb{B}[\bar{x}, t]) \quad \textit{for each} \quad t \in (0, \varepsilon). \tag{1.6}$$

The supremum of $c > 0$ for which there exists a constant $\varepsilon > 0$ such that (1.6) holds is called the modulus of openness *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\text{lopen}\, F(\bar{x}, \bar{y})$.

As we work with nonnegative quantities, we use the convention that $\sup \emptyset = 0$, that is, lopen $F(\bar{x}, \bar{y}) > 0$ if and only if F is open with a linear rate at $(\bar{x}, \bar{y})$.

Recession with a linear rate, introduced by A.D. Ioffe in [25], closes the first group of definitions. Note that this property is sometimes called pseudo-Calmness [19] or Lipschitz-lower semicontinuity [30].

Definition 1.2.3 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to* recede from $\bar{y}$ at $(\bar{x}, \bar{y})$ with a linear rate *when $\bar{y} \in F(\bar{x})$ and there is a constant $\mu > 0$ along with a neighborhood U of $\bar{x}$ in X such that*

$$\operatorname{dist}\big(\bar{y}, F(x)\big) \le \mu\, d(\bar{x}, x) \quad \text{for each} \quad x \in U. \tag{1.7}$$

The infimum of $\mu > 0$ for which there exists a neighborhood U of $\bar{x}$ in X such that (1.7) *holds is called the* speed of recession *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{recess} F(\bar{x}, \bar{y})$.

The mapping F recedes from $\bar{y}$ at $(\bar{x}, \bar{y})$ with a linear rate if and only if $\operatorname{recess} F(\bar{x}, \bar{y}) < \infty$. If, in addition, the space Y is a vector (linear) space, then for any $\mu > \operatorname{recess} F(\bar{x}, \bar{y})$ there is a neighborhood U of $\bar{x}$ in X such that

$$\bar{y} \in F(x) + \mu\, d(\bar{x}, x)\, \mathbb{B}_Y \quad \text{for each} \quad x \in U.$$

Example 1.2.1 *Consider a single-valued mapping $f : X \to Y$ which recedes from $f(\bar{x})$ at $\bar{x}$ with a linear rate. Then for any $\mu > \operatorname{recess} f(\bar{x})$ there is a neighborhood U of $\bar{x}$ in X such that*

$$\rho\big(f(\bar{x}), f(x)\big) \le \mu\, d(\bar{x}, x) \quad \text{for each} \quad x \in U.$$

This is the definition of calmness *of f at $\bar{x}$.*

The following theorem guarantees the above mentioned equivalence of metric semiregularity, openness with a linear rate at the reference point, and recession with a linear rate of the inverse, for the proof see [14, Proposition 2.1].

Theorem 1.2.1 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$. The following assertions are equivalent:*

(i) *F is metrically semiregular at $(\bar{x}, \bar{y})$;*

(ii) *F is open with a linear rate at $(\bar{x}, \bar{y})$;*

(iii) *F^{-1} recedes from $\bar{x}$ at $(\bar{y}, \bar{x})$ with a linear rate.*

In addition it holds

$$\operatorname{lopen} F(\bar{x}, \bar{y}) \cdot \operatorname{semireg} F(\bar{x}, \bar{y}) = 1 \quad \text{and} \quad \operatorname{semireg} F(\bar{x}, \bar{y}) = \operatorname{recess} F^{-1}(\bar{y}, \bar{x}), \tag{1.8}$$

under a convention $0 \cdot \infty = \infty \cdot 0 = 1$.

The above statement justifies the following definition.

Definition 1.2.4 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* semiregular at $(\bar{x}, \bar{y})$ *if and only if* $\operatorname{semireg} F(\bar{x}, \bar{y}) < \infty$ *if and only if* $\operatorname{lopen} F(\bar{x}, \bar{y}) > 0$ *if and only if* $\operatorname{recess} F^{-1}(\bar{y}, \bar{x}) < \infty$.

Further, by the term *subregularity* at the reference point we mean the group of three equivalent properties called metric subregularity, pseudo-openness with a linear rate at the reference point, and calmness of the inverse. Metric subregularity is entrenched in the literature [18].

Definition 1.2.5 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* metrically subregular at $(\bar{x}, \bar{y})$ *when $\bar{y} \in F(\bar{x})$ and there is a constant $\kappa > 0$ along with a neighborhood U of $\bar{x}$ in X such that*

$$\operatorname{dist}\big(x, F^{-1}(\bar{y})\big) \le \kappa \operatorname{dist}\big(\bar{y}, F(x)\big) \quad \text{for every} \quad x \in U. \tag{1.9}$$

The infimum of $\kappa > 0$ for which there exists a neighborhood U of $\bar{x}$ in X such that (1.9) *holds is called the* subregularity modulus *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{subreg} F(\bar{x}, \bar{y})$.

The mapping F is metrically subregular at $(\bar{x}, \bar{y})$ if and only if $\operatorname{subreg} F(\bar{x}, \bar{y}) < \infty$. Note that metric subregularity does not guarantee solvability of (1.1) and (1.2), respectively, as in the case of semiregularity.

Example 1.2.2 *Consider a single-valued mapping $f : X \to Y$ which is metrically subregular at a point $\bar{x} \in X$. Then for any $\kappa >$ subreg $f(\bar{x})$ there is a neighborhood of U of $\bar{x}$ such that for a fixed $x \in U$ there is $x' \in X$ such that*

$$\bar{y} = f(x') \quad \textit{and} \quad d\big(x, x'\big) \leq \kappa \rho\big(\bar{y}, f(x)\big).$$

In other words, if $x \in U$ is an approximate solution of (1.1) with $y := \bar{y}$, then we can estimate the distance from x to the solution set $f^{-1}(\bar{y})$ by the residuum $\rho\big(\bar{y}, f(x)\big)$. The same is true for set-valued mappings.

The following proposition shows us two more equivalent properties to metric subregularity.

Proposition 1.2.1 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$. The following assertions are equivalent:*

(i) *F is metrically subregular at $(\bar{x}, \bar{y})$;*

(ii) *there is a constant $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\big(x, F^{-1}(\bar{y})\big) \leq \kappa \operatorname{dist}(\bar{y}, F(x) \cap V) \quad \textit{for each} \quad x \in U;$$

(iii) *there is a constant $\kappa > 0$ along with a neighborhood U of $\bar{x}$ in X such that*

$$\operatorname{dist}\big(x, F^{-1}(\bar{y})\big) \leq \operatorname{dist}_{1,\kappa}((x, \bar{y}), \operatorname{Graph} F) \quad \textit{for each} \quad x \in U,$$

where for a subset $A \subset X \times Y$ and a point $(u, v) \in X \times Y$ we define

$$\operatorname{dist}_{1,\kappa}((u, v), A) := \inf\{d(u, u') + \kappa \rho(v, v') : (u', v') \in A\}. \tag{1.10}$$

The equivalence (i) $\Leftrightarrow$ (ii) was showed in [18, Exercise 3H.4]. The property (iii) is called *graph-subregularity* of F at $(\bar{x}, \bar{y})$ and was proved to be equivalent to (i) in [26]. It uses the graph of F instead of the values of F. The mapping $X \times Y \ni (x, y) \longmapsto \operatorname{dist}_{1,\kappa}((x, y), \operatorname{Graph} F)$ is Lipschitz continuous whereas the mapping $X \times Y \ni (x, y) \longmapsto \operatorname{dist}(y, F(x))$ may be not even continuous. Therefore sometimes it is convenient to work with the latter property.

Next property is pseudo-openness that is defined and proved to be equivalent to metric subregularity and calmness in [1].

Definition 1.2.6 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* pseudo-open with a linear rate at $(\bar{x}, \bar{y})$ *when $\bar{y} \in F(\bar{x})$ and there are positive constants c and ε along with a neighborhood U of $\bar{x}$ in X such that*

$$\bar{y} \in F(\mathbb{B}[x, t]) \quad \textit{whenever} \quad x \in U \cap F^{-1}(\mathbb{B}[\bar{y}, ct]) \quad \textit{and} \quad t \in (0, \varepsilon). \tag{1.11}$$

The supremum of $c > 0$ for which there exist a constant $\varepsilon > 0$ and a neighborhood U of $\bar{x}$ in X such that (1.11) holds is called the modulus of pseudo-openness *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{popen} F(\bar{x}, \bar{y})$.

The mapping F is pseudo-open at $(\bar{x}, \bar{y})$ with a linear rate if and only if $\operatorname{popen} F(\bar{x}, \bar{y}) > 0$.

Calmness is entrenched in literature [25, 18] and closes the second group of definitions.

Definition 1.2.7 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* calm at $(\bar{x}, \bar{y})$ *when $\bar{y} \in F(\bar{x})$ and there is a constant $\mu > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\big(y, F(\bar{x})\big) \leq \mu\, d(x, \bar{x}) \quad \textit{whenever} \quad x \in U \quad \textit{and} \quad y \in F(x) \cap V. \tag{1.12}$$

The infimum of $\mu > 0$ for which there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.12) *holds is called the* calmness modulus *of F at $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{calm} F(\bar{x}, \bar{y})$.

Hence the mapping F is calm at $(\bar{x}, \bar{y})$ if and only if $\operatorname{calm} F(\bar{x}, \bar{y}) < \infty$. If, in addition, the space Y is a vector space, then for any $\mu > \operatorname{calm} F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$F(x) \cap V \subset F(\bar{x}) + \mu\, d(x, \bar{x})\, \mathbb{B}_Y \quad \text{for each} \quad x \in U.$$

Example 1.2.3 *Consider a single-valued mapping $f : X \to Y$ which is calm at a point $\bar{x} \in X$. Then for any $\mu >$* calm $F(\bar{x}, \bar{y})$ *there is a neighborhood U of $\bar{x}$ such that*

$$\rho\big(f(x), f(\bar{x})\big) \leq \mu\, d(x, \bar{x}) \quad \textit{for each} \quad x \in U.$$

In this case, calmness and recession with a linear rate coincide.

The following theorem, established in [30], guarantees the equivalence of metric subregularity, pseudo-openness with a linear rate, and calmness of the inverse.

Theorem 1.2.2 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in$* Graph F. *The following assertions are equivalent:*

(i) *F is metrically subregular at $(\bar{x}, \bar{y})$;*

(ii) *F is pseudo-open with a linear rate at $(\bar{x}, \bar{y})$;*

(iii) *F^{-1} is calm at $(\bar{y}, \bar{x})$.*

In addition, it holds

$$\operatorname{popen} F(\bar{x}, \bar{y}) \cdot \operatorname{subreg} F(\bar{x}, \bar{y}) = 1 \quad \textit{and} \quad \operatorname{subreg} F(\bar{x}, \bar{y}) = \operatorname{calm} F^{-1}(\bar{y}, \bar{x}). \tag{1.13}$$

The above statement justifies the following definition.

Definition 1.2.8 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* subregular at $(\bar{x}, \bar{y})$ *if and only if* subreg $F(\bar{x}, \bar{y}) < \infty$ *if and only if* popen $F(\bar{x}, \bar{y}) > 0$ *if and only if* calm $F^{-1}(\bar{y}, \bar{x}) < \infty$.

We have seen that semiregularity of the mappings appearing in (1.1) or (1.2) gives us solvability of these problems as well as stability of a solution with respect to small perturbations of the right-hand side. On the other hand, subregularity provides an estimate of the error of an approximate solution via the residuum. Now we present a property which guarantees both the previous ones. By the term *regularity* around the reference point we mean the group of equivalent properties called metric regularity, openness with a linear rate around the reference point, and Aubin property of the inverse.

The name metric regularity was suggested by J.M. Borwein [7] in 1986.

Definition 1.2.9 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* metrically regular *around $(\bar{x}, \bar{y})$ when $\bar{y} \in F(\bar{x})$ and there is a constant $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\big(x, F^{-1}(y)\big) \leq \kappa \operatorname{dist}(y, F(x)) \quad \textit{for every} \quad (x, y) \in U \times V. \tag{1.14}$$

The infimum of $\kappa > 0$ for which there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.14) *holds is called the* regularity modulus *of F around $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{reg} F(\bar{x}, \bar{y})$.

The mapping F is metrically regular at $(\bar{x}, \bar{y})$ if and only if $\operatorname{reg} F(\bar{x}, \bar{y}) < \infty$. In this case, for any $\kappa > \operatorname{reg} F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.14) holds. Letting $x := \bar{x}$, we get

$$\operatorname{dist}\left(\bar{x}, F^{-1}(y)\right) \leq \kappa \operatorname{dist}(y, F(\bar{x})) \leq \kappa\, \rho(y, \bar{y}) \quad \text{for every} \quad y \in V.$$

We derived (1.5), hence F is semiregular at $(\bar{x}, \bar{y})$. Further, letting $y := \bar{y}$ in (1.14), we get (1.9), which means F is subregular at $(\bar{x}, \bar{y})$.

There are several equivalent definitions in the literature.

Proposition 1.2.2 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$. The following assertions are equivalent:*

(i) *F is metrically regular around $(\bar{x}, \bar{y})$;*

(ii) *there is $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\left(x, F^{-1}(y)\right) \leq \kappa \operatorname{dist}(y, F(x) \cap V) \quad \textit{for each} \quad (x, y) \in U \times V;$$

(iii) *there is $\kappa > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\left(x, F^{-1}(y)\right) \leq \operatorname{dist}_{1,\kappa}((x, y), \operatorname{Graph} F) \quad \textit{for each} \quad (x, y) \in U \times V, \tag{1.15}$$

where $\operatorname{dist}_{1,\kappa}$ is defined in (1.10).

The equivalence (i) $\Leftrightarrow$ (ii) was showed in [18, Proposition 5H.1]. The property (iii) is called *graph-regularity at $(\bar{x}, \bar{y})$* in [39], where the equivalence (i) $\Leftrightarrow$ (iii) was proved.

Openness with a linear rate around the reference point is a stronger concept than openness with a linear rate at the reference point defined above.

Definition 1.2.10 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* open with a linear rate *around $(\bar{x}, \bar{y})$ when $\bar{y} \in F(\bar{x})$ and there are positive constants c and ε along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\mathbb{B}[y, ct] \subset F(\mathbb{B}[x, t]) \quad \textit{whenever} \quad (x, y) \in U \times V, \quad y \in F(x), \quad \textit{and} \quad t \in (0, \varepsilon). \tag{1.16}$$

The supremum of $c > 0$ for which there exist a constant $\varepsilon > 0$ and a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.16) holds is called the modulus of surjection *of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{sur} F(\bar{x}, \bar{y})$.*

The mapping F is open around $(\bar{x}, \bar{y})$ with a linear rate if and only if $\operatorname{sur} F(\bar{x}, \bar{y}) > 0$. The following statement, proved in [18, Theorem 5H.3], contains another equivalent definition of linear openness around the reference point.

Proposition 1.2.3 *A set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) is open with a linear rate around $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$ if and only if there are $c > 0$ and $\varepsilon > 0$ such that*

$$\mathbb{B}(y, ct) \cap \mathbb{B}(\bar{y}, \varepsilon) \subset F(\mathbb{B}(x, t)) \ \textit{whenever}\ (x, y) \in \operatorname{Graph} F, \quad d(x, \bar{x}) < \varepsilon, \quad \textit{and} \quad t \in (0, \varepsilon).$$

Aubin property, introduced by J.-P. Aubin in [4] under the name pseudo-Lipschitz property, closes the third group of definitions. We can also find a term Lipschitz-like property in literature [31].

Definition 1.2.11 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to have* Aubin property *around $(\bar{x}, \bar{y})$ when $\bar{y} \in F(\bar{x})$ and there is a constant $\mu > 0$ along with a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that*

$$\operatorname{dist}\big(y, F(x')\big) \leq \mu\, d(x, x') \quad \textit{whenever} \quad x, x' \in U \quad \textit{and} \quad y \in F(x) \cap V. \tag{1.17}$$

The infimum of $\mu > 0$ for which there exists a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that (1.17) *holds is called the Lipschitz modulus of F around $(\bar{x}, \bar{y})$ and is denoted by* $\operatorname{lip} F(\bar{x}, \bar{y})$.

The mapping F has Aubin property around $(\bar{x}, \bar{y})$ if and only if $\operatorname{lip} F(\bar{x}, \bar{y}) < \infty$. If, in addition, the space Y is a vector space, then for any $\mu > \operatorname{lip}\, F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that

$$F(x) \cap V \subset F(x') + \mu\, d(x, x')\, \mathbb{B}_Y \quad \text{for each} \quad x, x' \in U.$$

As in the case of metric regularity and openness with a linear rate around the point, letting $x' := \bar{x}$ in (1.17), we conclude that F is calm at $(\bar{x}, \bar{y})$ and, letting $y := \bar{y}$ and $x := \bar{x}$, we conclude that F recedes from $\bar{y}$ at $(\bar{x}, \bar{y})$ with a linear rate.

