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Jednoznačnost a existence řešení parabolické PDR s p-Laplaceovým operátorem a zobecněné trigonometrické a hyperbolické funkce

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On Uniqueness and Existence of Solution of Parabolic PDE Involving p-Laplacian and Generalized Trigonometric and Hyperbolic Functions

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Declaration

I hereby declare that this thesis is my own work, unless clearly stated otherwise[1].

Plzeň, ...

Lukáš Kotrla

[1] Most of the results have been obtained jointly with co-authors. For that reason, each result contains reference to the list of references at the end of the thesis. Moreover, the name of all co-authors together with matching results are mentioned in Section 1.3 (Organization of thesis).

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Abstrakt

Náplní disertační práce je studium kvazilineárních parabolických a eliptických úloh s p-Laplaciánem popisujících difuzní proces. Text je rozdělen do dvou částí. V první části se zabýváme Cauchyovou úlohou pro parabolický p-Laplacián. Zaměříme se na otázku jednoznačnosti/nejednoznačnosti řešení a platnosti principu maxima. V druhé části zkoumáme eliptickou okrajovou úlohu v jedné dimenzi. Detailně se zajímáme o p-trigonometrické funkce, které se využívají v teoretickém i numerickém studiu parabolických i eliptických úloh s p-Laplaciánem.

Klíčová slova p-Laplacián, kvasilineární, parabolické PDR, reakčně difuzní rovnice, existence, jednoznačnost, kompaktní nosič, silný princip maxima, dolní řešení, horní řešení, p-trigonometrické funkce, p-hyperbolické funkce, aproximace, analytické funkce, diferenciální rovnice v komplexním oboru, rozšíření do komplexního oboru

Abstract

The Thesis is devoted to the study of quasilinear parabolic and elliptic problems with diffusion driven by the p-Laplacian. The Thesis is divided into two parts. The first part concerns uniqueness/nonuniqueness and validity/nonvalidity of the strong maximum principle of the solution of the Cauchy problem for the parabolic p-Laplacian. The second part concerns elliptic boundary value problems in one dimension. In particular, we provide detailed study of p-trigonometric functions which are usefull in theoretical and numerical treatment of parabolic and elliptic problems with the p-Laplacian.

Keywords p-Laplacian, Quasilinear, parabolic PDE, reaction-diffusion equation, existence, uniqueness, compact support, strong maximum principle, subsolution, supersolution, p-trigonometric functions, p-hyperbolic functions, approximation, analytic functions, differential equation in complex domain, extension to complex domain

Zusammenfassung

Die Dissertation widmet sich der Untersuchung von quasilinearen parabolischen und elliptischen Problemen mit Diffusion durch den p-Laplace-Operator. Die Arbeit gliedert sich in zwei Teile. Der erste Teil betrifft die Eindeutigkeit/Uneindeutigkeit und Gültigkeit/Ungültigkeit des starken Maximumprinzips der Lösung des Anfangswertproblems für parabolische Probleme. Der zweite Teil betrifft elliptische Randwertprobleme in einer Dimension. Insbesondere bieten wir detaillierte Studien von p-trigonometrischen Funktionen, die in der theoretischen und numerischen Behandlung parabolischer und elliptischer Probleme mit dem p-Laplace-Operator nützlich sind.

Schlüsselwörter p-Laplacian, quasilinearen, parabolische PDG, Reaktionsdiffusionsgleichung, Existenz, Eindeutigkeit, kompaktem Träger, starke Maximumprinzip, Unterlösung, Oberlösung, p-trigonometrischen Funktionen, p-Hyperbelfunktionen, Approximation, Analytische Funktionen, Differentialgleichung im Komplexen, Analytische Fortsetzung

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CHAPTER I

Introduction

The nonlinear operator $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ with $p > 1$ and the related problems have attracted a lot of attention over the last decades. The operator Δ_p is a generalization of the classical Laplacian ($\Delta_2 \equiv \Delta$) and hence it is usually called the p-Laplacian. Let us first consider general problem

$$\left\{\begin{array}{rcll} \frac{\partial}{\partial t} b(u) - \Delta_p u & = & h & \text{in } (0,T) \times \Omega\,, \\ u & = & 0 & \text{on } (0,T) \times \partial\Omega\,, \\ u(0,x) & = & u_0(x) & \text{in } \Omega\,. \end{array}\right. \tag{1.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$, $N \in \mathbb{N}$, with $C^{1+\mu}$ (Hölder) boundary $\partial\Omega$ (an interval for $N = 1$), $\mu \in (0,1)$, $T > 0$, $b : \mathbb{R}_+ \to \mathbb{R}_+$, $b \in C^1(\mathbb{R}_+)$, and $b'(s) > 0$ for all $s > 0$. The functions h and u_0 as well as the definition of solution of (1.1) will be specified later in the special cases.

In the context of Physics, problem (1.1) can be interpreted, among others, as a model of fluid flow through porous media in turbulent regime (see Leibenson [39] or Diaz and De Thelin [23]). This phenomenon arises in many different fields of human activity, e.g. food industry (sugar processing, Missbach [46–49]), civil engineering (building of rockfill dams, Wilkins [56]), and/or extraction of natural resources (water Smreker [55]; oil and natural gas, Leibenson [39]). In other context, problem (1.1) appears also in mathematical models of sandpile growth (Aronsson et al. [5] and Evans et al. [26]), image analysis (Kuijper [35]), and climatology (Bermejo et al. [12]).