Example 1.2.4 *Consider a single-valued mapping $f : X \to Y$ which has Aubin property around $\bar{x}$. Then for any $\mu >$ lip $f(\bar{x})$ there is a neighborhood U of $\bar{x}$ in X such that*

$$\rho\big(f(x), f(x')\big) \leq \mu\, d(x, x') \quad \textit{for each} \quad x, x' \in U.$$

The last inequality is the definition of Lipschitz continuity of f on U and therefore Aubin property of f around $\bar{x}$ means local Lipschitz continuity of f around $\bar{x}$.

The following theorem guarantees the equivalence of metric regularity, openness with a linear rate around the reference point, and Aubin property of the inverse, and gives us relations among the corresponding moduli. The equivalence of openness with a linear rate and metric regularity was mentioned, probably for the first time, by Dmitruk, Milyutin, and Osmolowski [16] in 1980. In late 80s, Borwein-Zhuang [8] and Penot [33] proved (along with the equivalence with Aubin property) the full statement.

Theorem 1.2.3 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in$ Graph F. The following assertions are equivalent:*

(i) *F is metrically regular around $(\bar{x}, \bar{y})$;*

(ii) *F is open with a linear rate around $(\bar{x}, \bar{y})$;*

(iii) *F^{-1} has Aubin property around $(\bar{y}, \bar{x})$.*

In addition, it holds

$$\operatorname{sur} F(\bar{x}, \bar{y}) \cdot \operatorname{reg} F(\bar{x}, \bar{y}) = 1 \quad \textit{and} \quad \operatorname{reg} F(\bar{x}, \bar{y}) = \operatorname{lip} F^{-1}(\bar{y}, \bar{x}). \tag{1.18}$$

The above statement justifies the following definition.

Definition 1.2.12 *Consider a set-valued mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ρ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* regular *around $(\bar{x}, \bar{y})$ if and only if* reg $F(\bar{x}, \bar{y}) < \infty$ *if and only if* sur $F(\bar{x}, \bar{y}) > 0$ *if and only if* lip $F^{-1}(\bar{y}, \bar{x}) < \infty$.

We close this section by the group of stronger versions of the previous properties. For this purpose we need the notion of a *localization* of a set-valued mapping $F : X \rightrightarrows Y$ around the reference point $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$, which is any mapping $\widetilde{F} : X \rightrightarrows Y$ such that $\operatorname{Graph} \widetilde{F} = \operatorname{Graph} F \cap (U \times V)$ for some neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$, see Figure 1.2.

We start with strong semiregularty, for example see [2].

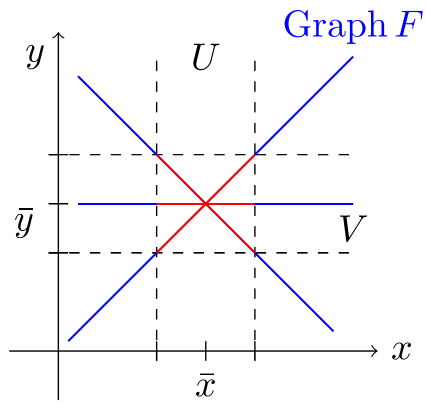

Figure 1.2: A localization (in red) of the set-valued mapping F (in blue).

Definition 1.2.13 *Consider a mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ϱ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* strongly semiregular at $(\bar{x}, \bar{y})$ *when F is metrically semiregular at $(\bar{x}, \bar{y})$ and F^{-1} has a localization around $(\bar{y}, \bar{x})$ which is nowhere multivalued.*

Let $F : X \rightrightarrows Y$ be strongly semiregular at $(\bar{x}, \bar{y})$ Then for any $\ell >$ semireg $F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that the mapping $V \ni y \longmapsto F^{-1}(y) \cap U$ is single-valued on V and calm at $\bar{y}$ with the constant ℓ.

Strong subregularity is entrenched in the literature [12].

Definition 1.2.14 *Consider a mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ϱ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* strongly subregular at $(\bar{x}, \bar{y})$ *when F is subregular at $(\bar{x}, \bar{y})$ and F^{-1} has no localization around $(\bar{y}, \bar{x})$ that is multivalued at $\bar{y}$.*

Let $F : X \rightrightarrows Y$ be strongly subregular at $(\bar{x}, \bar{y})$. Then for any $\ell >$ subreg $F(\bar{x}, \bar{y})$ there is a neighborhood U of $\bar{x}$ such that

$$d\big(x, \bar{x}\big) \leq \ell \operatorname{dist}(\bar{y}, F(x)) \quad \text{whenever} \quad x \in U,$$

that is, F^{-1} has *isolated calmness property at* $(\bar{y}, \bar{x})$, see [18].

Strong regularity was introduced by S.M. Robinson in [38] for generalized equations. This property is related to the (local) inverse function theorem and the implicit function theorem.

Definition 1.2.15 *Consider a mapping $F : X \rightrightarrows Y$ between metric spaces (X, d) and (Y, ϱ) and a point $(\bar{x}, \bar{y}) \in X \times Y$. The mapping F is said to be* strongly regular around $(\bar{x}, \bar{y})$ *when F is regular around $(\bar{x}, \bar{y})$ and F^{-1} has a localization around $(\bar{y}, \bar{x})$ which is nowhere multivalued.*

Let $F : X \rightrightarrows Y$ be strongly regular around $(\bar{x}, \bar{y})$. Then for any $\ell >$ reg $F(\bar{x}, \bar{y})$ there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that the mapping $V \ni y \longmapsto F^{-1}(y) \cap U$ is single-valued on V and Lipschitz continuous on V with the constant ℓ.

The section closes with several examples.

Example 1.2.5 1) *Let $f_1 : \mathbb{R} \to \mathbb{R}$ be defined by $f_1(x) := |x|$, $x \in \mathbb{R}$. Obviously for each $y < 0$ there is no $x \in \mathbb{R}$ such $f_1(x) = y$, hence f_1 is not semiregular at* 0*. On other hand, for each $x \in \mathbb{R}$ it holds*

$$\operatorname{dist}\big(x, f_1^{-1}(0)\big) = |x| = \operatorname{dist}(0, f_1(x)).$$

Therefore f_1 is subregular at 0 *with the constant* 1*. The graph of f_1 is in Figure 1.3a;*

2) *Let* $f_2 : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_2(x) := \begin{cases} x^2 \sin(1/x), & \text{for} \quad x \neq 0, \\ 0, & \text{for} \quad x = 0. \end{cases}$$

Then f_2 *is subregular and open at* 0 *but it is not semiregular at* 0*. The graph of* f_2 *is in Figure 1.3b;*

3) *Let* $f_3 : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_3(x) := \begin{cases} x + x|x \sin(1/x)|, & \text{for} \quad x \neq 0, \\ 0, & \text{for} \quad x = 0. \end{cases}$$

Then f_3 *is semiregular (not strongly) at* 0 *and strongly subregular at* 0*. This example is from [14] and for the graph of* f_3 *see Figure 1.3c;*

4) *Let* $f_4 : \mathbb{R} \to \mathbb{R}$ *be defined by* $f_4(x) := \sqrt[3]{x}$*,* $x \in \mathbb{R}$*. Then* f_4 *is strongly regular at any* $x \in \mathbb{R}$*. Moreover, the inverse is* $f_4^{-1}(x) = x^3$*, for* $x \in \mathbb{R}$*. The graph of* f_4 *is in Figure 1.3d;*

5) *Let* $f_5 : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_5(x) := \begin{cases} x, & \text{for} \quad x \in \mathbb{Q}, \\ -x, & \text{for} \quad x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then f_5 *is strongly semiregular at* 0 *and strongly subregular at* 0*, but it is not regular around* 0*.*

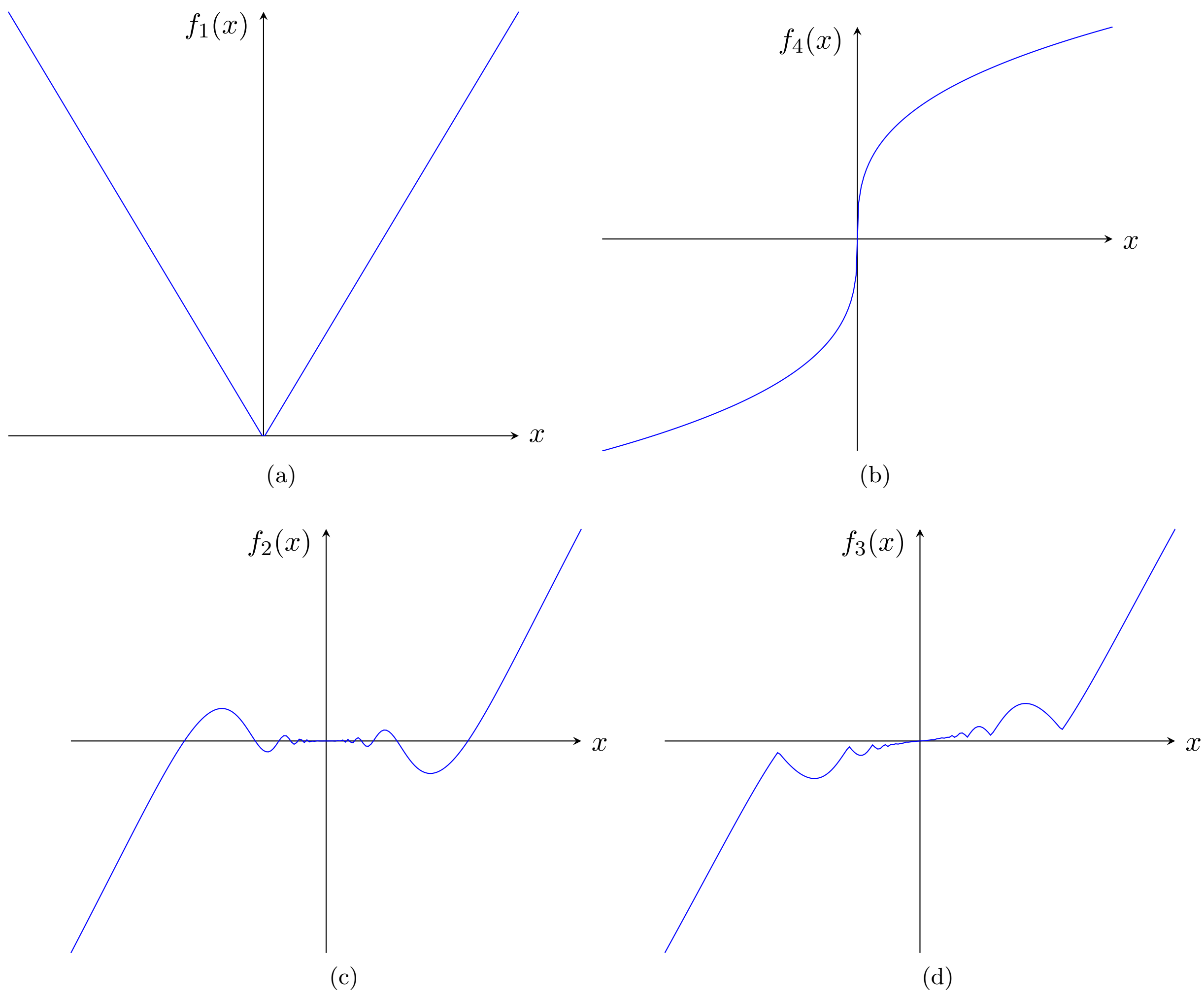

Figure 1.3: Graphs of functions from Example 1.2.5.

Chapter 2

Regularity criteria

In this chapter we present some criteria which guarantee regularity, subregularity and semiregularity and their stronger version.

2.1 Historical background

We begin with Banach open mapping theorem, which is also known as Banach–Schauder theorem and guarantees regularity of a linear continuous mapping between Banach spaces.

Theorem 2.1.1 (Banach open mapping theorem) *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $A \in \mathcal{L}(X, Y)$. Then the following assertions are equivalent:*

(i) $\operatorname{sur} A > 0$;

(ii) $A(X) = Y$;

(iii) $0 \in \operatorname{int} A(\mathbb{B}_X)$;

(iv) *A is open at 0;*

(v) *the adjoint (dual) operator $A^* : Y^* \to X^*$ is injective.*

Moreover,

$$\operatorname{sur} A = \operatorname{lopen} A = \sup\{c > 0 : A(\mathbb{B}_X) \supset c\,\mathbb{B}_Y\} = \inf\{\|A^* y^*\|_{X^*} : y^* \in \mathbb{S}_{Y^*}\}.$$

Example 2.1.1 *Consider a matrix $A \in \mathbb{R}^{n\times m}$ with $n \le m$. Then the mapping $\mathbb{R}^m \ni x \longmapsto Ax$ is regular if and only if the rows of A are linearly independent. Moreover, $\operatorname{sur} A$ equals to the least singular value of A.*

In 1950 L.M. Graves [21] published a sufficient condition for semiregularity of a nonlinear mapping at the reference point, which generalizes Banach open mapping theorem to nonlinear mappings.

Theorem 2.1.2 (Graves theorem) *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $\bar{x} \in X$ be given. Consider a mapping $f : X \to Y$ such that there is $A \in \mathcal{L}(X, Y)$ with $\operatorname{sur} A > \operatorname{lip}(f - A)(\bar{x})$. Then f is semiregular at $\bar{x}$ and $\operatorname{lopen} f(\bar{x}) \ge \operatorname{sur} A - \operatorname{lip}(f - A)(\bar{x})$.*

Another generalization of Banach open mapping theorem was proved by S.M. Robinson [37] and independently by C. Ursescu [41] for set-valued mappings with a closed convex graph. This statement follows, for example, from a constrained version of Banach open mapping theorem applied to the restriction of the canonical projection from $X \times Y$ onto Y to the graph of the mapping under consideration, that is, the assignment $\operatorname{Graph} F \ni (x, y) \longmapsto y \in Y$.

Theorem 2.1.3 (Robinson–Ursescu theorem) *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $\bar{y} \in Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ having a closed convex graph. Then the following assertions are equivalent:*

(i) $\bar{y} \in \operatorname{int} \operatorname{rge} F$;

(ii) *for each $\bar{x} \in F^{-1}(\bar{y})$, the mapping F is open at $(\bar{x}, \bar{y})$;*

(iii) *for each $\bar{x} \in F^{-1}(\bar{y})$, we have* $\operatorname{sur} F(\bar{x}, \bar{y}) > 0$.

We say that a mapping $f : X \to Y$ between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is *Fréchet differentiable* at a point $\bar{x} \in X$ if there is $A \in \mathcal{L}(X, Y)$ such that $\operatorname{calm}(f - A)(\bar{x}) = 0$, that is, for each $\mu > 0$ there is $\delta > 0$ such that

$$\|f(x) - f(\bar{x}) - A(x - \bar{x})\|_Y \leq \mu \|x - \bar{x}\|_X \quad \text{for each} \quad x \in \mathbb{B}(\bar{x}, \delta).$$

Such a mapping A is called the *Fréchet derivative* of f at $\bar{x}$ and denoted by $f'(\bar{x})$. The mapping f is said to be *continuously (Fréchet) differentiable* at $\bar{x}$ if f is Fréchet differentiable on a neighborhood U of $\bar{x}$ in X and the mapping $U \ni x \longmapsto f'(x) \in \mathcal{L}(X, Y)$ is continuous at $\bar{x}$.

In 1970 S.M. Robinson [38] studied the solution stability of the so-called *generalized equation*, which is the problem to find $x \in X$ such that

$$f(x) + F(x) \ni 0, \tag{2.1}$$

with given mappings $f : X \to Y$ and $F : X \rightrightarrows Y$. He proved a sufficient condition for strong regularity in case that f is continuously Fréchet differentiable and F is a *normal cone mapping* N_K associated with a closed convex subset K of X, which is the mapping

$$X \ni x \longmapsto N_K(x) := \{x^* \in X^* : \langle x^*, x' - x \rangle \leq 0 \text{ for each } x' \in K\}.$$

More precisely, Robinson proved the implicit function theorem for generalized equations, where $f : P \times X \to X^*$ with a parameter space P.

Theorem 2.1.4 (Robinson theorem) *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ and a single-valued mapping $f : X \to Y$ which is continuously Fréchet differentiable at $\bar{x}$ and $\bar{y} \in f(\bar{x}) + F(\bar{x})$. If the mapping $f(\bar{x}) + f'(\bar{x})(\cdot - \bar{x}) + F$ is strongly regular around $(\bar{x}, \bar{y})$, then $f + F$ is strongly regular around $(\bar{x}, \bar{y})$.*

In 1996 A.L. Dontchev [17] proved a generalization of Theorem 2.1.2. We need one more definition, we say that a set-valued mapping $F : X \rightrightarrows Y$ has a *locally closed graph* around $(\bar{x}, \bar{y}) \in \operatorname{Graph} F$ if there is a neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ in $X \times Y$ such that the set $\operatorname{Graph} F \cap \big(U \times V\big)$ is closed.

Theorem 2.1.5 *Let (X, d) be a complete metric space, (Y, ρ) be a linear metric space with a shift-invariant metric, and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ with a locally closed graph around $(\bar{x}, \bar{y})$ and a single-valued mapping $f : X \to Y$ such that* $\operatorname{lip} f(\bar{x}) = 0$, *that is, for each $\mu > 0$ there is $\delta > 0$ such that*

$$\rho(f(x), f(x')) \leq \mu \, d(x, x') \quad \textit{for each} \quad x, x' \in \mathbb{B}(\bar{x}, \delta). \tag{2.2}$$

Then $\operatorname{sur} F(\bar{x}, \bar{y}) = \operatorname{sur}(f + F)(\bar{x}, f(\bar{x}) + \bar{y})$.

We say that a mapping $f : X \to Y$ between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is *strictly differentiable* at a point $\bar{x} \in X$ if there is $A \in \mathcal{L}(X, Y)$ such that $\operatorname{lip}(f - A)(\bar{x}) = 0$, that is, for each $\mu > 0$ there is $\delta > 0$ such that

$$\|f(x) - f(x') - A(x - x')\|_Y \le \mu \|x - x'\|_X \quad \text{for each} \quad x, x' \in \mathbb{B}(\bar{x}, \delta).$$

Such a mapping A is called the *strict derivative* of f at $\bar{x}$. Note that the existence of the strict derivative of f at $\bar{x}$ implies that f is Fréchet differentiable at $\bar{x}$, continuous on a neighborhood of $\bar{x}$, and locally Lipschitz continuous around $\bar{x}$. Clearly, (2.2) means that f is strictly differentiable at $\bar{x}$ and the strict derivative is zero. The following example shows that a strictly differentiable mapping is regular around the reference point if and only if its strict derivative at this point is surjective.

Example 2.1.2 *Let $g : X \to Y$ be a single-valued mapping between Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$. Suppose that g is strictly differentiable at $\bar{x} \in X$, then Theorem 2.1.5, with $F := g$ and $f := g(\bar{x}) - g + g'(\bar{x})(\cdot - \bar{x})$, implies that* $\operatorname{sur} g(\bar{x}) = \operatorname{sur}\big(g'(\bar{x})\big)$.

2.2 Regularity

In this section we present some statements which guarantee (strong) regularity of mappings. For more criteria of regularity see [25] and [13].

In 1987 M. Fabian and D. Preiss [20, Corollary 1] proved a sufficient condition for semiregularity of both single-valued and set-valued mappings at the reference point via Caristi's principle. Thirteen years later, A.D. Ioffe [24, Theorem 1b] proved independently a necessary and sufficient condition for regularity of a set-valued mapping around the reference point via Ekeland's variational principle. This statement will be called *Ioffe's regularity criterion.*

Theorem 2.2.1 *Let (X, d) be a complete metric space, (Y, ρ) be a metric space, and $\bar{x} \in X$ be given. Consider a continuous single-valued mapping $f : X \to Y$ whose domain is all of X. Then* $\operatorname{sur} f(\bar{x})$ *equals to the supremum of all $c > 0$ for which there is $r > 0$ such that for all $(x, y) \in \mathbb{B}[\bar{x}, r] \times \big(\mathbb{B}[f(\bar{x}), r] \setminus \{f(x)\}\big)$ there is $x' \in X$ satisfying*

$$c\, d(x', x) < \rho(f(x), y) - \rho(f(x'), y). \tag{2.3}$$

It is a well-known fact that a study of regularity properties for a set-valued mapping $F : X \rightrightarrows Y$ can be reduced to the study of the corresponding property for the restriction of the canonical projection from $X \times Y$ onto Y, which is the mapping $\operatorname{Graph} F \ni (x, y) \longmapsto y \in Y$, for the proof and more details see [24, Proposition 3].

Theorem 2.2.2 *Let (X, d), (Y, ρ) be metric spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ whose graph is complete in a vicinity of $(\bar{x}, \bar{y})$. Then* $\operatorname{sur} F(\bar{x}, \bar{y})$ *equals to the supremum of all $c > 0$ for which there are $r > 0$ and $\alpha \in (0, 1/c)$ such that for any $(x, v) \in \operatorname{Graph} F \cap \big(\mathbb{B}[\bar{x}, r] \times \mathbb{B}[\bar{y}, r]\big)$ and any $y \in \mathbb{B}[\bar{y}, r] \setminus \{v\}$ there is a pair $(x', v') \in \operatorname{Graph} F$ such that*

$$c \max\{d(x, x'), \alpha \rho(v, v')\} < \rho(v, y) - \rho(v', y). \tag{2.4}$$

We apply Theorem 2.2.2 to show that (strong) regularity is stable with respect to Lipschitz single-valued perturbations.

Theorem 2.2.3 *Let (X, d) be a complete metric space, (Y, ρ) be a linear metric space with a shift-invariant metric, and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a single-valued mapping $f : X \to Y$ which is* *both defined and continuous around $\bar{x}$ and a set-valued mapping $F : X \rightrightarrows Y$ which has a locally closed graph around $(\bar{x}, \bar{y})$. Then*

$$\operatorname{sur}(F + f)(\bar{x}, \bar{y} + f(\bar{x})) \geq \operatorname{sur} F(\bar{x}, \bar{y}) - \operatorname{lip} f(\bar{x}). \tag{2.5}$$

In particular, if F is (strongly) regular around $(\bar{x}, \bar{y})$ and $\operatorname{sur} F(\bar{x}, \bar{y}) > \operatorname{lip} f(\bar{x})$, then $F+f$ is (strongly) regular around $(\bar{x}, \bar{y} + f(\bar{x}))$ and (2.5) holds.

Proof. If $\operatorname{lip} f(\bar{x}) \geq \operatorname{sur} F(\bar{x}, \bar{y})$ then (2.5) holds trivially. Suppose that this is not the case and fix constants c, c', and ℓ such that

$$\operatorname{lip} f(\bar{x}) < \ell < c < c' < \operatorname{sur} F(\bar{x}, \bar{y}).$$

Find $\varepsilon > 0$ such that for each $t \in (0, 2\varepsilon)$ and each $(x, w) \in \big(\mathbb{B}[\bar{x}, 2\varepsilon] \times \mathbb{B}[\bar{y}, 2\varepsilon]\big) \cap \operatorname{Graph} F$ we have

$$\mathbb{B}[w, c't] \subset F(\mathbb{B}[x, t]). \tag{2.6}$$

By assumptions on f there is $r \in (0, \min\{\varepsilon, c'\varepsilon\})$ such that

$$f(x) \in \mathbb{B}[f(\bar{x}), \varepsilon] \quad \text{and} \quad \rho(f(x), f(x')) \leq \ell\, d(x, x') \quad \text{for each} \quad x, x' \in \mathbb{B}[\bar{x}, r(1 + 2/c')].$$

Let $(x, v) \in \big(\mathbb{B}[\bar{x}, r] \times \mathbb{B}[\bar{y} + f(\bar{x}), r]\big) \cap \operatorname{Graph}(F + f)$ and $y \in \mathbb{B}[\bar{y} + f(\bar{x}), r]$ with $y \neq v$ be arbitrary. Let

$$t := \rho(v, y)/c' \quad \text{and} \quad w := v - f(x).$$

Then $0 < t \leq 2r/c' < 2\varepsilon$. Clearly, $(x, w) \in \operatorname{Graph} F$ and $x \in \mathbb{B}[\bar{x}, 2\varepsilon]$. As $f(x) \in \mathbb{B}[f(\bar{x}), \varepsilon]$, we get

$$\begin{aligned} \rho(w, \bar{y}) &= \rho(v - f(x) + f(\bar{x}), \bar{y} + f(\bar{x})) \leq \rho(v - f(x) + f(\bar{x}), v) + \rho(v, \bar{y} + f(\bar{x})) \\ &= \rho(f(\bar{x}), f(x)) + \rho(v, \bar{y} + f(\bar{x})) \leq \varepsilon + r < 2\varepsilon. \end{aligned}$$

Therefore $w \in \mathbb{B}[\bar{y}, 2\varepsilon]$. Also $y - f(x) \in \mathbb{B}[w, c't]$ because

$$\rho(y - f(x), w) = \rho(y - f(x), v - f(x)) = \rho(y, v) = c't.$$

By (2.6), there is $x' \in \mathbb{B}[x, t]$ such that $y - f(x) \in F(x')$. Then

$$d(x', \bar{x}) \leq d(x', x) + d(x, \bar{x}) \leq t + r \leq 2r/c' + r = r(1 + 2/c'),$$

and consequently

$$\rho(f(x), f(x')) \leq \ell\, d(x, x') \leq \ell t. \tag{2.7}$$

Let $v' := y - f(x) + f(x')$. Then $(x', v') \in \operatorname{Graph}(F + f)$. Using (2.7), we get

$$\begin{aligned} \rho(v', y) &= \rho(y - f(x) + f(x'), y) = \rho(f(x), f(x')) \leq \ell t = c't - (c' - \ell)t \\ &= \rho(v, y) - (c' - \ell)t < \rho(v, y) - (c - \ell)t. \end{aligned}$$

Noting that $d(x, x') \leq t$ and

$$\begin{aligned} \rho(v, v') &= \rho(v, y - f(x) + f(x')) \leq \rho(v, y) + \rho(y, y - f(x) + f(x')) \\ &= \rho(v, y) + \rho(f(x), f(x')) \leq c't + \ell\, d(x, x') \leq (c' + \ell)t, \end{aligned}$$

we conclude that

$$\rho(v', y) < \rho(v, y) - (c - \ell) \max\{d(x, x'), \rho(v, v')/(c' + \ell)\}.$$

Theorem 2.2.2 with $\alpha := 1/(c'+\ell)$ implies that $\operatorname{sur}(F+f)(\bar{x},\bar{y}+f(\bar{x})) \geq c-\ell$. Letting $c \uparrow \operatorname{sur} F(\bar{x},\bar{y})$ and $\ell \downarrow \operatorname{lip} f(\bar{x})$ we finish the proof of (2.5).

If F is regular around $(\bar{x},\bar{y})$ and $\operatorname{sur} F(\bar{x},\bar{y}) > \operatorname{lip} f(\bar{x})$, then (2.5) implies that $\operatorname{sur}(F+f)(\bar{x},\bar{y}+f(\bar{x})) > 0$, that is, the mapping $F+f$ is regular around $(\bar{x},\bar{y}+f(\bar{x}))$.

Assume that F is strongly regular around $(\bar{x},\bar{y})$ and $\operatorname{sur} F(\bar{x},\bar{y}) > \operatorname{lip} f(\bar{x})$. Then $F+f$ is regular around $(\bar{x},\bar{y}+f(\bar{x}))$. It suffices to show that $(F+f)^{-1}$ has a localization around $(\bar{x},\bar{y}+f(\bar{x}))$ which is single-valued. Find c and ℓ such that $\operatorname{sur} F(\bar{x},\bar{y}) > c > \ell > \operatorname{lip} f(\bar{x})$. Then there is a neighborhood $U \times V$ of $(\bar{x},\bar{y})$ in $X \times Y$ such that the mapping $\sigma : V \ni y \longmapsto \sigma(y) := F^{-1}(y) \cap U$ is single-valued and Lipschitz continuous on V with the constant $1/c$, and also that f is Lipschitz continuous on U with the constant ℓ. Find a neighborhood $U' \times V'$ of $(\bar{x},\bar{y}+f(\bar{x}))$ in $X \times Y$ such that $y - f(x) \in V$ whenever $(x,y) \in U' \times V'$. Fix any $y \in V'$. Suppose on the contrary that there are two distinct $x, x' \in (F+f)^{-1}(y)$. Then we have $x = \sigma(y - f(x))$ and $x' = \sigma(y - f(x'))$. Hence

$$\begin{aligned} 0 < d(x,x') &= d(\sigma(y-f(x)),\sigma(y-f(x'))) \leq c^{-1}\rho(y-f(x),y-f(x')) \\ &= c^{-1}\rho(f(x),f(x')) \leq (\ell/c)d(x,x') < d(x,x'), \end{aligned}$$

a contradiction. ∎

The above statement fails if we replace a single-valued perturbation by a set-valued one as the following example from [18, Example 5I.1] shows.

Example 2.2.1 *Consider set-valued mappings F, $G : \mathbb{R} \rightrightarrows \mathbb{R}$ defined for each $x \in \mathbb{R}$ by*

$$F(x) = \{-2x, 1\} \quad \textit{and} \quad G(x) = \{x^2, -1\}.$$

Then $\operatorname{sur} F(0,0) = 2$ *and* $\operatorname{lip} G(0,0) = 0$. *But the mapping*

$$\mathbb{R} \ni x \longmapsto (F+G)(x) = \{x^2 - 2x, x^2+1, -2x-1, 0\}$$

is not regular around $(0,0)$.

Despite the above example, H.V. Ngai, N.H. Tron, and M. Théra proved regularity of the sum of two set-valued mappings [32] under the local sum-stability assumption around the reference point, the property introduced by M. Durea and R. Strugariu [19].

Definition 2.2.1 *Let (X,d), (Y,ρ) be metric spaces and $(\bar{x},\bar{y},\bar{z}) \in X \times Y \times Y$ be given. Consider set-valued mappings F, $G : X \rightrightarrows Y$ such that $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$. We say that the pair (F,G)* is sum-stable around *$(\bar{x},\bar{y},\bar{z})$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $x \in \mathbb{B}[\bar{x},\delta]$ and every $v \in (F+G)(x) \cap \mathbb{B}[\bar{y}+\bar{z},\delta]$, there exist $y \in F(x) \cap \mathbb{B}[\bar{y},\varepsilon]$ and $z \in G(x) \cap \mathbb{B}[\bar{z},\varepsilon]$ such that $v = y + z$.*

If the perturbing mapping is single-valued and continuous at the reference point then the local sum-stability holds (cf. [19] where the perturbation is assumed to be calm at the reference point).

Example 2.2.2 *Let (X,d) be a metric space, (Y,ρ) be a linear metric space with a shift-invariant metric, and $\bar{x} \in X$ be given. Consider a single-valued mapping $f : X \to Y$ and a set-valued mapping $F : X \rightrightarrows Y$ such that $0 \in f(\bar{x}) + F(\bar{x})$. If f is continuous at $\bar{x}$, then (F,f) is locally sum-stable around $(\bar{x}, -f(\bar{x}), f(\bar{x}))$. Indeed, fix any $\varepsilon > 0$ and find $\delta \in (0, \varepsilon/2)$ such that*

$$\rho(f(x), f(\bar{x})) < \varepsilon/2 \quad \textit{for each} \quad x \in \mathbb{B}[\bar{x},\delta].$$

Pick any $x \in \mathbb{B}[\bar{x},\delta]$ and any $v \in (F+f)(x) \cap \mathbb{B}[0,\delta]$, then $v - f(x) \in F(x)$. Further, $\rho(f(x), f(\bar{x})) \leq \varepsilon/2 < \varepsilon$, so $f(x) \in \mathbb{B}[f(\bar{x}),\varepsilon]$. Also, $\rho(v - f(x), -f(\bar{x})) = \rho(v, f(x) - f(\bar{x})) \leq \rho(v,0) + \rho(f(x), f(\bar{x})) < \delta + \varepsilon/2 < \varepsilon$.

As observed in [32], even without local sum-stability, we always have regularity of the so-called epigraphical multifunction introduced in [19].