In Section 1.1.1, we derive (1.1) from the continuity equation for compressible fluid (gas) using experimentally verified nonlinear generalization of Darcy constitutive law. Note that we follow approach of Leibenson [39] who modelled motion of natural gas in a porous medium. To the best of our knowledge, it is work [39] where the derivation of an equation of the type (1.1) appears for the first time in the history in all three dimensions.

The rest of the thesis is devoted to the existence and the uniqueness results for (1.1). Special attention is paid to generalized sine function denoted by $\sin_p$, which is the principal eigenfunction of the p-Laplacian in one dimension. More precisely, $\sin_p$ is absolutely continuous function such that $|\varphi'|^{p-2}\varphi'$ is also absolutely continuous and it satisfies following nonlinear eigenvalue problem

$$\begin{cases} -\left(|\varphi'|^{p-2}\varphi'\right)' &= (p-1)\,|\varphi|^{p-2}\,\varphi\,, \quad x \in (0,\pi_p) \text{ a.e.} \\ \varphi(0) = \varphi(\pi_p) &= 0\,, \end{cases} \tag{1.2}$$

and it is normalized as $\varphi'(0) = 1$. Here

$$\pi_p \stackrel{\text{def}}{=} \frac{2\pi}{p\sin\left(\pi/p\right)}\,. \tag{1.3}$$

1.1. Mathematical Models Involving p-Laplacian

1.1.1. Fluid Flow through Porous Medium

The aim of the Section 1.1.1 is to derive a problem of type (1.1). Leibenson [39, pp. 503 – 505] studied movement of natural gas in a porous medium in 1945. We will follow his work and hence we will assume that the porous medium is nondeformable, isotropic, and homogeneous at macroscopic scale and the gas is a homogeneous mixture. The conditions on the porous medium cause that the porosity n is constant and the condition on the gas ensures that its density depends on pressure only. We also suppose that the examined thermodynamic process is *polytropic*, i.e. it obeys the relation:

$$\frac{P}{\varrho^\gamma} = \beta^\gamma\,. \tag{1.4}$$

Here, $\varrho = \varrho(t,x)$ is the density, $P = P(t,x)$ is the pressure, $\gamma > 1$ is the *polytropic index* of the process, and $\beta > 0$ is a constant.

The flow of the gas (as of any fluid) in the porous medium is governed by continuity equation in the form

$$n\frac{\partial\varrho}{\partial t} + \operatorname{div}\left(\varrho q\right) = 0 \tag{1.5}$$

and an appropriate constitutive law which relates specific discharge vector $q = nv$ and pressure gradient ∇P. Specific discharge vector is volumetric flux per unit area and the term ϱq represents mass flux per unit area. Vector field $v(t,x)\colon (0,T) \times \Omega \to \mathbb{R}^3$

describes velocity distribution. We refer to BEAR [6, Section 6.2] for derivation of (1.5) for homogeneous mixture.

For real world case $N = 3$, the continuity equation (scalar equation) contains four variables and hence it is necessary to add other three equations. These equations provide relation between three components of the gradient of the state (scalar) variable ϱ and of the flux (vector) variable q. In most of real world problems, the constitutive law has to be obtained experimentally. Initial experimental work was done by DARCY [20] who studied filtration of water through pipe filled with sand (as one dimensional problem). He observed that

$$-P' = \frac{H}{L} = \text{const.}\, v\,. \tag{1.6}$$

Here, H/L is pressure slope (loss) and v is velocity. The constant depends on the physical properties of the porous medium and the fluid within (and does not depend on the velocity). This constant is obtained experimentally. To the best of our knowledge (see BENEDIKT et al. [11]), SMREKER [55] was the first one who questioned validity of *Darcy's law* (1.6) for large velocities. Based on observations on real water wells, he proposed the following constitutive law for turbulent flow of water in a porous medium

$$-P' = \frac{H}{L} = \text{const.}\, v^m\,, \quad v > 0\,, \tag{1.7}$$

with $1 < m < 2$. Constitutive law (1.7) was verified experimentally, e.g. by MISSBACH [49]. In Missbach's experiments, the porous medium was simulated by large pipe filling with tiny uniform glass balls. The fluid was pure water free of air bubbles (water was heated-up and subsequently cooled-down) and it flew through the porous medium under constant pressure until the volume of passed water was 1000 cm^3. Time was measured by stopwatch. Missbach studied how the fluid's velocity depends on the size of the glass balls, the height of the layer of glass balls, and, in particular, on the pressure slope. He confirmed the validity of (1.7) and the linear dependence of the velocity on the height of the layer of glass balls. He also found out that the exponent m in (1.7) decreased as the diameter of the glass balls decreased.

Similar power law,

$$\varrho v = -C \left|\frac{\partial P}{\partial x}\right|^{s-1} \frac{\partial P}{\partial x}\,,$$

$1/2 < s < 1$, holds also for compressible fluid. In isotropic homogeneous 3D porous medium, the constitutive law has the following form:

$$\varrho v = -C|\nabla P_1|^{s-1}\nabla P_1\,, \tag{1.8}$$

where $P_1 = P^{(\gamma+1)/\gamma}$ (see Leibenson [39]). Plugging (1.8) into (1.5), we obtain

$$n\frac{\partial}{\partial t}\left(\frac{P_1^{\frac{1}{\gamma+1}}}{\beta}\right) - C\,\mathrm{div}\left(|\nabla P_1|^{s-1}\nabla P_1\right) = 0$$

by (1.4). Setting $s = p-1$ we get (1.1) with $h \equiv 0$ and $b(u) = \dfrac{n}{C\beta}u^{\frac{1}{\gamma+1}}$.