Theorem 2.2.4 *Let (X,d) be a complete metric space (Y,ρ) be a linear metric space with a shift-invariant metric, and $(\bar{x},\bar{y},\bar{z}) \in X \times Y \times Y$ be given. Consider set-valued mappings $F,\ G : X \rightrightarrows Y$ such that F has a locally closed graph around $(\bar{x},\bar{y})$ and G has a locally closed graph around $(\bar{x},\bar{z})$. Define the mapping $\mathcal{E}_{F,G} : X \times Y \rightrightarrows Y$ by*

$$\mathcal{E}_{F,G}(x,z) := F(x) + z \quad \text{if} \quad z \in G(x) \quad \text{and} \quad \mathcal{E}_{F,G}(x,z) = \emptyset \quad \text{otherwise.}$$

Then

$$\operatorname{sur} \mathcal{E}_{F,G}(\bar{x},\bar{z},\bar{y}+\bar{z}) \geq \operatorname{sur} F(\bar{x},\bar{y}) - \operatorname{lip} G(\bar{x},\bar{z}). \tag{2.8}$$

If, in addition, the pair (F,G) is sum-stable around $(\bar{x},\bar{y},\bar{z})$, then

$$\operatorname{sur}(F+G)(\bar{x},\bar{y}+\bar{z}) \geq \operatorname{sur} F(\bar{x},\bar{y}) - \operatorname{lip} G(\bar{x},\bar{z}). \tag{2.9}$$

Proof. If $\operatorname{lip} G(\bar{x},\bar{z}) \geq \operatorname{sur} F(\bar{x},\bar{y})$ then we are done. If this is not the case then fix constants c, c', and ℓ such that

$$\operatorname{lip} G(\bar{x},\bar{z}) < \ell < c < c' < \operatorname{sur} F(\bar{x},\bar{y}).$$

Define the (equivalent) metric on $X \times Y$ by

$$\widetilde{d}((x,z),(x',z')) := \max\{d(x,x'),\rho(z,z')/\ell\}, \quad (x,z),(x',z') \in X \times Y.$$

As $\operatorname{Graph} F$ and $\operatorname{Graph} G$ are closed around $(\bar{x},\bar{y})$ and $(\bar{x},\bar{z})$, respectively, so is $\operatorname{Graph} \mathcal{E}_{F,G}$ around $(\bar{x},\bar{z},\bar{y}+\bar{z})$. Find $\varepsilon > 0$ such that for each $t \in (0,2\varepsilon)$ and each $(x,w) \in \big(\mathbb{B}[\bar{x},2\varepsilon] \times \mathbb{B}[\bar{y},2\varepsilon]\big) \cap \operatorname{Graph} F$ we have

$$\mathbb{B}[w,c't] \subset F(\mathbb{B}[x,t]). \tag{2.10}$$

Since $\operatorname{lip} G(\bar{x},\bar{z}) < \ell$, there is $r \in (0,\min\{\varepsilon,c'\varepsilon\})$ such that

$$G(x) \cap \mathbb{B}[\bar{z},r] \subset G(x') + \ell\, d(x,x')\, \mathbb{B}_Y \quad \text{for each} \quad x,x' \in \mathbb{B}[\bar{x},r(1+2/c')]. \tag{2.11}$$

Let $(x,z,v) \in \big(\mathbb{B}[\bar{x},r] \times \mathbb{B}[\bar{z},r] \times \mathbb{B}[\bar{y}+\bar{z},r]\big) \cap \operatorname{Graph} \mathcal{E}_{F,G}$ and $y \in \mathbb{B}[\bar{y}+\bar{z},r]$ with $v \neq y$ be arbitrary. Then there is $w \in F(x)$ such that $v = w + z$. Let

$$t := \rho(v,y)/c'.$$

Then $0 < t \leq 2r/c' < 2\varepsilon$. Also $(x,w) \in \big(\mathbb{B}[\bar{x},2\varepsilon] \times \mathbb{B}[\bar{y},2\varepsilon]\big) \cap \operatorname{Graph} F$, because

$$\rho(w,\bar{y}) = \rho(v-z,\bar{y}) \leq \rho(v-z,v-\bar{z}) + \rho(v-\bar{z},\bar{y}) = \rho(\bar{z},z) + \rho(v,\bar{y}+\bar{z}) \leq r + r < 2\varepsilon.$$

Moreover, $y - z \in \mathbb{B}[w,c't]$ since

$$\rho(y-z,w) = \rho(y-z,v-z) = \rho(y,v) = c't.$$

By (2.10), there is $x' \in \mathbb{B}[x,t]$ such that $y - z \in F(x')$. Then

$$d(x',\bar{x}) \leq d(x',x) + d(x,\bar{x}) \leq t + r \leq 2r/c' + r = r(1+2/c').$$

Since $z \in G(x) \cap \mathbb{B}[\bar{z},r]$, using (2.11) we find $z' \in G(x')$ such that

$$\rho(z,z') \leq \ell\, d(x,x') \leq \ell t.$$

Then $v' := y - z + z' \in \mathcal{E}_{F,G}(x', z')$ and we may estimate

$$\begin{array}{rcl}\rho(v', y) & = & \rho(y - z + z', y) = \rho(z, z') \le \ell t = c't - (c' - \ell)t = \rho(v, y) - (c' - \ell)t \\ & < & \rho(v, y) - (c - \ell)t.\end{array}$$

Remembering that $d(x', x) \le t$ and $\rho(z, z') \le \ell t$, as well as that

$$\rho(v, v') \;=\; \rho(v, y - z + z') \le \rho(v, y) + \rho(y, y - z + z') = \rho(v, y) + \rho(z, z') \le c't + \ell t = (c' + \ell)t,$$

we conclude that

$$\rho(v', y) < \rho(v, y) - (c - \ell) \max\{\widetilde{d}\,((x, z), (x', z')), \rho(v, v')/(c' + \ell)\}.$$

Theorem 2.2.2 with $\alpha := 1/(c' + \ell)$ and $(X, d) := \big(X \times Y, \widetilde{d}\,\big)$ implies that

$$b := \operatorname{sur} \mathcal{E}_{F,G}(\bar{x}, \bar{z}, \bar{y} + \bar{z}) \ge c - \ell.$$

Letting $c \uparrow \operatorname{sur} F(\bar{x}, \bar{y})$ and $\ell \downarrow \operatorname{lip} G(\bar{x}, \bar{z})$ we get (2.8).

Let $\lambda \in (0, b)$ be arbitrary. Then there exists $\varepsilon > 0$ such that for any $t \in (0, \varepsilon)$ and any $(x, z, v) \in \operatorname{Graph} \mathcal{E}_{F,G} \cap (\mathbb{B}[\bar{x}, \varepsilon] \times \mathbb{B}[\bar{z}, \varepsilon] \times \mathbb{B}[\bar{y} + \bar{z}, \varepsilon])$, that is, $x \in \mathbb{B}[\bar{x}, \varepsilon]$, $z \in G(x) \cap \mathbb{B}[\bar{z}, \varepsilon]$ and $v \in (F(x) + z) \cap \mathbb{B}[\bar{y} + \bar{z}, \varepsilon]$, we have

$$\mathbb{B}[v, \lambda t] \subset \mathcal{E}_{F,G}(\mathbb{B}[x, t] \times \mathbb{B}[z, t]).$$

Suppose that the pair (F, G) is sum-stable around $(\bar{x}, \bar{y}, \bar{z})$. Then there is $\delta \in (0, \varepsilon)$ such that for any $x \in \mathbb{B}[\bar{x}, \delta]$ and any $v \in (F + G)(x) \cap \mathbb{B}[\bar{y}+\bar{z}, \delta]$, there are $y \in F(x) \cap \mathbb{B}[\bar{y},\varepsilon]$ and $z \in G(x) \cap \mathbb{B}[\bar{z}, \varepsilon]$ such that $v = y + z$. Fix any $t \in (0, \delta)$ and $(x, v) \in (\mathbb{B}[\bar{x}, \delta] \times \mathbb{B}[\bar{y} + \bar{z}, \delta]) \cap \operatorname{Graph}(F + G)$. Then $v = y + z$ for some $y \in F(x) \cap \mathbb{B}[\bar{y},\varepsilon]$ and $z \in G(x) \cap \mathbb{B}[\bar{z}, \varepsilon]$. Given $v' \in \mathbb{B}[v, \lambda t]$, we find $x' \in \mathbb{B}[x, t]$ and $z' \in \mathbb{B}[z, t]$ such that $v' \in \mathcal{E}_{F,G}(x', z')$, that is, $v' \in F(x') + z' \subset (F + G)(x')$. Consequently,

$$\mathbb{B}[v, \lambda t] \subset (F + G)(\mathbb{B}[x, t]).$$

Letting $\lambda \uparrow b$, we get $\operatorname{sur}(F + G)(\bar{x}, \bar{y} + \bar{z}) \ge b$, which in view of (2.8) implies (2.9). ■

Let us point out that a direct application of Theorem 2.2.2 gives us a short and easy to read proof of Theorem 2.2.4 which is similar to the one of Theorem 2.2.3. The proof in [32] which is based on error bounds and slopes is more involved and longer. Example 2.2.2 shows that Theorem 2.2.4 implies Theorem 2.2.3 .

The following proposition reveals that regularity for a mapping F is still guaranteed if we replace the image of a ball under F in (1.16) by its closure. We need a definition of a ball around a set. For a subset K of a metric spaces (X, d) and a constant $\delta > 0$, the *open ball around K with the radius δ* is the set

$$\widehat{\mathbb{B}}(K, \delta) := \{x \in X : \ \operatorname{dist}(x, K) < \delta\}.$$

Proposition 2.2.1 *Let (X, d), (Y, ρ) be complete metric spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ which has a locally closed graph around $(\bar{x}, \bar{y})$. Then $\operatorname{sur} F(\bar{x}, \bar{y})$ is equal to the supremum of $c > 0$ such that there are $r > 0$ and $\varepsilon > 0$ such that*

$$\mathbb{B}(y, ct) \subset \overline{F(\mathbb{B}(x, t))}$$

for all $(x, y) \in \operatorname{Graph} F \cap \big(\mathbb{B}(\bar{x}, r) \times \mathbb{B}(\bar{y}, r)\big)$ and all $t \in (0, \varepsilon)$.

Proof. Denote by s the supremum from the statement. Clearly $s \ge \operatorname{sur} F(\bar{x}, \bar{y})$. Assume that $s > \operatorname{sur} F(\bar{x}, \bar{y})$. Fix an arbitrary $c \in (0, s)$. Find $\varepsilon > 0$ such that for each $(x, v) \in \operatorname{Graph} F \cap \big(\mathbb{B}[\bar{x}, \varepsilon] \times \mathbb{B}[\bar{y}, \varepsilon]\big)$, each $t \in (0, \varepsilon)$, and each $\delta > 0$ it holds

$$\mathbb{B}[v, ct] \subset \mathbb{B}(v, (c + \varepsilon)t) \subset \overline{F(\mathbb{B}(x, t))} \subset \overline{F(\mathbb{B}[x, t])} \subset \widehat{\mathbb{B}}(F(\mathbb{B}[x, t]), \delta t). \tag{2.12}$$

Let $r \in (0, \min\{\varepsilon, \varepsilon c/2\})$. Fix any $(x, v) \in \operatorname{Graph} F \cap \big(\mathbb{B}[\bar{x}, r] \times \mathbb{B}[\bar{y}, r]\big)$, any $y \in \mathbb{B}[\bar{y}, r] \setminus \{v\}$, and any $\delta \in (0, c)$. Define $t := \rho(v, y)/c$. Then $0 < t \le 2r/c < \varepsilon$. By (2.12), there are $x' \in \mathbb{B}[x, t]$ and $v' \in F(x')$ such that $\rho(v', y) < \delta t$. Hence

$$\rho(v', y) \quad < \quad \delta t = ct - (c - \delta)t = \rho(v, y) - (c - \delta)t.$$

As $d(x, x') \le t$ and

$$\rho(v, v') \le \rho(v, y) + \rho(y, v') < (c + \delta)t,$$

we get

$$\rho(v', y) < \rho(v, y) - (c - \delta) \max\{d(x, x'), \rho(v, v')/(c + \delta)\}.$$

Theorem 2.2.2 with $\alpha := 1/(c + \delta)$ says that $\operatorname{sur} F(\bar{x}, \bar{y}) \ge c - \delta$. Letting $c \uparrow s$ and $\delta \downarrow 0$, we conclude that $\operatorname{sur} F(\bar{x}, \bar{y}) \ge s > \operatorname{sur} F(\bar{x}, \bar{y})$, a contradiction. ■

Further, let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. We study a global version of strong regularity of a set-valued mapping $F : H \rightrightarrows H$, which means, that F is strongly regular around any point in its graph and the corresponding neighborhood $U \times V$ in (1.14) equals $H \times H$. This means that F^{-1} is Lipschitz continuous on the whole of H. For this purpose we need two more definitions. The set-valued mapping $F : H \rightrightarrows H$ is said to be *monotone* if

$$\langle y - y', x - x'\rangle \ge 0 \quad \text{for each} \quad (x, y), (x', y') \in \operatorname{Graph} F;$$

and F is said to be *maximal monotone* if it is monotone and there is no other monotone mapping whose graph strictly contains the graph of F. Note that by [5, Theorem 3.5.9], if the mapping F is maximal monotone then for each $\mu > 0$ the mapping $(I + \mu F)^{-1}$ is single-valued and Lipschitz continuous on H with the constant 1, where I is the identity mapping on H. We will follow the proof from [40, Lemma 2.2].

Theorem 2.2.5 *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Consider a maximal monotone mapping $F : H \rightrightarrows H$ and $A \in \mathcal{L}(H, H)$ such that there is $c > 0$ such that*

$$\langle Ax, x\rangle \ge c\|x\|^2 \quad \textit{for each } x \in H. \tag{2.13}$$

Then the mapping $(A + F)^{-1}$ is single-valued and Lipschitz continuous on H with the constant $1/c$.

Proof. Let $\mu := c/\|A\|^2$. Since the mapping F is maximal monotone, the mapping $(I + \mu F)^{-1}$ is Lipschitz continuous on H with the constant 1. We show that $\operatorname{rge}(A + F) = H$, that is, for each $y \in H$ we find $x \in H$ with

$$y \in Ax + F(x). \tag{2.14}$$

Fix any $y \in H$. Consider the mapping $H \ni u \longmapsto h(u) := (I + \mu F)^{-1}(\mu y + u - \mu A u)$. We will find a fixed point $x \in H$ of h, which automatically satisfies (2.14). By (2.13), for arbitrary $x, x' \in H$, we have

$$\begin{aligned}
\|h(x) - h(x')\|^2 &\le \|x - x' - \mu A(x - x')\|^2 \\
&= \|x - x'\|^2 - 2\mu\langle A(x - x'), x - x'\rangle + \mu^2\|A(x - x')\|^2 \\
&\le \big(1 - 2c\mu + \mu^2\|A\|^2\big)\|x - x'\|^2 \\
&= \big(1 - c^2/\|A\|^2\big)\|x - x'\|^2.
\end{aligned}$$

By (2.13), we get $0 \le 1 - c^2/\|A\|^2 < 1$. Then by Banach contraction theorem, we conclude the first part of the proof.

Now we show that the mapping $(A+F)^{-1}$ is Lipschitz continuous. Fix any $(x,v),(x',v') \in \mathrm{Graph}(A+F)$. Then there are $y \in F(x)$ and $y' \in F(x')$ such that $v = Ax + y$ and $v' = Ax' + y'$. Using the monotonicity of F and (2.13), we have

$$\langle v - v', x - x' \rangle = \langle y - y', x - x' \rangle + \langle A(x - x'), x - x' \rangle \geq \langle A(x - x'), x - x' \rangle \geq c\|x - x'\|^2.$$

Therefore

$$c\|x - x'\|^2 \leq \langle v - v', x - x' \rangle \leq \|v - v'\|\|x - x'\|.$$

Hence $\|x - x'\| \leq \|v - v'\|/c$. Consequently, $(A+F)^{-1}$ is single-valued and Lipschitz continuous on H with the constant $1/c$. ■

2.3 Subregularity

In this section we present criteria for (strong) subregularity of mappings. The first statement is an analogue Ioffe's criterion and we prove the statement by the iterative process, which is a modification of the proof from [13].

Theorem 2.3.1 *Let (X,d) be a complete metric space, (Y,ρ) be a metric space, and $\bar{x} \in X$ be given. Consider a continuous mapping $f : X \to Y$ whose domain is all of X. Then* popen $f(\bar{x})$ *equals to the supremum of $c > 0$ for which there is $r > 0$ such that for all $x \in \mathbb{B}[\bar{x}, r]$ with $f(x) \neq f(\bar{x})$ there is a point $x' \in X$ satisfying*

$$c\,d(x,x') < \rho(f(x), f(\bar{x})) - \rho(f(x'), f(\bar{x})). \tag{2.15}$$

Proof. Let $b :=$ popen $f(\bar{x})$ and s be the supremum from the statement. Fix any $c > 0$ for which there is $r > 0$ such that for any $x \in \mathbb{B}[\bar{x}, r]$ with $f(x) \neq f(\bar{x})$ there exists $x' \in X$ such that (2.15) holds. Let $\varepsilon := r/2$. Then

$$\mathbb{B}[u,\varepsilon] \subset \mathbb{B}[\bar{x}, r] \quad \text{whenever } u \in \mathbb{B}[\bar{x},\varepsilon]. \tag{2.16}$$

Fix any $t \in (0,\varepsilon)$ and any $u \in \mathbb{B}[\bar{x},\varepsilon] \cap f^{-1}(\mathbb{B}[f(\bar{x}), ct])$. Then $\rho(f(u), f(\bar{x})) \leq ct$. We have to show that $f(\bar{x}) \in f(\mathbb{B}[u,t])$, that is, to find $x \in \mathbb{B}[u,t]$ such that $f(\bar{x}) = f(x)$. If $f(\bar{x}) = f(u)$, take $x := u$ and we are done. Assume further that $f(\bar{x}) \neq f(u)$. We will construct a sequence $x_1, x_2, \ldots$ in $\mathbb{B}[u,t]$ satisfying

$$c\,d(x_m, u) \leq \rho(f(u), f(\bar{x})) - \rho(f(x_m), f(\bar{x})), \quad m \in \mathbb{N}. \tag{2.17}$$

Clearly, the point $x_1 := u$ satisfies (2.17) with $m = 1$. Let $n \in \mathbb{N}$ and assume that $x_n \in \mathbb{B}[u,t]$ satisfying (2.17) with $m = n$ was already found. If $f(x_n) = f(\bar{x})$, then take $x := x_n$, and stop the construction. Assume further that $f(x_n) \neq f(\bar{x})$. Then by (2.16) and (2.17), we find $x_{n+1} \in X$ such that

$$c\,d(x_n, x_{n+1}) < \rho(f(x_n), f(\bar{x})) - \rho(f(x_{n+1}), f(\bar{x})) \quad \text{and that} \quad d(x_n, x_{n+1}) \geq \tfrac{1}{2} s_n \tag{2.18}$$

where

$$s_n := \sup\left\{ d(x_n, x') : \ x' \in X \ \text{ and } \ c\,d(x_n, x') < \rho(f(x_n), f(\bar{x})) - \rho(f(x'), f(\bar{x})) \right\}. \tag{2.19}$$

Note that $0 \leq s_n \leq \frac{1}{c}\rho(f(x_n), f(\bar{x})) < \infty$. Using the first inequality in (2.18), and (2.17) with $m := n$, we get

$$c\,d(u, x_{n+1}) \leq c\,d(u, x_n) + c\,d(x_n, x_{n+1}) < \rho(f(u), f(\bar{x})) - \rho(f(x_{n+1}), f(\bar{x})),$$

which is (2.17) with $m := n+1$. In particular, we have $c\,d(u, x_{n+1}) < \rho(f(u), f(\bar{x})) \le ct$; thus $x_{n+1} \in \mathbb{B}\,[u,t]$. If the process stops at some $n \in \mathbb{N}$, we are done.