1.1.2. Nonlinear Reaction-Diffusion

We interpret (1.1) as the problem of fluid flow through porous media only in [11], where we study the origin of a problem of type (1.1). In the rest of our work, problem (1.1) is a model of nonlinear (slow or fast) diffusion where u is concentration. The diffusion process is governed by continuity equation

$$\frac{\partial u}{\partial t} + \mathrm{div}\, j = h\,, \tag{1.9}$$

where $j = j(t,x,u,\nabla u)\colon (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is the flux of diffusing material and $h = h(t,x,u,\nabla u)\colon (0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$. The constitutive relation for diffusion processes (Fick's law) states

$$j = -D\nabla u\,, \tag{1.10}$$

where $D = D(t,x)$ is the diffusion coefficient which depends on the diffusing material (see Drábek and Holubová [24] for more details). In some circumstances, the diffusion coefficient depends also on u and/or ∇u, see [5] and/or [26]. We suppose that $D(t,x,u,\nabla u) \overset{\mathrm{def}}{=} |\nabla u|^{p-2}$. Combining (1.9) and (1.10), we get problem (1.1) for $b(u) \equiv u$. If $1 < p < 2$, the diffusion coefficient is high for small $|\nabla u|$ and, hence, diffusion is fast in this case. On the contrary, the diffusion coefficient is low for small $|\nabla u|$ and hence diffusion is slow, see Figure 1.1. We will address this problem in Section 2.2 in detail. Let us note that we considered stationary problem

$$\begin{cases} -\Delta_p u &= h\,(x,u,\nabla u) & \text{in } \Omega\,, \\ u &= 0 & \text{on } \partial\Omega \end{cases}$$

in master thesis Kotrla [33].

1.2. From Parabolic to Eigenvalue Problem

Problem (1.1) has attracted a lot of attention of mathematicians for many decades. Let us mention the classical work of Lions [41] where (1.1) is introduced as a suitable

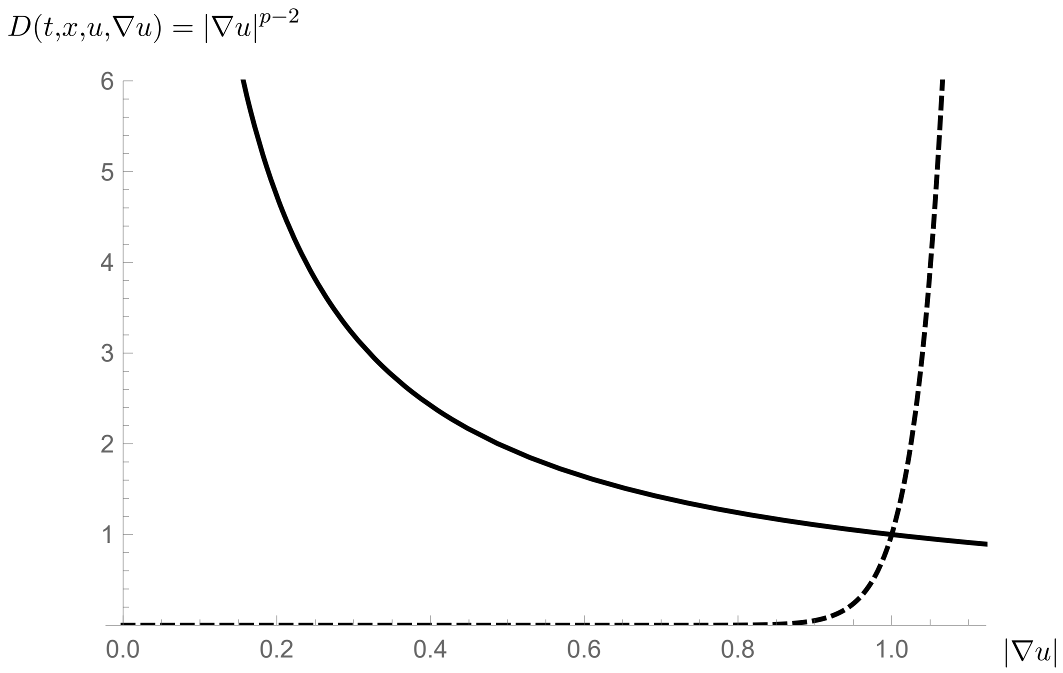

Figure 1.1: Diffusion coefficient $D(t,x,u,\nabla u) = |\nabla u|^{p-2}$ restricted to the plain $|\nabla u| \times D$ for $p = 30$ (dashed line) and $p = \frac{30}{29}$ (continuous line).

representative of a wider class of quasilinear parabolic problems. At first, assume that $b(u) \equiv u$, $h(t,x,u) \equiv h(t,x)$, and $u_0 \in L^2(\Omega)$. Then, problem (1.1) possesses the unique solution $u \in L^p\left((0,T) \to W_0^{1,p}(\Omega)\right)$ (or $u \in L^p((0,T) \to V)$, $V \stackrel{\text{def}}{=} W_0^{1,p}(\Omega) \cap L^2(\Omega)$ for $1 < p < 2$) under the conditions $h \in L^{p'}\left((0,T) \to W^{-1,p'}(\Omega)\right)$ (or $h \in L^{p'}((0,T) \to V')$ for $1 < p < 2$). We refer the reader to [41], Théorème 1.1, p. 156, Théorème 1.2, p. 162 and Section 1.5.2, p. 166. Bochner spaces $L^p((0,T) \to X)$ will be defined in Section 2.1. Padial et al. [51] widely discuss the question of the existence and the uniqueness of weak solution of problem (1.1) with $b(u) \equiv u$ and $h(t,x,u) = \lambda|u|^{p-2}u + f(x,t)$ in [51, Appendix A]. It follows from the validity of weak comparison principle (see [57, Proposition 2.3.1, p. 190] for appropriate version) that the solution is unique when the right $h(t,x,u)$ is Lipschitz function in u.