Assume that this was not the case, that is, $f(x_n) \neq f(\bar{x})$ for every $n \in \mathbb{N}$. From (2.18) we have, for all $1 \le n < m$, that

$$\begin{aligned} 0 \le c\,d(x_n, x_m) &\le c\,d(x_n, x_{n+1}) + \cdots + c\,d(x_{m-1}, x_m) \\ &< \big(\rho(f(x_n), f(\bar{x})) - \rho(f(x_{n+1}), f(\bar{x}))\big) + \cdots + \big(\rho(f(x_{m-1})f(\bar{x})) - \rho(f(x_m), f(\bar{x}))\big) \\ &= \rho(f(x_n), f(\bar{x})) - \rho(f(x_m), f(\bar{x})), \end{aligned} \tag{2.20}$$

and so, $\rho(f(x_n), f(\bar{x})) > \rho(f(x_m), f(\bar{x}))$. Thus $\ell := \lim_{n\to\infty} \rho(f(x_n), f(\bar{x}))$ exists and is finite, and hence (x_n) is a Cauchy sequence in the complete metric space X. Put $x := \lim_{n\to\infty} x_n$. Then $x \in \mathbb{B}\,[u,t]$. Suppose that $f(\bar{x}) \neq f(x)$. By the assumption, there is $x' \in X$ such that

$$c\,d(x, x') < \rho(f(x), f(\bar{x})) - \rho(f(x'), f(\bar{x})). \tag{2.21}$$

Note that $x \neq x'$ by (2.21). Since $x_n \to x$ and $f(x_n) \to f(x)$, and the function $\rho(f(\cdot), f(\bar{x}))$ is continuous, for any $n \in \mathbb{N}$ sufficiently large, we have

$$c\,d(x_n, x') < \rho(f(x_n), f(\bar{x})) - \rho(f(x'), f(\bar{x})),$$

hence $s_n \ge d(x_n, x')$. Thus $\limsup_{n\to\infty} s_n \ge d(x, x') > 0$. But we know by (2.18) that

$$s_n \le 2d(x_n, x_{n+1}) \to 0 \quad \text{as} \quad n \to \infty,$$

hence we obtain a contradiction. So $f(\bar{x}) = f(x)$. Therefore $c \le b$, and thus $s \le b$.

Assume now that $s < b$. Fix any $c \in (s, b)$. Find $\varepsilon > 0$ such that for any $t \in (0, \varepsilon)$ and any $x \in \mathbb{B}\,[\bar{x}, \varepsilon] \cap f^{-1}(\mathbb{B}\,[f(\bar{x}), ct])$ one has

$$f(\bar{x}) \in f(\mathbb{B}\,[x,t]).$$

By the continuity of f, there is $r \in (0, \varepsilon)$ such that $\rho(f(x), f(\bar{x})) < c\varepsilon$ for each $x \in \mathbb{B}\,[\bar{x}, r]$. Fix any $x \in \mathbb{B}\,[\bar{x}, r]$ with $f(x) \neq f(\bar{x})$. Let $t := \rho(f(x), f(\bar{x}))/c$. Then $t \in (0, \varepsilon)$, and hence there is $x' \in \mathbb{B}\,[x,t]$ such that $f(\bar{x}) = f(x')$. Noting that $x' \neq x$ because $t > 0$, we get

$$0 < c\,d(x, x') \le ct = \rho(f(x), f(\bar{x})) = \rho(f(x), f(\bar{x})) - \rho(f(x'), f(\bar{x})).$$

Hence $s \ge c'$ for any $c' \in (s, c)$, a contradiction. ∎

Again, using the canonical projection we obtain a set-valued version of the previous theorem.

Theorem 2.3.2 *Let (X,d), (Y,ρ) be metric spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a set-valued mapping $F : X \rightrightarrows Y$ whose graph is complete around $(\bar{x}, \bar{y})$. Then* popen $F(\bar{x}, \bar{y})$ *equals to the supremum of all $c > 0$ for which there are $r > 0$ and $\alpha \in (0, 1/c)$ such that any $(x,y) \in$* Graph $F \cap (\mathbb{B}\,[\bar{x}, r] \times \mathbb{B}\,[\bar{y}, r])$ *with $y \neq \bar{y}$ there is a pair $(x', y') \in$* Graph F *such that*

$$c \max\big\{d(x, x'), \alpha\rho(y, y')\big\} < \rho(y, \bar{y}) - \rho(y', \bar{y}). \tag{2.22}$$

Proof. Let $b := \operatorname{popen} F(\bar{x}, \bar{y})$ and denote by s the supremum from the statement.

First, we show that $b \ge s$. Fix an arbitrary $c \in (0, s)$ (if there is any). Find $\alpha \in (0, 1/c)$ and $r > 0$ such that the property involving (2.22) holds. Define the (compatible) metric $\widetilde{d}$ on $X \times Y$ by $\widetilde{d}((u,v),(u',v')) := \max\{d(u,u'), \alpha\rho(v,v')\}$ for any $(u,v), (u',v') \in X \times Y$. Fix $r' \in (0, r)$ such that $\widetilde{X} := \operatorname{Graph} F \cap (\mathbb{B}\,[\bar{x}, r'] \times \mathbb{B}\,[\bar{y}, r'/\alpha])$ is a complete metric space. Consider $f := p_Y|_{\widetilde{X}}$, where p_Y denotes the canonical projection of $X \times Y$ onto Y. Obviously, f is continuous on $\widetilde{X}$. Let $\widetilde{r} > 0$ be such that $\widetilde{r}(1 + \alpha c) < \alpha c r'$. In particular, $\widetilde{r} < r'$. Fix any

$$(x,y) \in \mathbb{B}_{\widetilde{X}}\,[(\bar{x}, \bar{y}), \widetilde{r}] = \operatorname{Graph} F \cap (\mathbb{B}\,[\bar{x}, \widetilde{r}] \times \mathbb{B}\,[\bar{y}, \widetilde{r}/\alpha]) \subset \operatorname{Graph} F \cap \big(\mathbb{B}\,[\bar{x}, r'] \times \mathbb{B}\,[\bar{y}, r'/\alpha]\big)$$

such that $y \neq \bar{y}$. By assumption, there is a pair $(x',y') \in \operatorname{Graph} F$ such that (2.22) holds. Then

$$\widetilde{d}((x',y'),(\bar{x},\bar{y})) \leq \widetilde{d}((x',y'),(x,y)) + \widetilde{r} < \frac{\rho(y,\bar{y})}{c} + \widetilde{r} \leq \widetilde{r}\left(\frac{1}{\alpha c}+1\right) < r',$$

hence $(x',y') \in \widetilde{X}$. Theorem 2.3.1 with $(X,d) := \big(\widetilde{X},\widetilde{d}\big)$ says that $\operatorname{popen} f(\bar{x},\bar{y}) \geq c$. Fix an arbitrary $c' \in (0,c)$. Find $\varepsilon > 0$ such that for any $t \in (0,\varepsilon)$ and any $(x,y) \in \mathbb{B}_{\widetilde{X}}[(\bar{x},\bar{y}),\varepsilon] \cap f^{-1}(\mathbb{B}[\bar{y},c't])$ one has $\bar{y} \in f\big(\mathbb{B}_{\widetilde{X}}[(x,y),t]\big)$. Fix an arbitrary $t \in (0,\varepsilon)$ and $x \in \mathbb{B}[\bar{x},\varepsilon] \cap F^{-1}(\mathbb{B}[\bar{y},c't])$. As $c' < c < 1/\alpha$, there is $y \in \mathbb{B}[\bar{y},c't] \subset \mathbb{B}[\bar{y},\varepsilon/\alpha]$ such that $y \in F(x)$. Thus $(x,y) \in \operatorname{Graph} F \cap (\mathbb{B}[\bar{x},\varepsilon] \times \mathbb{B}[\bar{y},\varepsilon/\alpha]) = \mathbb{B}_{\widetilde{X}}[(\bar{x},\bar{y}),\varepsilon]$ and $(x,y) \in f^{-1}(\mathbb{B}[\bar{y},c't])$. Find a pair $(u,v) \in \operatorname{Graph} F \cap (\mathbb{B}[x,t] \times \mathbb{B}[y,t/\alpha])$ such that $\bar{y} = f(u,v) = v$. Then $\bar{y} = v \in F(u) \subset F(\mathbb{B}[x,t])$. Thus $b \geq c'$. Letting $c' \uparrow c$ and then $c \uparrow s$, we get $b \geq s$.

Suppose that $b > s$. Fix an arbitrary $c \in (s,b)$. Find $\varepsilon > 0$ such that $\operatorname{Graph} F \cap (\mathbb{B}[\bar{x},\varepsilon] \times \mathbb{B}[\bar{y},\varepsilon])$ is complete and such that for any $t \in (0,\varepsilon)$ and any $x \in \mathbb{B}[\bar{x},\varepsilon] \cap F^{-1}(\mathbb{B}[\bar{y},ct])$ one has $\bar{y} \in F(\mathbb{B}[x,t])$. Pick $\alpha \in (0,1/c)$ and let $\widetilde{d}$ be as above. Set $\widetilde{X} := \operatorname{Graph} F \cap (\mathbb{B}[\bar{x},r'] \times \mathbb{B}[\bar{y},r'/\alpha])$ with $r' \in (0,\min\{\varepsilon,\alpha\varepsilon\})$. Then $\widetilde{X} \subset \operatorname{Graph} F \cap (\mathbb{B}[\bar{x},\varepsilon] \times \mathbb{B}[\bar{y},\varepsilon])$, hence $\widetilde{X}$ is a complete metric space.

Fix any $t \in (0,r')$ and any $(x,y) \in \mathbb{B}_{\widetilde{X}}[(\bar{x},\bar{y}),r'] \cap f^{-1}(\mathbb{B}[\bar{y},ct])$. Then $(x,y) \in \operatorname{Graph} F \cap (\mathbb{B}[\bar{x},r'] \times \mathbb{B}[\bar{y},r'/\alpha])$ with $y \in \mathbb{B}[\bar{y},ct]$. This implies, on one hand, that $\bar{y} \in \mathbb{B}[y,ct] \subset \mathbb{B}[y,t/\alpha]$. On the other hand, it follows that $x \in \mathbb{B}[\bar{x},\varepsilon] \cap F^{-1}(\mathbb{B}[\bar{y},ct])$, hence there is $u \in \mathbb{B}[x,t]$ such that $\bar{y} \in F(u)$. Then

$$(u,\bar{y}) \in \operatorname{Graph} F \cap (\mathbb{B}[x,t] \times \mathbb{B}[y,t/\alpha]) = \mathbb{B}_{\widetilde{X}}[(x,y),t].$$

Thus $\bar{y} = f(u,\bar{y}) \in f\big(\mathbb{B}_{\widetilde{X}}[(x,y),t]\big)$. It follows that $\operatorname{popen} f(\bar{x},\bar{y}) \geq c > s$. By Theorem 2.3.1 for any $c' \in (s,c)$ there is $\widetilde{r} \in (0,r')$ such that for all $(x,y) \in \mathbb{B}_{\widetilde{X}}[(\bar{x},\bar{y}),\widetilde{r}]$ with $y \neq \bar{y}$, there is a point $(x',y') \in \widetilde{X} \subset \operatorname{Graph} F$ such that

$$c' \max\big\{d(x,x'),\alpha\rho(y,y')\big\} < \rho(y,\bar{y}) - \rho(y',\bar{y}).$$

Take $r \in (0,\min\{\widetilde{r},\widetilde{r}/\alpha\})$. As $\alpha < 1/c < 1/c'$, we get $s \geq c' > s$, a contradiction. ■

Strong subregularity is stable under calm single-valued perturbations, see [14].

Proposition 2.3.1 *Let (X,d) be a complete metric space, (Y,ρ) be a linear metric space with a shift-invariant metric, and $(\bar{x},\bar{y}) \in X \times Y$ be given. Consider a mapping $f : X \to Y$ defined around $\bar{x}$ and a mapping $F : X \rightrightarrows Y$ such that $\bar{y} \in F(\bar{x})$. If F is strongly subregular at $(\bar{x},\bar{y})$ and* $\operatorname{calm} f(\bar{x}) < \operatorname{popen} F(\bar{x},\bar{y})$*, then $F+f$ is strongly subregular at $(\bar{x},\bar{y}+f(\bar{x}))$ and*

$$\operatorname{popen}(F+f)(\bar{x},\bar{y}+f(\bar{x})) \geq \operatorname{popen} F(\bar{x},\bar{y}) - \operatorname{calm} f(\bar{x}) > 0.$$

Example 2.3.1 *Let $g : X \to Y$ be a single-valued mapping between Banach spaces $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$. Suppose that g is Fréchet differentiable and strongly subregular at $\bar{x} \in X$, then Proposition 2.3.1, with $F := g$ and $f := g(\bar{x}) - g + g'(\bar{x})(\cdot - \bar{x})$, implies* $\operatorname{popen} g(\bar{x}) \leq \operatorname{popen}\big(g'(\bar{x})\big)$*. On the other hand, suppose that* $\operatorname{popen}\big(g'(\bar{x})\big) > 0$*, then Proposition 2.3.1, with $F := g(\bar{x}) + g'(\bar{x})(\cdot - \bar{x})$ and $f := g - g(\bar{x}) - g'(\bar{x})(\cdot - \bar{x})$, implies* $\operatorname{popen}\big(g'(\bar{x})\big) \leq \operatorname{popen} g(\bar{x})$*. Hence* $\operatorname{popen} g(\bar{x}) = \operatorname{popen}\big(g'(\bar{x})\big)$.

2.4 Semiregularity

In this section we will focus on criteria for (strong) semiregularity. The following theorem provides an answer to the question what happens if we sum a semiregular set-valued mapping with a set-valued mapping which has Aubin property, see [14]. This assertion follows either from [32, Corollary 3.1] or Theorem 2.2.4.

Theorem 2.4.1 *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$ be given. Consider set-valued mappings F, $G : X \rightrightarrows Y$ such that F has a locally closed graph around $(\bar{x}, \bar{y})$ and G has a locally closed graph around $(\bar{x}, \bar{z})$. Then*

$$\operatorname{lopen}(F + G)(\bar{x}, \bar{y} + \bar{z}) \geq \operatorname{sur} F(\bar{x}, \bar{y}) - \operatorname{lip} G(\bar{x}, \bar{z}).$$

In the case of a single-valued perturbation we get:

Corollary 2.4.1 *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces and $(\bar{x}, \bar{y}) \in X \times Y$ be given. Consider a single-valued mapping $f : X \to Y$ which is continuous around $\bar{x}$ and a set-valued mapping $F : X \rightrightarrows Y$ which has a locally closed graph around $(\bar{x}, \bar{y})$. Then*

$$\operatorname{lopen}(F + f)(\bar{x}, \bar{y} + f(\bar{x})) \geq \operatorname{sur} F(\bar{x}, \bar{y}) - \operatorname{lip} f(\bar{x}).$$

The following example shows that semiregularity is unstable with respect calm (even differentiable at $\bar{x}$) single-valued perturbations.

Example 2.4.1 *Consider single-valued mappings $f, g : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) := x, \quad x \in \mathbb{R}, \quad \text{and} \quad g(x) := \begin{cases} x^2, & \text{for} \quad x \in \mathbb{Q}, \\ 0, & \text{for} \quad x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $\operatorname{sur} f(0) = 1$ and $\operatorname{calm} g(0) = 0$ but the mapping $f + g$ is not semiregular at 0.