Later the sufficient conditions for the existence and the uniqueness of (the certain type of) a solution of (1.1) is studied by, e.g Ladyzhenskaya at al. [36], Alt and Luckhaus [2], and Diaz and De Thelin [23]. On the other hand, nonuniqueness results are obtained, for instance employing reaction term $h = h(u)$ which is not a Lipschitz function near $u = 0$, in the following works: Guedda [30] in one-dimensional case, Bobkov and Takáč [14], and Merchán et al. [44].

In Section 2.1, we provide a nonuniqueness result for the following special case of (1.1):

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p u = q(x)u^\alpha & \text{in } (0,T)\times\Omega\,, \\ u = 0 & \text{on } (0,T)\times\partial\Omega\,, \\ u(0,x) = 0 & \text{in } \Omega\,, \end{cases} \tag{1.11}$$

where $\alpha \in (0,1)$ and $q \in C(\overline{\Omega})$, $q \geq 0$, $q \not\equiv 0$. It is easy to see that problem (1.11) has at least the trivial solution. We take advantage of method of monotone iterations to show that problem (1.11) possesses (under some restriction on α) a nontrivial, nonnegative, weak solution

$$u \in C\left([0,T] \to L^2(\Omega)\right) \cap L^p\left((0,T) \to W^{1,p}(\Omega)\right)$$

in both, singular case ($1 < p < 2$) and degenerate case ($2 < p < +\infty$). Moreover, we are able to construct a nontrivial solution with compact support in the degenerate case. In particular, problem (1.11) with $p > 2$ exhibits the finite speed of propagation.

By our assumption on q, there exists $x_0 \in \Omega$ such that $q(x_0) > 0$. Moreover, q is continuous and hence we are able to choose $R > 0$ such that $q \geq q_0 \equiv \text{const} > 0$ on $B_R(x_0) \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^N : |x - x_0| < R\right\}$ and $B_R(x_0) \subset \Omega$. It appears that $\underline{u} = \theta(t)\widetilde{\varphi}_1^\beta(x)$ with some $\beta > 1$ is a suitable subsolution for problem (1.11) (see Benedikt et al. [7]). Function $\theta(t)$ is the unique solution of

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\theta(t) = \frac{q_0}{2}\theta^\alpha(t) & \text{in } (0,T)\,, \\ \theta(0) = 0\,, \\ \theta(t) > 0\,, & \text{for } t > 0 \end{cases}$$

and

$$\widetilde{\varphi}_1(x) = \begin{cases} \varphi_1(x) & \text{for } x \in B_R(x_0)\,, \\ 0 & \text{for } x \in \overline{\Omega} \setminus B_R(x_0)\,. \end{cases}$$

Here, $\varphi_1 \in W_0^{1,p}(B_R(x_0))$ is the first eigenfunction of the following eigenvalue problem:

$$\begin{cases} -\Delta_p \varphi = \lambda\, |\varphi|^{p-2}\, \varphi & \text{in } B_R(x_0)\,, \\ \varphi = 0 & \text{on } \partial B_R(x_0)\,. \end{cases} \tag{1.12}$$

Note that φ_1 is normalized by $\varphi_1(x_0) = 1$. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of (1.12) if there exists a nontrivial solution of (1.12), which is called an eigenfunction.

The first eigenfunction φ_1 is associated with the least eigenvalue λ_1. The structure of the spectrum of (1.12) is still a challenging open problem unless $N = 1$. ANANE [3] proved that the first eigevalue λ_1 is simple and isolated for the general domain Ω, and the corresponding eigenfunction φ_1 is positive on Ω.

We use method of monotone iterations in BENEDIKT at el. [7] and [9] as was mentioned before. More precisely, we use iteration scheme

$$\begin{cases} \frac{\partial u_n}{\partial t} - \Delta_p u_n = q(x)u_{n-1}^{\alpha} & \text{in } (0,T) \times \Omega\,, \\ u_n = 0 & \text{on } (0,T) \times \partial\Omega\,, \\ u_n(0,x) = 0 & \text{in } \Omega \end{cases} \tag{1.13}$$

for $n \in \mathbb{N}$ with $u_0 = \underline{u} = \theta(t)\widetilde{\varphi}_1^{\beta}(x)$. The method is constructive and, hence, we may use (1.13) also to find a numerical approximation of solution of (1.11). In one space dimension, BOULTON and LORD [15] employ Galerkin method to solve

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u}{\partial x}\right|^{p-2}\dfrac{\partial u}{\partial x}\right) = g\,, \\ u(0,x) = 0\,,\ x \in (0,1)\,, \\ u(t,0) = u(t,1) = 0\,, \qquad t > 0\,, \end{cases} \tag{1.14}$$

where $g \in L^2(0,1)$. Besides the usual *sine* basis, they also consider the basis $\{\sin_p(k\pi_p\, x)\}_{k=1}^{+\infty}$ in their experiments. BINDING et al. [13] established the existence of $p_0 > 1$ such that, for $p > p_0$, the system of functions $\{\sin_p(k\pi_p\, x)\}_{k=1}^{+\infty}$ forms a Riesz basis of $L^2(0,1)$ and a Schauder basis of $L^r(0,1)$ for any $1 < r < +\infty$. The procedure how to find an appropriate number p_0 was corrected and improved by BUSHELL and EDMUNDS [18], where the value p_0 was established as the solution of the transcendent equation

$$\frac{2\pi}{p_0 \sin(\pi/p_0)} = \frac{2\pi^2}{\pi^2 - 8}\,.$$

The results obtained for $p = 10$ in [15] are visualized for two special choices of g in [15, Figure 9, p. 2708]. It is assumed there that

$$g(x) = \begin{cases} 1\,, & x \in \left(\frac{1}{4},\frac{3}{4}\right)\,, \\ 0\,, & \text{otherwise} \end{cases}$$

and/or $g(x) \equiv 1$ for $x \in (0,1)$. In both cases, the optimal basis for Galerkin method is not the usual *sine* basis, but there exists p_1 such that the application of the basis $\{\sin_{p_1}(k\pi_{p_1}\, x)\}_{k=1}^{+\infty}$ provides the smallest spatial error under the assumption that the solution approaches steady state (t is sufficiently large). Nevertheless, we are interested in a solution for small t, where the computation advantage of the basis $\{\sin_{p_1}(k\pi_{p_1} x)\}_{k=1}^{+\infty}$ over the classical *sine* basis is not clear. Another disadvantage of the basis $\{\sin_{p_1}(k\pi_{p_1} x)\}_{k=1}^{+\infty}$ is a computational overhead in obtaining the basis. In [15], the inverse function of $\sin_p$,

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}} \, \mathrm{d}s \,, \qquad x \in [0,1] \,,$$

is used to obtain $\sin_p$. Then, the function $\sin_p$ on $[0,\pi_p/2]$ is evaluated using numerical inverse of the function arcsin_p which is a time consuming process. Hence, it is important to find a new more efficient numerical implementation of $\sin_p$. We address the task in Chapter 3, where we extend $\sin_p$ into Maclaurin series convergent on $(-\pi_p/2,\pi_p/2)$ under the assumption p is an even integer.

1.3. Organization of Thesis

Chapter 2 is based on the joint articles of the author with Jiří Benedikt, Petr Girg, and Peter Takáč [9], [10] and also the joint article by above stated authors with Vladimir E. Bobkov [7]. We study a quasilinear parabolic problem (1.1) involving the p-Laplacian on space-time domain $(0,T) \times \Omega$ with Ω bounded in $\mathbb{R}^N$. In the introduction of Chapter 2, we formulate two model cases which exhibit strikingly different behaviour. In Section 2.1, the existence of a nontrivial, nonnegative, weak solution is obtained for a reaction function $h(t,x,u) = q(x)u^\alpha$ which does not satisfy a local Lipschitz condition (problem (1.11)). Section 2.2 is motivated by very classical property of the linear heat equation (namely, the infinite speed of propagation). Let $u(t,x) \in C^\infty([0,T) \times \overline{\Omega})$ is a positive solution of heat equation ($p = 2$) with an appropriate initial and boundary conditions. Then, the solution admits an *infinite speed of propagation*, i.e. $u(t,x) \equiv 0$ on $[0,T) \times \overline{\Omega}$ or there exists $\tau \in (0,T)$ such that $u(t,x) > 0$ for any $(t,x) \in (0,\tau) \times \overline{\Omega}$. We will study such phenomenon for $p \neq 2$. At first, we show that there exists a solution of (1.11) with compact support provided $p > 2$. Hence, the solution possesses a *finite speed of propagation* (see Definition 2.8). Conversely, suppose that $1 < p < 2$ and there exists a continuous, nonnegative, weak solution of parabolic problem (1.1) with $h(t,x,u) \equiv f(t,x)$. Then, the solution possesses an *infinite speed of propagation* (see Definition 2.12).

Chapter 3 is based on the results on generalized *sine* (see (1.2)), which appears in article by the author [34] and in the joint work with Petr Girg [27], [28], and [29]. Beside the Maclaurin series of $\sin_p$, it is also devoted to some other properties of p-trigonometric and p-hyperbolic functions in real and complex domain. In the introduction of Chapter 3, we discuss the property of solution of prototypical initial value problem which is used to define generalized *sine* and *hyperbolic sine* functions. Moreover, we explain how generalized *sine* functions can be used in numerical methods to treat certain parabolic problems of type (1.1) as well as certain boundary value problems in one dimension steady states of (1.1) for $N = 1$). In Section 3.1, the main emphasis is laid on differentiability of generalized *sine* since $\sin_p$ possesses different order of differentiability from *sine* in general. The most interesting result is that $\sin_p(\cdot) \in C^\infty(-\pi_p/2, \pi_p/2)$ for $p > 1$ even. In Section 3.2, we obtain desired Maclaurin series and generalized Maclaurin series of $\sin_p$ for $p > 1$ be an even integer and an odd integer, respectively. Then, local convergence of Maclaurin series of $\sin_p$ around $x = 0$ follows from Paredes and Uchiyama [52]. We prove that the (generalized) Maclaurin series converge toward $\sin_p$ on $(-\pi_p/2, \pi_p/2)$ for $p > 1$ be an integer. Finally, we use Maclaurin series to extend $\sin_p$ to complex domain in Section 3.3. We also study generalization of *hyperbolic sine* and its properties in complex domain. In particular, we suppose p be an even integer and provide a generalization of the well-known identity

$$\sin(z) = -\mathrm{i}\sinh(\mathrm{i}z)\,.$$

CHAPTER II

Quasilinear Parabolic Problem

Let us recall two special cases of problem (1.1) which are considered in Chapter 2. In Section 2.1, we are interested in the existence nontrivial, nonnegative, weak solution (see Definition 2.5) of problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta_p\, u &= q(x)u^\alpha & \text{in } (0,T)\times\Omega\,, \\ u &= 0 & \text{on } (0,T)\times\partial\Omega\,, \\ u(0,x) &= 0 & \text{in } \Omega\,. \end{cases} \tag{2.1}$$

Here $\alpha \in (0,1)$ and potential q satisfies

(Q) $q \in C(\overline{\Omega})$, $q \geq 0$, and $q(x_0) > 0$ for some $x_0 \in \Omega$.