Indeed, assume that $f + g$ is semiregular at 0. For $k = 5, 6, \ldots,$ consider a problem to find $x_k \in (-0.5, 0.5)$ such that

$$f(x_k) + g(x_k) = -\frac{1}{k}.$$

But there is no $x_k \in (-0.5, 0.5) \cap \mathbb{Q}$ such that

$$x_k^2 + x_k = -\frac{1}{k},$$

because the only possible solution has the form

$$x_k = -\frac{1}{2} + \frac{\sqrt{k-4}\sqrt{k}}{2k}$$

but such $x_k \in \mathbb{R} \setminus \mathbb{Q}$ for $k = 5, 6, \ldots,$ a contradiction.

Chapter 3

Newton-type methods

Consider a generalized equation: find $x \in X$ such that

$$f(x) + F(x) \ni 0, \tag{3.1}$$

where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, $f : X \to Y$ is a single-valued mapping, and $F : X \rightrightarrows Y$ is a set-valued mapping. We investigate a Newton-type method for solving (3.1) in the form

$$f(x_k) + A_k(x_{k+1} - x_k) + F(x_{k+1}) \ni 0 \quad \text{for} \quad k = 0, 1, 2, \dots, \tag{3.2}$$

where an initial point $x_0 \in X$ and a sequence of $A_k \in \mathcal{L}(X, Y)$, $k = 0, 1, 2, \dots$, are given. If $F \equiv 0$, then (3.1) reduces to the equation: find $x \in X$ such that

$$f(x) = 0. \tag{3.3}$$

Then (3.2) reads as

$$f(x_k) + A_k(x_{k+1} - x_k) = 0 \quad \text{for} \quad k = 0, 1, 2, \dots. \tag{3.4}$$

We are going to discuss three types of theorems - local convergence results, Dennis-Moré theorems and Kantorovich theorems.

3.1 Historical background

We distinguish the following convergence rates of a sequence (x_k) converging to $\bar{x}$ in a Banach space $(X, \|\cdot\|_X)$. We say that the sequence (x_k) converges:

1. *q-linearly* to $\bar{x}$ if there is $\mu \in (0, 1)$ such that
$$\|x_{k+1} - \bar{x}\|_X \leq \mu \|x_k - \bar{x}\|_X$$
for all sufficiently large $k \in \mathbb{N}$;

2. *q-superlinearly* to $\bar{x}$ if there is a sequence of positive numbers (μ_k) converging to 0 such that
$$\|x_{k+1} - \bar{x}\|_X \leq \mu_k \|x_k - \bar{x}\|_X$$
for all sufficiently large $k \in \mathbb{N}$;

3. *r-superlinearly* to $\bar{x}$ if there are sequences of positive numbers (η_k) and (μ_k) such that $\eta_k \to 0$ and

$$\|x_k - \bar{x}\|_X \leq \mu_k \quad \text{and} \quad \mu_{k+1} \leq \eta_k \mu_k$$

for all sufficiently large $k \in \mathbb{N}$;

4. *q-quadratically* to $\bar{x}$ if there is $\mu > 0$ such that

$$\|x_{k+1} - \bar{x}\|_X \leq \mu \|x_k - \bar{x}\|_X^2$$

for all sufficiently large $k \in \mathbb{N}$.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a single-valued mapping and consider the problem (3.3). If f is differentiable, then we can take $A_k := \nabla f(x_k)$ in (3.4) and we get Newton iteration in the form

$$f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots. \tag{3.5}$$

For this method we have the following local convergence result, see [29, p. 71].

Theorem 3.1.1 *Consider a single-valued mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ and a point $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) = 0$. Suppose that f is twice continuously differentiable in a neighborhood of $\bar{x}$ and the matrix $\nabla f(\bar{x})$ is nonsingular. Then there is a neighborhood O of $\bar{x}$ such that for each $x_0 \in O$ the sequence (x_k) generated by (3.5) exists and converges q-quadratically to $\bar{x}$.*

The following example shows that a sequence generated by (3.5) may converge for any initial point, even though some assumptions of Theorem 3.1.1 are not satisfied.

Example 3.1.1 *Consider a problem to find $x \in \mathbb{R}$ such that*

$$x^2 = 0.$$

Clearly the solution is $\bar{x} = 0$ and the Newton iteration reads as

$$(x_k)^2 + 2x_k(x_{k+1} - x_k) = 0 \quad \textit{for} \quad k = 0, 1, 2, \ldots. \tag{3.6}$$

Hence

$$x_{k+1} = x_k/2.$$

Then $x_k = 2^{-k}x_0$ for $k = 0, 1, 2, \ldots$ and any fixed initial point $x_0 \in \mathbb{R}$. The sequence (x_k) converges q-linearly to $\bar{x}$ no matter how far from $\bar{x}$ the initial point is.

Sometimes it is difficult to compute $\nabla f(x_k)$ or this takes too much time. For this reason Chord method is introduced. It uses $\nabla f(x_0)$ instead of $\nabla f(x_k)$, that is, we set $A_k := \nabla f(x_0)$ for each $k = 0, 1, 2, \ldots$ in (3.4), and we get the iteration scheme

$$f(x_k) + \nabla f(x_0)(x_{k+1} - x_k) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, \tag{3.7}$$

where $x_0 \in \mathbb{R}^n$ is given. The following statement guarantees the convergence of the Chord method, see [29, p. 76].

Theorem 3.1.2 *Consider a single-valued mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ and a point $\bar{x} \in \mathbb{R}^n$ such that $f(\bar{x}) = 0$. Suppose that f is twice continuously differentiable in a neighborhood of $\bar{x}$ and the matrix $\nabla f(\bar{x})$ is nonsingular. Then there is a neighborhood O of $\bar{x}$ such that for each $x_0 \in O$ the sequence (x_k) generated by (3.7) exists and converges q-linearly to $\bar{x}$.*

Statements as Theorem 3.1.1 and Theorem 3.1.2 which impose assumptions on the derivative at the unknown solution $\bar{x}$ and guarantee the convergence of the sequence (x_k) to a solution $\bar{x}$ are called *local convergence theorems.*

In 1974 J.E. Dennis and J.J. Moré [15] proved the theorem characterizing the superlinear convergence of the sequence generated by (3.4) to a solution of (3.3).

Theorem 3.1.3 (Dennis-Moré theorem) *Consider a single-valued mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ and a point $\bar{x} \in \mathbb{R}^n$. Suppose that f is differentiable on a neighborhood U of $\bar{x}$, the derivative ∇f is continuous at $\bar{x}$ and the matrix $\nabla f(\bar{x})$ is nonsingular. Let (A_k) be a sequence of nonsingular matrices and let for some initial point $x_0 \in U$ the sequence (x_k) be generated by* (3.4) *and converge to $\bar{x}$. Then (x_k) converges q-superlinearly to $\bar{x}$ and $f(\bar{x}) = 0$ if and only if*

$$\lim_{k\to\infty} \frac{\|\big(A_k - \nabla f(\bar{x})\big)(x_{k+1} - x_k)\|}{\|x_{k+1} - x_k\|} = 0.$$

Statements, which assume that there is a sequence converging to a solution and guarantee a certain rate of convergence of the sequence, we call *Dennis-Móre theorems.*

While there is some disagreement among historians who actually invented the Newton method, see [42] for an excellent reading about early history of the method, it is well documented in the literature that L.V. Kantorovich [27] was the first to obtain convergence of the method on assumptions involving the point where iterations begin. Specially, Kantorovich considered the Newton method for solving the equation (3.3) and proved convergence by imposing conditions on the derivative $\nabla f(x_0)$ of the function f and the residual $\|f(x_0)\|$ at the initial point x_0. These conditions can be actually checked, in contrast to the conventional approach in local convergence theorems. For this reason Kantorovich-type theorems are usually called *semi-local convergence theorems*[1] whereas conventional convergence theorems are described as local theorems.

Theorem 3.1.4 (Kantorovich theorem) *Consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$, a point $x_0 \in \mathbb{R}^n$, and positive constants a, ℓ, κ, and μ. Suppose that f is continuously differentiable in an open neighborhood of the ball $\mathbb{B}[x_0, a]$, the derivative ∇f is Lipschitz continuous in $\mathbb{B}[x_0, a]$ with the constant ℓ, and we have*

$$\|\nabla f(x_0)^{-1}\| \le \kappa \quad \textit{and} \quad \|\nabla f(x_0)^{-1} f(x_0)\| < \mu.$$

If $\alpha := \kappa\ell\mu a < 1/2$ and $a \ge a_0 := \dfrac{1-\sqrt{1-2\alpha}}{\kappa\ell}$, then there exists a unique sequence (x_k) generated by (3.5) *with the initial point x_0; this sequence converges to a point $\bar{x} \in \mathbb{B}[x_0, a_0]$ with $f(\bar{x}) = 0$. In addition, $\bar{x}$ is the unique solution of* (3.3) *in $\mathbb{B}[x_0, a_0]$ and*

$$\|x_k - \bar{x}\| \le \frac{\mu}{\alpha}(2\alpha)^{2^k} \quad \textit{for} \quad k = 0, 1, 2, \ldots.$$

In [28, Chapter 18] Kantorovich showed that under exactly same assumptions as in Theorem 3.1.4, the sequence (x_k) generated by the scheme (3.7) converges q-linearly to the solution $\bar{x}$.

In 1955 R.G. Bartle [6] studied the iteration

$$f(x_k) + \nabla f(z_k)(x_{k+1} - x_k) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots, \tag{3.8}$$

where $z_k \in \mathbb{R}^n$ are, to quote [6], "arbitrarily selected points ... sufficiently close to the solution desired." For $z_k := x_k$ one obtains the usual Newton method, and for $z_k := x_0$ the Chord method, but z_k may be chosen in other ways. For example as x_0 for the first s iterations and then the derivative could be calculated again every s iterations, obtaining in this way a hybrid version of the method.

[1]Some authors prefer the name global convergence theorems.

If computing the derivatives, in particular in the case they are obtained numerically, involves time consuming procedures, it is quite plausible to expect that for large scale problems the chord method or a hybrid version of it would possibly be faster than the usual method. We present here the following somewhat modified statement of Bartle theorem which fits our purposes.

Theorem 3.1.5 (Bartle theorem) *Consider a single-valued mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, a point $x_0 \in \mathbb{R}^n$, and positive constants a and κ. Suppose that f is continuously differentiable on $\mathbb{B}(x_0, 2a)$, for any three points $x_1, x_2, x_3 \in \mathbb{B}[x_0, a]$ we have*

$$\|\nabla f(x_1)^{-1}\| \leq \kappa \quad \text{and} \quad \|f(x_1) - f(x_2) - \nabla f(x_3)(x_1 - x_2)\| \leq \frac{1}{2\kappa}\|x_1 - x_2\|, \tag{3.9}$$

and also

$$\|f(x_0)\| < \frac{a}{2\kappa}. \tag{3.10}$$

Then for every sequence (z_k) with $z_k \in \mathbb{B}[x_0, a]$, $k = 0, 1, 2, \ldots,$ there exists a unique sequence (x_k) generated by (3.8) with the initial point x_0; this sequence converges to a point $\bar{x} \in \mathbb{B}[x_0, a]$ with $f(\bar{x}) = 0$. In addition, $\bar{x}$ is the unique solution of (3.3) in $\mathbb{B}[x_0, a]$ and

$$\|x_k - \bar{x}\| \leq 2^{-k}a \quad \text{for each} \quad k = 0, 1, 2, \ldots.$$

Let us point out that all the previous results were originally proved in infinite-dimensional spaces.

3.2 Local convergence theorems

The set-valued mapping $\mathcal{H}$ between Banach spaces is said to be *outer semicontinuous* at a point $\bar{x} \in \operatorname{dom}\mathcal{H}$ if for every open set O containing $\mathcal{H}(\bar{x})$ there exists a neighborhood U of $\bar{x}$ such that $\mathcal{H}(x) \subset O$ for each $x \in U$. For a given subset $\mathcal{A}$ in $\mathcal{L}(X, Y)$, *the Kuratowski measure of noncompactness* $\chi(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$\chi(\mathcal{A}) := \inf\{r > 0 : \mathcal{A} \subset \mathcal{B} + r\,\mathbb{B}_{\mathcal{L}(X,Y)} \text{ for some } \mathcal{B} \subset \mathcal{A} \text{ finite}\}.$$

The following theorems from [11] (accommodated a bit to our purposes) generalize Theorem 3.1.1 for a generalized equation (3.1) and regularity plays a crucial role in them.

Theorem 3.2.1 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Consider a single-valued continuous mapping $f : X \to Y$ along with set-valued mappings $F : X \rightrightarrows Y$ with a closed graph, $\mathcal{H} : X \rightrightarrows \mathcal{L}(X, Y)$, and a point $\bar{x} \in X$ such that $f(\bar{x}) + F(\bar{x}) \ni 0$. Suppose that $\mathcal{H}$ is outer semicontinuous at $\bar{x}$ and that for each $\varepsilon > 0$ there is a neighborhood U of $\bar{x}$ such that*

$$\|f(x) - f(\bar{x}) - A(x - \bar{x})\|_Y \leq \varepsilon\|x - \bar{x}\|_X \quad \text{for every} \quad x \in U \quad \text{and} \quad A \in \mathcal{H}(x). \tag{3.11}$$

Define

$$G_A : X \ni x \longmapsto f(\bar{x}) + A(x - \bar{x}) + F(x) \quad \text{for} \quad A \in \mathcal{H}(\bar{x}) \tag{3.12}$$

and let

$$\chi(\mathcal{H}(\bar{x})) < \inf_{A \in \mathcal{H}(\bar{x})} \operatorname{sur} G_A(\bar{x}, 0).$$

Then there is a neighborhood O of $\bar{x}$ such that for any $x_0 \in O$ there exists a sequence (x_k) generated by (3.2) with $A_k \in \mathcal{H}(x_k)$ for $k = 0, 1, 2, \ldots,$ converging q-linearly to $\bar{x}$.

Suppose that f is continuously Fréchet differentiable at $\bar{x}$, F has a closed graph, and $\operatorname{sur}(f+F)(\bar{x},0)>0$. Then f satisfies (3.11) with $\mathcal{H}:=f'$. By Theorem 2.1.5, with $f:=f(\bar{x})-f+f'(\bar{x})(\cdot-\bar{x})$ and $F:=f+F$, we get $\operatorname{sur} G_A(\bar{x},0)>0$ with $A:=f'(\bar{x})$. Moreover, $\mathcal{H}$ is outer semicontinuous at $\bar{x}$ and $\chi(\mathcal{H}(\bar{x}))=0<\operatorname{sur} G_A(\bar{x},0)$, with $A:=f'(\bar{x})$. Therefore the assumptions of Theorem 3.2.1 are satisfied.

If we assume that for each $A\in\mathcal{H}(\bar{x})$ the mapping G_A is strongly regular at $(\bar{x},0)$, then we get the superlinear or even the quadratic convergence of the sequence in (3.2).

Theorem 3.2.2 *Suppose that the assumptions of Theorem 3.2.1 are satisfied. In addition, suppose that for every $A\in\mathcal{H}(\bar{x})$ the mapping G_A defined in* (3.12) *is strongly regular around $(\bar{x},0)$. Then every sequence (x_k) generated by* (3.2) *with $A_k\in\mathcal{H}(x_k)$ for $k=0,1,2,\ldots$, which converges to $\bar{x}$ is in fact q-superlinearly convergent.*

Moreover, suppose that there are $\mu>0$ and a neighborhood V of $\bar{x}$ such that

$$\|f(x)-f(\bar{x})-A(x-\bar{x})\|_Y\le\mu\|x-\bar{x}\|_X^2 \quad \textit{for every} \quad x\in V \textit{ and } A\in\mathcal{H}(x). \tag{3.13}$$

Then every sequence (x_k) generated by (3.2) *with $A_k\in\mathcal{H}(x_k)$ for $k=0,1,2,\ldots$, which converges to $\bar{x}$ is in fact q-quadratically convergent.*

Suppose that assumptions of Theorem 3.1.1 are satisfied, that is, $f:\mathbb{R}^n\to\mathbb{R}^n$ is twice continuously differentiable around $\bar{x}\in\mathbb{R}^n$ with $f(\bar{x})=0$ and the matrix $\nabla f(\bar{x})$ is nonsingular. Then (3.13) holds with $\mathcal{H}:=\nabla f$, $\mathcal{H}$ is outer semicontinuous at $\bar{x}$, $\mathcal{H}(\bar{x})=\nabla f(\bar{x})$, and $\chi(\mathcal{H}(\bar{x}))=0$. The mapping G_A from (3.12) with $A:=\nabla f(\bar{x})$ and $F\equiv 0$, is strongly regular around $(\bar{x},0)$. Consequently, Theorem 3.1.1 follows from Theorem 3.2.2.

3.3 Dennis-Moré theorems

The following two statements come from [11] and strong subregularity plays crucial role in them.