In Section 2.2, we show that (2.1) possesses a (possibly different) solution with compact support in $(0,T)\times\Omega$ provided $p > 2$. Later, we consider problem

$$\begin{cases} \frac{\partial}{\partial t} b(u) - \Delta_p\, u &= f(t,x) & \text{in } (0,T)\times\Omega\,, \\ u &= 0 & \text{on } (0,T)\times\partial\Omega\,, \\ u(0,x) &= u_0(x) & \text{in } \Omega\,, \end{cases} \tag{2.2}$$

where $f = f(t,x)\colon (0,T)\times\Omega \to \mathbb{R}$ and $u_0\colon \Omega \to \mathbb{R}$ are continuous and nonnegative. We assume that there exists a continuous solution of (2.2) and prove that it exhibits *infinite speed of propagation* (see Definition 2.12).

2.1. Nonuniqueness Results

We recall a concept of weak (or generalized) solution at first. Let X be Banach space equipped with the norm $\|\cdot\|_X$. In PDE theory, we search for a solution $u\colon [0,T]\times$

$\Omega \to \mathbb{R}$. Hence, X will be a function space (Lebesgue or Sobolev space). A function $v(t)\colon [0,T] \to X$ assigns to any $t \in [0,T]$ a function $w(x)$ from appropriate Lebesgue or Sobolev space. We need to introduce a Bochner integral which generalized Lebesgue integral for functions with values in Banach space (see, e.g. Zeidler [58, Appendix, p. 1009])

Definition 2.1. A function $v_s\colon [0,T] \to X$ is called a step function if there exists $m \in \mathbb{N}$, $v_i \in X$ and (Lebesgue) measurable disjoint sets $M_i \subset [0,T]$, $|M_i| < +\infty$, $1 \le i \le m$ such that

$$v_s(t) = \sum_{i=1}^{m} \chi_{M_i}(t) v_i \,.$$

Function $v\colon [0,T] \to X$ is Bochner measurable if there exists a sequence $\{v_n\}_{n=1}^{+\infty}$ of step functions such that $\lim_{n\to+\infty} v_n(t) = v(t)$ for almost all $t \in [0,T]$.

Any almost everywhere continuous function v is Bochner measurable provided X is separable. Now we are able to define a Bochner integral.

Definition 2.2. Let $v_s\colon [0,T] \to X$ be step function. Then

$$\int_0^T v_s(t)\,\mathrm{d}t = \sum_{i=1}^{m} |M_i| v_i \,.$$

A Bochner measurable function $v\colon [0,T] \to X$ is Bochner integrable if there exists a sequence $\{v_n\}_{n=1}^{+\infty}$ of step functions such that

$$\lim_{n\to+\infty} \int_0^T \|v - v_n\|_X \,\mathrm{d}t = 0 \,.$$

Let v be Bochner integrable function and $\{v_n\}_{n=1}^{+\infty}$ be corresponding sequence of step functions. Then

$$\int_0^T v(t)\,\mathrm{d}t \overset{\mathrm{def}}{=} \lim_{n\to+\infty} \int_0^T v_n(t)\,\mathrm{d}t \,.$$

Finally, we define function spaces $L^p\left((0,T) \to W^{1,p}(\Omega)\right)$ and $L^p\left((0,T) \to W^{-1,p'}(\Omega)\right)$.

Definition 2.3. Bochner space $L^p\left((0,T) \to W^{1,p}(\Omega)\right)$ contains all Bochner measurable functions $v(t) : [0,T] \to W^{1,p}(\Omega)$ such that

$$\|v\|_{L^p((0,T)\to W^{1,p}(\Omega))} \overset{\mathrm{def}}{=} \left(\int_0^T \|v(t)\|^p_{W^{1,p}(\Omega)} \,\mathrm{d}t \right)^{\frac{1}{p}} < +\infty \,.$$

Analogously, Bochner space $L^p\left((0,T) \to W^{-1,p'}(\Omega)\right)$ contains all Bochner measurable functions $v(t) : [0,T] \to W^{-1,p'}(\Omega)$ such that

$$\|v\|_{L^p\left((0,T)\to W^{-1,p'}(\Omega)\right)} \overset{\mathrm{def}}{=} \left(\int_0^T \|v(t)\|^p_{W^{-1,p'}(\Omega)} \,\mathrm{d}t \right)^{\frac{1}{p}} < +\infty \,.$$

Further, we also need to define space $C\left([0,T] \to L^2(\Omega)\right)$

Definition 2.4. Space of continuous functions $C\left([0,T] \to L^2(\Omega)\right)$ is set of all functions $v(t,\cdot) : [0,T] \to L^2(\Omega)$ which satisfy

$$\lim_{n\to+\infty} |t_n - t| = 0 \Rightarrow \lim_{n\to+\infty} \|v(t_n,\cdot) - v(t,\cdot)\|_{L^2(\Omega)} = 0$$

for any sequences $\{t_n\}_{n=1}^{+\infty}$, $t_n \in [0,T]$ and $t \in [0,T]$.

We are ready to define a weak solution (following Padial et al. [51, Definition 2.1, p. 605]).

Definition 2.5. A function $u(t,x)$ is called a weak solution of the problem (2.1) if

$$u \in C\left([0,\tau] \to L^2(\Omega)\right) \cap L^p\left((0,\tau) \to W_0^{1,p}(\Omega)\right)$$

for every $\tau \in (0,T)$ and $u(t,x)$ satisfies (2.1) in the weak sense, i.e.

$$\begin{aligned}
&\int_\Omega u(\tau,x)\phi(\tau,x)\,\mathrm{d}x - \int_0^\tau \left\langle u(s,\cdot), \frac{\partial \phi}{\partial s}(s,\cdot) \right\rangle \mathrm{d}s \\
&+ \int_0^\tau \int_\Omega |\nabla u(s,x)|^{p-2} \nabla u(s,x) \cdot \nabla \phi(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^\tau \int_\Omega q(x) u^\alpha(s,x) \phi(s,x)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{2.3}$$

for all $\tau \in (0,T]$ and all test functions

$$\phi \in L^p\left((0,T) \to W_0^{1,p}(\Omega)\right) \cap W^{1,p'}\left((0,T) \to W^{-1,p'}(\Omega)\right).$$

Here, $\langle \cdot,\cdot \rangle$ stands for duality paring between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

Remark 2.6. A function u satisfying Definition 2.5 with "$\leq$" and "$\geq$" instead of "$=$" in (2.3) is called a weak subsolution and supersolution, respectively.

Once we define the solution we are able to study the existence of nontrivial solution of (2.1). Since q is continuous and positive at least at some $x_0 \in \Omega$ by (Q), there exists $R > 0$ such that $q(x) \geq q_0 \equiv \text{const} > 0$ for all $x \in B_R(x_0)$. Then we denote by $\varphi_{1,R}$ the normalized ($\varphi_{1,R}(x_0) = 1$) eigenfunction corresponding to the first eigenvalue λ_1 of $-\Delta_p \colon W_0^{1,p}(B_R(x_0)) \to W^{-1,p'}(B_R(x_0))$ and by

$$\widetilde{\varphi}_{1,R}(x) \stackrel{\text{def}}{=} \begin{cases} \varphi_{1,R}(x) & \text{for } x \in B_R(x_0), \\ 0 & \text{for } x \in \overline{\Omega} \setminus B_R(x_0), \end{cases} \tag{2.4}$$

the natural zero extension of $\varphi_{1,R}$ from $B_R(x_0)$ to the whole of $\overline{\Omega}$.

Theorem 2.7 ([7], Theorem 1.1, p. 2)**.** *Assume that* $0 < \alpha < \min\{1,p-1\}$ *and* (Q) *are satisfied. Then there exists* $T > 0$ *small enough, such that problem* (2.1) *possesses (besides the trivial solution* $u \equiv 0$*) a nontrivial, nonnegative weak solution which is bounded below by a subsolution* $\underline{u} : (0,T) \times \Omega \to \mathbb{R}_+$ *of type*

$$\underline{u}(x,t) = \theta(t)\widetilde{\varphi}_{1,R}(x)^{\beta} \geq 0 \quad \text{in } (0,T) \times \Omega\,, \tag{2.5}$$

where $\theta\colon [0,T] \to \mathbb{R}_+$ *is a strictly increasing, continuously differentiable function with* $\theta(0) = 0$*, and* $\beta \in (1,\infty)$ *is a suitable number.*

Sketch of proof: It was shown in [7] that (2.5) is a subsolution of problem (2.1) with $\theta : [0,S] \to \mathbb{R}_+$ which is the positive solution of the Cauchy problem

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}(t) = \frac{q_0}{2}\theta^{\alpha}(t) \quad \text{for } t \in (0,S)\,; \quad \theta(0) = 0\,,$$

such that $0 < \theta(t) < \infty$ for every $t \in (0,S)$. We choose the supersolution $\overline{u} = \|q\|_{L^\infty(\Omega)}^{\frac{1}{1-\alpha}} t$ and show that $\underline{u} < \overline{u}$ for all $x \in \Omega$ and all $t > 0$ small enough. Finally, the existence of a weak solution is obtained via method of monotone iterations (see Derlet and Takáč [22]). ■

2.2. Speed of Propagation

In Section 2.1, the existence of a nontrivial, nonnegative weak solution of problem (2.1) was obtained via method of monotone iterations, where we used positive spatially constant supersolution

$$\overline{u}(t,x) = \|q\|_{L^\infty(\Omega)}^{\frac{1}{1-\alpha}} t\,. \tag{2.6}$$

Now, our aim is to show that there exists a solution of (2.1) with a compact support in $(0,T) \times \Omega$ provided $p > 2$. In other words, the solution admits *finite speed of propagation* in the following sense.

Definition 2.8. Let a weak positive solution of (2.1) have a compact support at some time $0 < t_0 < T$. We say that u possesses a finite speed of propagation if there exists $0 < \tau < T - t_0$ such that u has compact support on $[t_0, t_0 + \tau) \times \Omega$.