Theorem 3.3.1 *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces. Consider a single-valued continuous mapping $f:X\to Y$ along with set-valued mappings $F:X\rightrightarrows Y$ and $\mathcal{H}:X\rightrightarrows\mathcal{L}(X,Y)$, and a point $\bar{x}\in X$. Suppose that F has a closed graph, the set $\mathcal{H}(\bar{x})$ is bounded and $\bar{x}\in\operatorname{int}\operatorname{dom}\mathcal{H}$, and that for each $\varepsilon>0$ there is a neighborhood U of $\bar{x}$ such that for every $x,x'\in U$ there is $A\in\mathcal{H}(\bar{x})$ satisfying*

$$\|f(x)-f(x')-A(x-x')\|_Y\le\varepsilon\|x-x'\|_X. \tag{3.14}$$

Consider a sequence (x_k) generated by (3.2) *with $A_k\in\mathcal{L}(X,Y)$ and $x_k\neq\bar{x}$ for every $k=0,1,2,\ldots$, which converges to $\bar{x}$. Let (B_k) be a sequence in $\mathcal{H}(\bar{x})$ satisfying*

$$\lim_{k\to\infty}\frac{\|f(x_{k+1})-f(x_k)-B_k(x_{k+1}-x_k)\|_Y}{\|x_{k+1}-x_k\|_X}=0 \tag{3.15}$$

with $x_{k+1}\neq x_k$ for every $k=0,1,2,\ldots$.

(i) *If (x_k) converges q-superlinearly to $\bar{x}$, then*

$$\lim_{k\to\infty}\frac{\operatorname{dist}\big(0,f(\bar{x})+(A_k-B_k)(x_{k+1}-x_k)+F(x_{k+1})\big)}{\|x_{k+1}-x_k\|_X}=0. \tag{3.16}$$

(ii) *If*

$$\lim_{k\to\infty}\frac{\|(A_k-B_k)(x_{k+1}-x_k)\|_Y}{\|x_{k+1}-x_k\|_X}=0, \tag{3.17}$$

then $f(\bar{x})+F(\bar{x})\ni 0$. If, in addition, the mapping $f+F$ is strongly subregular at $(\bar{x},0)$ then (x_k) converges q-superlinearly to $\bar{x}$.

Due to [12, Theorem 3.1], the condition that $f+F$ is strongly subregular at $(\bar{x},0)$, can be substituted by the condition that for every $A\in\mathcal{H}(\bar{x})$ the mapping G_A is strongly subregular at $(\bar{x},0)$, and that $\chi(\mathcal{H}(\bar{x}))$ is smaller than $\operatorname{popen} G_A(\bar{x},0)$ for each $A\in\mathcal{H}(\bar{x})$.

Theorem 3.3.2 *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces. Consider a single-valued continuous mapping $f: X\to Y$ along with set-valued mappings $F: X\rightrightarrows Y$ and $\mathcal{H}: X\rightrightarrows \mathcal{L}(X,Y)$, and a point $\bar{x}\in X$. Suppose that F has a closed graph and $\mathcal{H}$ is outer semicontinuous at $\bar{x}\in \operatorname{int}\ \operatorname{dom}\mathcal{H}$, satisfies (3.11), and that the set $\mathcal{H}(\bar{x})$ is bounded. Consider a sequence (x_k) generated by (3.2) with $A_k\in\mathcal{L}(X,Y)$ and $x_k\neq\bar{x}$ for every $k=0,1,2,\dots$, which converges to $\bar{x}$.*

(i) *If (x_k) converges q-superlinearly to $\bar{x}$, then for every sequence (B_k) with $B_k\in\mathcal{H}(x_k)$ for all sufficiently large $k\in\mathbb{N}$, the condition (3.16) holds.*

(ii) *If there exists a sequence (B_k) such that $B_k\in\mathcal{H}(x_k)$ for all sufficiently large $k\in\mathbb{N}$ and that the condition (3.17) holds, then $f(\bar{x})+F(\bar{x})\ni 0$. If, in addition, for every $A\in\mathcal{H}(\bar{x})$ the mapping G_A defined in (3.12) is strongly subregular at $(\bar{x},0)$ and*

$$\chi(\mathcal{H}(\bar{x}))<\inf_{A\in\mathcal{H}(\bar{x})}\operatorname{popen} G_A(\bar{x},0).$$

Then (x_k) converges q-superlinearly to $\bar{x}$.

Suppose that assumptions of Theorem 3.1.3 are satisfied, that is, $f:\mathbb{R}^n\to\mathbb{R}^n$ is differentiable around a point $\bar{x}\in\mathbb{R}^n$, ∇f is continuous at $\bar{x}$, $\nabla f(\bar{x})$ is nonsingular, (A_k) is a sequence of nonsingular matrices in $\mathbb{R}^{n\times n}$, and (x_k) is a sequence in $\mathbb{R}^n$ converging to $\bar{x}$, generated by (3.4) with an initial point x_0 close enough to $\bar{x}$. Set $\mathcal{H}\equiv\nabla f(\bar{x})$ and $B_k:=\nabla f(\bar{x})$ for every $k=0,1,2,\dots$. Then $\mathcal{H}$ is outer semicontinuous at $\bar{x}$, $\mathcal{H}(\bar{x})$ is bounded, $\chi(\mathcal{H}(\bar{x}))=0$, and the conditions (3.11), (3.14) and (3.15) hold. Consequently, the mappings f and G_A from (3.12), with $A:=\nabla f(\bar{x})$ and with $F\equiv 0$, are strongly subregular[2] at $\bar{x}$ and $\chi(\mathcal{H}(\bar{x}))=0<\operatorname{popen} G_A$. Consequently, Theorem 3.1.3 follows either from Theorem 3.3.1 or Theorem 3.3.2.

3.4 Semilocal convergence theorems

We present semilocal convergence theorems, which were proved in [10]. In the following assertions, (strong) regularity plays a crucial role. The first theorem is a generalization of Bartle theorem and [35, Theorem 3.3].

Theorem 3.4.1 *Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces. Consider a single-valued continuous mapping $f: X\to Y$, positive constants a,κ,δ, and a point $x_0\in X$ such that*

$$\kappa\delta<1\quad\textit{and}\quad \|f(x_0)\|_Y<(1-\kappa\delta)\frac{a}{\kappa}.$$

Suppose that there exists a sequence (A_k) in $\mathcal{L}(X,Y)$ such that, for every $k=0,1,2,\dots$ and every $x,x'\in\mathbb{B}[x_0,a]$, we have

$$\|A_k^{-1}\|_{\mathcal{L}(Y,X)}\le\kappa\quad\textit{and}\quad\|f(x)-f(x')-A_k(x-x')\|_Y\le\delta\|x-x'\|_X.$$

Then there exists a unique sequence (x_k) generated by (3.4) with the initial point x_0; this sequence remains in $\mathbb{B}(x_0,a)$ and converges to a point $\bar{x}\in\mathbb{B}(x_0,a)$ with $f(\bar{x})=0$. In addition, $\bar{x}$ is the unique solution of (3.3) in $\mathbb{B}[x_0,a]$ and for each $\alpha\in(\kappa\delta,1)$ we have

$$\|x_k-\bar{x}\|_X<\alpha^k a\quad\textit{for every}\quad k=0,1,2,\dots.$$

[2]The mappings are in fact strongly regular around $\bar{x}$, see Example 2.1.2 and Theorem 2.2.3.

It follows a generalization of the previous theorem for the generalized equation with the iteration (3.2).

Theorem 3.4.2 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Consider a single-valued continuous mapping $f : X \to Y$ and a set-valued mapping $F : X \rightrightarrows Y$ with a closed graph along with positive constants $a, b, \kappa, \delta,$ and points $x_0 \in X$, $y_0 \in f(x_0) + F(x_0)$ such that*

(i) $\kappa\delta < 1$ *and* $\|y_0\|_Y < (1 - \kappa\delta) \min\{\frac{a}{\kappa}, b\}$.

Suppose that there exists a function $\omega : [0, a] \to [0, \delta]$ such that for every $k = 0, 1, 2, \ldots$ and every $x_1, \ldots, x_k \in \mathbb{B}[x_0, a]$ the operator $A_k := A_k(x_0, \ldots, x_k) \in \mathcal{L}(X, Y)$ appearing in (3.2) has the following properties:

(ii) *the mapping*

$$X \ni x \longmapsto G_{A_k}(x) := f(x_0) + A_k(x - x_0) + F(x) \tag{3.18}$$

is metrically regular around (x_0, y_0) with constant κ and neighborhoods $\mathbb{B}[x_0, a]$ and $\mathbb{B}[y_0, b]$;

(iii) $\|f(x) - f(x_k) - A_k(x - x_k)\|_Y \le \omega(\|x - x_k\|_X)\, \|x - x_k\|_X$ *for every* $x \in \mathbb{B}[x_0, a]$.

Then for every $\alpha \in (\kappa\delta, 1)$ there exists a sequence (x_k) generated by (3.2) with the initial point x_0; this sequence remains in $\mathbb{B}(x_0, a)$ and converges to a point $\bar{x} \in \mathbb{B}(x_0, a)$ with $f(\bar{x}) + F(\bar{x}) \ni 0$. In addition, the convergence is r-linear; specifically

$$\|x_k - \bar{x}\|_X < \alpha^k a \quad \textit{and} \quad \operatorname{dist}(0, f(x_k) + F(x_k)) \le \alpha^k \|y_0\|_Y \quad \textit{for every} \quad k = 0, 1, 2, \ldots.$$

If $\lim_{\xi \to 0} \omega(\xi) = 0$, then the sequence (x_k) converges r-superlinearly to $\bar{x}$.

If there exists a constant $\ell > 0$ such that $\omega(\xi) \le \ell\xi$ for each $\xi \in [0, a]$, then (x_k) converges r-quadratically to $\bar{x}$; specifically, there exists a sequence of positive numbers (ε_k) such that for any $C > \frac{\alpha\ell}{\delta}$ we have $\varepsilon_{k+1} < C\varepsilon_k^2$ for all sufficiently large $k \in \mathbb{N}$.

If the mapping $\mathbb{B}[y_0, b] \ni y \longmapsto G_{A_k}^{-1}(y) \cap \mathbb{B}[x_0, a]$ is single-valued, then there is no other sequence (x_k) satisfying (3.2) starting from x_0 which stays in $\mathbb{B}[x_0, a]$.

Suppose that the assumptions of Theorem 3.1.5 are satisfied, that is, $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open neighborhood O of a point $x_0 \in \mathbb{R}^n$, there are positive constants a and κ, such that $\mathbb{B}[x_0, 2a] \subset O$ and (3.9) and (3.10) hold. Then for each $x \in \mathbb{B}[x_0, a]$ the mapping $f(x_0) + \nabla f(x)(\cdot - x_0)$ is strongly metrically regular with the constant κ and neighborhoods $\mathbb{R}^n$ and $\mathbb{R}^n$. Let (z_k) be an arbitrary sequence in $\mathbb{B}[x_0, a]$. Set $\omega \equiv \delta := 1/(2\kappa)$, $y_0 := f(x_0)$, $A_k := \nabla f(z_k)$ for every $k = 0, 1, 2, \ldots$. Then the conditions (i), (ii), and (iii) in Theorem 3.4.2 hold. Consequently, Theorem 3.1.5 follows from Theorem 3.4.2.

3.5 Numerical experiments

Suppose that K is a nonempty subset of $\mathbb{R}^m$ and let $F(x) := K$ for each $x \in \mathbb{R}^n$. Then the generalized equation (3.1) reads as

$$f(x) + K \ni 0. \tag{3.19}$$

When $f : \mathbb{R}^n \to \mathbb{R}^m$ and $K := \mathbb{R}^m_+$ then the above inclusion corresponds to a system of m nonlinear (possibly non-smooth) inequalities: find $x \in \mathbb{R}^n$ such that

$$f_1(x) \le 0, \quad f_2(x) \le 0, \quad \ldots, \quad f_m(x) \le 0.$$

Kantorovich-type theorems for Newton method for solving (3.19) with K being a closed convex cone and f being smooth can be found in [3, Chapter 2.6] and [36]. The paper [34] deals with a generalized equation of the form

$$g(x) + h(x) + K \ni 0, \tag{3.20}$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ is a smooth function having a Lipschitz derivative on a neighborhood U of the initial point $x_0 \in \mathbb{R}^n$ and the function $h : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on U. The algorithm proposed therein reads as: given $x_k \in \mathbb{R}^n$ find x_{k+1} satisfying

$$g(x_k) + h(x_k) + \nabla g(x_k)(x_{k+1} - x_k) + K \ni 0. \tag{3.21}$$

Clearly, (3.21) corresponds to our iteration scheme with $f := g + h$ and $A_k := \nabla g(x_k)$, and, since A_k does not take into account the non-smooth part, it is expected to be slower in general (or not even applicable) as we will show on two toy examples below.

Consider a sequence (A_k) in $\mathbb{R}^{m\times n}$ and an initial point $x_0 \in \mathbb{R}^n$. Given $k \in \mathbb{N}_0$, $x_k \in \mathbb{R}^n$, and A_k, let

$$\Omega_k := \{u \in \mathbb{R}^n : \ g(x_k) + h(x_k) + A_k(u - x_k) + K \ni 0\}.$$

For the already computed x_k, the next iterate x_{k+1} can be found as a solution of the problem:

$$\text{minimize} \quad \varphi_k(x) \quad \text{subject to} \quad x \in \Omega_k,$$

where $\varphi_k : \mathbb{R}^n \to [0,\infty)$ is a suitably chosen function. In [34], the function $\varphi_k = \|\cdot - x_k\|$ is used. In the following examples we solve the linearized problem in MATLAB using either function *fmincon* for $\varphi_k = \|\cdot - x_k\|^2$ or *quadprog* for $\varphi_k(x) := \frac{1}{2}\langle x, x\rangle - \langle x_k, x\rangle$ for $x \in \mathbb{R}^n$. We will compare the following three versions of (3.2) for solving (3.20) with different choices of A_k at the step $k \in \mathbb{N}_0$ and the current iterate x_k:

(C1) $A_k := \nabla g(x_k)$;

(C2) $A_k := \nabla g(x_k) + j(x_k)$, where $j : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ is specified later;

(C3) $A_k := A_0 := \nabla g(x_0) + j(x_0)$.

Example 3.5.1 Consider the system from [34]:

$$\begin{aligned} x^2 + y^2 - |x - 0.5| - 1 &\le 0, \\ x^2 + (y-1)^2 - |x - 0.5| - 1 &\le 0, \\ (x-1)^2 + (y-1)^2 - 1 &= 0. \end{aligned}$$

Observe that the solutions are given by $y = 1 \pm \sqrt{2x - x^2}$ if $0 \le x \le (11 - 6\sqrt{3})/26$ and $y = 1 - \sqrt{2x - x^2}$ when $(11 - 6\sqrt{3})/26 \le x \le 1/2$, in particular, the points $(x_1^*, y_1^*) := (0.5, 1 - \sqrt{3}/2)$ and $(x_2^*, y_2^*) = (1 - \sqrt{2}/2, 1 - \sqrt{2}/2)$ solve the problem. Then setting $K := \mathbb{R}^2_+ \times \{0\}$ and $g(x,y) := (x^2 + y^2 - 1, x^2 + (y-1)^2 - 1, (x-1)^2 + (y-1)^2 - 1)$, $h(x,y) := (-|x - 0.5|, -|x - 0.5|, 0)$ for each $(x,y) \in \mathbb{R}^2$, we arrive at (3.20). Then

$$\nabla g(x,y) = \begin{pmatrix} 2x & 2y \\ 2x & 2(y-1) \\ 2(x-1) & 2(y-1) \end{pmatrix}, \quad \text{for each} \quad (x,y) \in \mathbb{R}^2.$$

Let the function $j : \mathbb{R}^2 \to \mathbb{R}^{3\times 2}$ appearing in (C2) and (C3) be for each $(x,y) \in \mathbb{R}^2$ defined by

$$j(x,y) := \begin{pmatrix} -\operatorname{sgn}(x - 0.5) & 0 \\ -\operatorname{sgn}(x - 0.5) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad \operatorname{sgn}(u) := \begin{cases} 1 & \text{if } u > 0, \\ -1 & \text{otherwise.} \end{cases}$$

Step k	fmincon			quadprog		
	(C1)	(C2)	(C3)	(C1)	(C2)	(C3)
0	5.0×10^{-2}	5.0×10^{-2}	5.0×10^{-2}	5.0×10^{-2}	5.0×10^{-2}	5.0×10^{-2}
1	2.4×10^{-2}	2.0×10^{-3}	2.0×10^{-3}	2.5×10^{-2}	2.0×10^{-3}	2.0×10^{-3}
2	1.2×10^{-2}	2.3×10^{-6}	2.3×10^{-6}	1.3×10^{-3}	2.3×10^{-6}	2.3×10^{-6}
4	3.1×10^{-3}	1.0×10^{-8}	1.0×10^{-8}	3.1×10^{-3}	6.5×10^{-9}	6.5×10^{-9}

Table 3.1: $\|(x_1^*, y_1^*) - (x_k, y_k)\|_\infty$ in Example 3.5.1 for $(x_0, y_0) = (0.55, 0.1)$.

Step k	fmincon			quadprog		
	(C1)	(C2)	(C3)	(C1)	(C2)	(C3)
0	2.9×10^{-1}	2.9×10^{-1}	2.9×10^{-1}	2.9×10^{-1}	2.9×10^{-1}	2.9×10^{-1}
1	4.2×10^{-2}	4.2×10^{-2}	4.2×10^{-2}	4.2×10^{-2}	4.2×10^{-2}	4.2×10^{-2}
2	1.2×10^{-3}	1.2×10^{-3}	1.2×10^{-3}	1.2×10^{-3}	1.2×10^{-3}	1.2×10^{-3}
4	1.1×10^{-10}	5.2×10^{-10}	5.2×10^{-10}	7.9×10^{-13}	7.9×10^{-13}	5.2×10^{-13}
7	1.1×10^{-10}	5.2×10^{-10}	5.2×10^{-10}	1.6×10^{-16}	1.1×10^{-16}	1.1×10^{-16}

Table 3.2: $\|(x_2^*, y_2^*) - (x_k, y_k)\|_\infty$ in Example 3.5.1 for $(x_0, y_0) = (0, 0)$.

For an error estimate we use the norm $\|z\|_\infty := \max\{|z_1|, |z_2|, \dots, |z_n|\}$ for $z \in \mathbb{R}^n$.

From Table 3.1 we see that the convergence of (3.2) with the choice (C1) and the initial point $(0.55, 0.1)$ is much slower than (3.2) with the choice (C3). Both *quadprog* and *fmincon* are of almost the same efficiency.

From Table 3.2 we see that for the initial point $(0, 0)$ all the choices (C1)–(C3) provide similar accuracy but we get substantially better results when *quadprog* is used to solve the linearized problem.

Example 3.5.2 Consider the system

$$x^2 + y^2 - 1 \le 0 \quad \text{and} \quad -|x| - |y| + \sqrt{2} \le 0$$

having four distinct solutions. Setting $K := \mathbb{R}^2_+$ and $g(x, y) := (x^2 + y^2 - 1, 0)$, $h(x, y) := (0, -|x| - |y| + \sqrt{2})$ for each $(x, y) \in \mathbb{R}^2$, we arrive at (3.20). Then

$$\nabla g(x, y) = \begin{pmatrix} 2x & 2y \\ 0 & 0 \end{pmatrix}, \quad \text{for every} \quad (x, y) \in \mathbb{R}^2.$$

Let the function $j : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ appearing in (C2) and (C3) be for each $(x, y) \in \mathbb{R}^2$ defined by

$$j(x, y) := \begin{pmatrix} 0 & 0 \\ -\mathrm{sgn}(x) & -\mathrm{sgn}(y) \end{pmatrix}.$$

For the initial point $(0, 0)$ the method (3.2) with (C1) fails. The convergence for the remaining two choices (C2) and (C3) can be found in Table 3.3. Note that using *quadprog* we find a solution (up to a machine epsilon) after one step and the iteration using *fmincon* gives the precision 10^{-9} at most.

For the initial point $(99, -999)$ the method (3.2) with (C1) and (C3) does not converge - see Table 3.4. The only convergent scheme is (3.2) with (C2) (note that we start far away from the solution).

Step k	fmincon		quadprog	
	(C2)	(C3)	(C2)	(C3)
0	7.0×10^{-1}	7.0×10^{-1}	7.0×10^{-1}	7.0×10^{-1}
1	2.5×10^{-9}	2.5×10^{-9}	0	0
2	7.5×10^{-8}	7.5×10^{-8}	0	0
4	1.2×10^{-8}	1.2×10^{-8}	0	0
7	8.5×10^{-8}	8.5×10^{-8}	0	0
10	8.5×10^{-9}	3.7×10^{-9}	0	0

Table 3.3: $\|(-\sqrt{2}/2, -\sqrt{2}/2) - (x_k, y_k)\|_\infty$ in Example 3.5.2 for $(x_0, y_0) = (0, 0)$.

Step k	fmincon			quadprog		
	(C1)	(C2)	(C3)	(C1)	(C2)	(C3)
0	9.9×10^{2}	9.9×10^{2}	9.9×10^{2}	9.9×10^{2}	9.9×10^{2}	9.9×10^{2}
1	4.9×10^{2}	4.9×10^{2}	4.9×10^{2}	–	4.9×10^{2}	4.9×10^{2}
4	6.1×10^{1}	6.1×10^{1}	6.1×10^{1}	–	6.1×10^{1}	6.1×10^{1}
10	5.0×10^{-1}	6.0×10^{-1}	6.0×10^{-1}	–	5.8×10^{-1}	8.3×10^{-1}
21	7.0×10^{-1}	3.0×10^{-4}	1.5×10^{-1}	–	2.8×10^{-4}	1.4×10^{0}
40	7.0×10^{-1}	5.3×10^{-9}	1.5×10^{-1}	–	1.0×10^{-8}	1.4×10^{0}

Table 3.4: $\|(-\sqrt{2}/2, \sqrt{2}/2) - (x_k, y_k)\|_\infty$ in Example 3.5.2 for $(x_0, y_0) = (99, -999)$.

Open problems

(i) Let (P, ρ) be a metric space, let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces, and $\Omega \subset X \times P$. Let a single-valued mapping $f : X \times P \to Y$ and a set-valued $F : X \rightrightarrows Y$ be given. Suppose that for each $(x, p) \in \Omega$ the mapping $f(\cdot, p) + F$ is regular around $(x, 0)$.

The question is: is it possible to find assumptions on f, F and Ω such that the regularity is uniform, that is, there are positive constants κ, a, and b such that for each $(x, p) \in \Omega$ we have

$$\operatorname{dist}(x', (f(\cdot, p) + F)^{-1}(y)) \le \kappa \operatorname{dist}(f(x', p) + F(x'), y) \quad \text{for every} \quad (x', y) \in \mathbb{B}[x, a] \times \mathbb{B}[y, b].$$

(ii) We can study a *differential generalized equation (DGE)*, a model introduced in [9], that is, a problem to find a pair of functions $x : [0, \varepsilon] \to \mathbb{R}^n$ and $u : [0, \varepsilon] \to \mathbb{R}^m$ such that

$$\begin{cases} \dot{x}(t) &=& g(x(t), u(t)), \\ 0 &\in& f(x(t), u(t)) + F(u(t)), \qquad \text{for all} \quad t \in [0, \varepsilon], \\ x(0) &=& x_I, \end{cases}$$

with a fixed $\varepsilon > 0$, single-valued functions $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^d$, a set-valued mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^d$, and a given initial state $x_I \in \mathbb{R}^n$.

In [9], a numerical method for solving DGE is introduced and is based on Euler method and Euler-Newton continuation method for tracking a solution trajectory. The authors derived that the error of the numerical solution is of order $O(h)$. It is an open question how to improve this result. One possibility is the following scheme, which is based on a combination of Runge–Kutta method, Euler method and Euler-Newton continuation method.

Given $N \in \mathbb{N}$ and (x_0, u_0) close to $(x_I, u(0))$, consider an iteration

$$\begin{cases} \widetilde{x}_{i+1} &=& x_i + h g(x_i, u_i), \\ e_i &\in& f(\widetilde{x}_{i+1}, u_i) + \nabla_u f(\widetilde{x}_{i+1}, u_i)(u_{i+1} - u_i) + F(u_{i+1}), \\ x_{i+1} &=& x_i + \dfrac{h}{2}(g(x_i, u_i) + g(\widetilde{x}_{i+1}, u_{i+1})), \end{cases}$$

where $i = 0, 1, 2, \ldots, N-1$, $h := \varepsilon/N$ is a discretization step, and (e_i) is a sequence of sufficiently small numbers representing errors.

(iii) It seems to be an open question, whether it is possible to prove criteria for semiregularity in the spirit of Theorem 2.2.1 , Theorem 2.2.2, Theorem 2.3.1, and Theorem 2.3.2. We can find some attempts in [14].

Resumé (CZ)

V této práci jsme si vzali za úkol shrnout základní teorii o regularitě zobrazení a metodách Newtonově typu pro řešení (zobecněných) rovnic.

Kapitola 1 je rozdělená do dvou podkapitol. První podkapitola obsahuje motivaci pro studium regularity zobrazení skrze řešitelnost rovnic/inkluzí při malé perturbaci pravé strany. Dále jsme ukázali, že nutnou podmínku pro (lokální) minimum/maximum lze odvodit negací postačujících podmínek regularity. V druhé podkapitole definujeme metrickou regularitu, metrickou subregularitu a metrickou semiregularitu. Také je zde uvedeno několik ekvivalentních vlastností.

Kapitola 2 je rozdělena do čtyř podkapitol. První kapitola obsahuje stručný historický vývoj kritérií metrické regularity, metrické subregularity a semiregularity. Další kapitoly obsahují kritéria každé z těchto vlastnosti.

Kapitola 3 je zaměřena na metody Newtonova typu a je rozdělena do pěti podkapitol. První podkapitola obsahuje stručný historický vývoj Newtonovy metody. Druhá podkapitola je zaměřena na věty o lokální konvergenci a třetí obsahuje věty typu Dennis-Moré. Ve čtvrté kapitole najdeme věty o semilokální konvergenci, které jsou zobecnění Bartleho věty. Všechny tyto výsledky jsou založeny na vlastnostech regularity zobrazení. V poslední podkapitole jsou metody Newtonova typu aplikovány na problém nehladkých nerovnic.

Resume (EN)

In this thesis we set ourselves the task to present regularity properties of mappings, basic results for them, and Newton-type methods for solving (generalized) equations.

The Chapter 1 is divided into two sections. In the first section, we motivated our considerations by a solvability of equations/inclusions under small perturbations of the right hand side. Moreover, we showed that necessary conditions for (local) minimum/maximum can be derived by negating sufficient conditions of regularity. In the second section, we defined metric regularity, metric subregularity, and metric semiregularity. Several equivalent properties were presented.

The Chapter 2 is divided into four sections. The first section contains a brief historical development of criteria of metric regularity, metric subregularity, and metric semiregularity. In the remaining sections, criteria for each property are given.

The Chapter 3 is focused on Newton-type methods and is divided into five sections. In the first section, we presented a brief historical development of the Newton method. The second section is focused on local convergence theorems and the third one contains Dennis-Moré theorems. The fourth section contains semilocal convergence theorems, which generalize Bartle theorem. All these results are based on various combinations of regularity properties. In the last section Newton-type methods are applied to non-smooth inequalities.

Bibliography

[1] M. Apetrii, M. Durea, and R. Strugariu. On subregularity properties of set-valued mappings. *Set-Valued Var. Anal.*, 21(1):93–126, 2013.

[2] F. J. Aragón Artacho and B. S. Mordukhovich. Enhanced metric regularity and Lipschitzian properties of variational systems. *J. Global Optim.*, 50(1):145–167, 2011.

[3] I. K. Argyros. *Convergence and applications of Newton-type iterations*. Springer, New York, 2008.

[4] J.-P. Aubin. Lipschitz behavior of solutions to convex minimization problems. *Math. Oper. Res.*, 9(1):87–111, 1984.

[5] J.-P. Aubin and H. Frankowska. *Set-valued analysis*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 1990 edition [MR1048347].

[6] R. G. Bartle. Newton's method in Banach spaces. *Proc. Amer. Math. Soc.*, 6:827–831, 1955.

[7] J. M. Borwein. Stability and regular points of inequality systems. *J. Optim. Theory Appl.*, 48(1):9–52, 1986.

[8] J. M. Borwein and D. M. Zhuang. Verifiable necessary and sufficient conditions for openness and regularity of set-valued and single-valued maps. *J. Math. Anal. Appl.*, 134(2):441–459, 1988.

[9] R. Cibulka, A. Dontchev, M. Krastanov, and V. Veliov. Metrically regular differential generalized equations. *To appear in SIAM Journal Of Optimization.*

[10] R. Cibulka, A. Dontchev, J. Preininger, T. Roubal, and V. Veliov. Kantorovich-type theorems for generalized equations. *J. Convex Anal.*, 25, (2018).

[11] R. Cibulka, A. L. Dontchev, and M. H. Geoffroy. Inexact Newton methods and Dennis-Moré theorems for nonsmooth generalized equations. *SIAM J. Control Optim.*, 53(2):1003–1019, 2015.

[12] R. Cibulka, A. L. Dontchev, and A. Y. Kruger. Strong metric subregularity of mappings in variational analysis and optimization. *J. Math. Anal. Appl.*, 457(2):1247–1282, 2018.

[13] R. Cibulka and M. Fabian. On primal regularity estimates for set-valued mappings. *J. Math. Anal. Appl.*, 438(1):444–464, 2016.

[14] R. Cibulka, M. Fabian, and A. Y. Kruger. On semiregularity of mappings. *Submitted to Journal of Mathematical Analysis and Applications.*

[15] J. E. Dennis, Jr. and J. J. Moré. A characterization of superlinear convergence and its application to quasi-Newton methods. *Math. Comp.*, 28:549–560, 1974.

[16] A. Dmitruk, A. Milyutin, and N. P. Osmolovskii. Lyusternik's theorem and the theory of extrema. *Russian Mathematical Surveys*, 35(6):11, 1980.

[17] A. L. Dontchev. The Graves theorem revisited. *J. Convex Anal.*, 3(1):45–53, 1996.

[18] A. L. Dontchev and R. T. Rockafellar. *Implicit functions and solution mappings*. Springer Series in Operations Research and Financial Engineering. Springer, New York, second edition, 2014. A view from variational analysis.

[19] M. Durea and R. Strugariu. Openness stability and implicit multifunction theorems: applications to variational systems. *Nonlinear Anal.*, 75(3):1246–1259, 2012.

[20] M. Fabian and D. Preiss. A generalization of the interior mapping theorem of Clarke and Pourciau. *Comment. Math. Univ. Carolin.*, 28(2):311–324, 1987.

[21] L. M. Graves. Some mapping theorems. *Duke Math. J.*, 17:111–114, 1950.

[22] A. D. Ioffe. Necessary conditions in nonsmooth optimization. *Math. Oper. Res.*, 9(2):159–189, 1984.

[23] A. D. Ioffe. A Lagrange multiplier rule with small convex-valued subdifferentials for nonsmooth problems of mathematical programming involving equality and nonfunctional constraints. *Math. Programming*, 58(1, Ser. A):137–145, 1993.

[24] A. D. Ioffe. Metric regularity and subdifferential calculus. *Uspekhi Mat. Nauk*, 55(3(333)):103–162, 2000.

[25] A. D. Ioffe. Metric regularity—a survey Part 1. Theory. *J. Aust. Math. Soc.*, 101(2):188–243, 2016.

[26] A. D. Ioffe. Implicit functions: a metric theory. *Set-Valued and Variational Analysis*, Jun 2017.

[27] L. V. Kantorovich. On Newton's method for functional equations. *Doklady Akad. Nauk SSSR (N.S.)*, 59:1237–1240, 1948.

[28] L. V. Kantorovich and G. P. Akilov. *Functional analysis*. Pergamon Press, Oxford-Elmsford, N.Y., second edition, 1982. Translated from the Russian by Howard L. Silcock.

[29] C. T. Kelley. *Iterative methods for linear and nonlinear equations*, volume 16 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995. With separately available software.

[30] A. Y. Kruger. About stationarity and regularity in variational analysis. *Taiwanese J. Math.*, 13(6A):1737–1785, 2009.

[31] B. S. Mordukhovich. *Variational analysis and generalized differentiation. I*, volume 330 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006. Basic theory.

[32] H. V. Ngai, N. H. Tron, and M. Théra. Metric regularity of the sum of multifunctions and applications. *J. Optim. Theory Appl.*, 160(2):355–390, 2014.

[33] J.-P. Penot. Metric regularity, openness and Lipschitzian behavior of multifunctions. *Nonlinear Anal.*, 13(6):629–643, 1989.

[34] A. Pietrus. Non differentiable perturbed Newton's method for functions with values in a cone. *Investigación Oper.*, 35(1):58–67, 2014.

[35] L. Q. Qi and J. Sun. A nonsmooth version of Newton's method. *Math. Programming*, 58(3, Ser. A):353–367, 1993.

[36] S. M. Robinson. Extension of Newton's method to nonlinear functions with values in a cone. *Numer. Math.*, 19:341–347, 1972.

[37] S. M. Robinson. Regularity and stability for convex multivalued functions. *Math. Oper. Res.*, 1(2):130–143, 1976.

[38] S. M. Robinson. Strongly regular generalized equations. *Math. Oper. Res.*, 5(1):43–62, 1980.

[39] L. Thibault. Various forms of metric regularity. *unpublished note, Univ. de Montpellier*, 1999.

[40] L. U. Uko. Generalized equations and the generalized Newton method. *Math. Programming*, 73(3, Ser. A):251–268, 1996.

[41] C. Ursescu. Multifunctions with convex closed graph. *Czechoslovak Math. J.*, 25(100)(3):438–441, 1975.

[42] T. J. Ypma. Historical development of the Newton-Raphson method. *SIAM Rev.*, 37(4):531–551, 1995.