We prove the existence of the solution by the same method as in Section 2.1, but we use Barenblatt-type supersolution

$$\overline{u}(t,x) = (1+\sigma t)\left[1 - \left(\frac{|x-\xi|}{\varepsilon + \varrho t}\right)^2\right]_+^{\frac{1}{1-\alpha}}\,, \qquad \xi \in \Omega \text{ is fixed}\,, \tag{2.7}$$

instead of (2.6). Here the symbol $[v(t,x)]_+ \stackrel{\text{def}}{=} \max\{v(t,x),0\}$ for any $(t,x) \in (0,T) \times \Omega$. Let $K = \|q\|_{L^\infty(\Omega)}$ and $0 < T < 1/(K\alpha)$. Then $\varepsilon > 0$ is chosen such that $B_\varepsilon(\xi) \subset \Omega$ and $\sigma = \sigma(K,T,\alpha)$ and $\varrho = \varrho(T,\alpha,p,\sigma,\varepsilon)$ are chosen such that $\overline{u}$ is a supersoluiton of the problem (2.1) (see [9, Theorem 2.1, p. 996] for more details). Let us note that the choice of supersolution (2.7) guarantees the existence of a solution with compact support only for $t \in (0,T_0)$ such that $B_{\varepsilon+\varrho t}(\xi) \subset \Omega$. We may also construct a *multi-bump* solution using (2.7) as a supersolution.

Definition 2.9. A function $u(t,x)$ is an m-bump solution of the problem (2.1) if it is a weak solution of the problem (2.1) and it satisfies following properties:

1. $u\colon [0,T] \times \Omega \to \mathbb{R}$ is continuous, $u \geq 0$ in $[0,T] \times \Omega$, and $u \not\equiv 0$;
2. u has a compact support
$$\operatorname{supp}(u) \stackrel{\text{def}}{=} \overline{\{(t,x) \in [0,T] \times \Omega \colon u(t,x) > 0\}} \quad \text{in } [0,T] \times \mathbb{R}^N ,$$
$\operatorname{supp}(u) \subset [0,T] \times \Omega$;
3. there exist precisely m ($m \in \mathbb{N} = \{1,2,3,\dots\}$) pairwise disjoint, connected compact subsets K_k ($k = 1,2,\dots,m$) of Ω such that
$$\operatorname{supp}(u) \subset [0,T] \times (\cup_{k=1}^m K_k) ;$$
4. for all $t \in (0,T]$ and $k = 1,2,\dots,m$,
$$\{x \in K_k \colon u(t,x) > 0\}$$
is a nonempty connected open subset of Ω.

We find a solution with one bump around some $\xi \in \Omega$ such that $q(\xi) > 0$. The existence of point ξ is ensured by hypothesis (Q).

Theorem 2.10 ([9], Theorem 1.3, p. 995)**.** *Let $2 < p < \infty$, $1/(p-1) < \alpha < 1$, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Assume that q satisfies hypothesis (Q), $\xi \in \Omega$ is such that $q(\xi) > 0$, $r > 0$ satisfies $\overline{B}_r(\xi) \subset \Omega$, and $0 < T_0 < \infty$. Then there exists some $T \in (0,T_0]$ such that the initial-boundary value problem (2.1) possesses a nontrivial, nonnegative solution $u\colon (0,T) \times \Omega \to \mathbb{R}_+$ such that*

1. *$u(t,\xi) > 0$ for all $t \in (0,T)$;*
2. *$u(t,x) = 0$ for all $x \in \Omega \setminus B_r(\xi)$ and all $t \in (0,T)$.*

In addition, if $\Omega = B_R(\xi)$ is a ball with radius R centered at ξ, $0 < r < R < \infty$, and q is radially symmetric about ξ, i.e. $q(x) \equiv q(|x-\xi|)$ for $x \in \Omega$, then the nontrivial solution u above can be constructed radially symmetric about ξ in the space variable $x \in \Omega$, i.e. $u(t,x) \equiv u(t,|x-\xi|)$.

Sketch of proof: We choose $\varepsilon = r/4 > 0$ and $0 < T < \min\{1/(K\alpha),T_0\}$. Then we obtain that (2.7) is a supersolution of the problem (2.1) such that $\overline{u}(t,\xi) = 1 + \sigma t \geq 1$ and $\overline{u}(t,x) = 0$ for all $x \in \Omega \setminus B_{r/2}(\xi)$, both for all $t \in [0,T']$, $T' = \min\{r/(4\varrho),T\}$. The supersolution is also positive on $B_{r/4}(\xi)$ for any time $t \in [0,T']$ and, moreover,

$$\overline{u}(t,x) \geq (1+\sigma t)2^{-\frac{1}{1-\alpha}} \geq 2^{-\frac{1}{1-\alpha}} \qquad \text{on } B_{\frac{\sqrt{2}r}{8}}(\xi)\,.$$

Then we use a subsolution (2.5) with $R = r\sqrt{2}/8 < r$. Since $\theta(t)$ is continuous and $\theta(0) = 0$ in (2.5), we may choose $\underline{T}$ such that

$$\underline{u}(t,x) \leq 2^{-\frac{1}{1-\alpha}}$$

for all $x \in B_r(\xi)$ and $t \in [0,\underline{T}]$. In particular, we find the subsolution $\underline{u}$ and the supersolution $\overline{u}$ such that $\underline{u} \leq \overline{u}$ for all $x \in B_r(\xi)$ and $t \in [0, \min\{T',\underline{T}\}]$. Let us redefine $T = \min\{T',\underline{T}\}$ due to the statement of Theorem 2.10. Now it remains to apply monotone iteration method to obtain desired solution with compact support in Ω. ■

We use Theorem 2.10 to prove an existence of an m-bump solution (see Figure 2.1).

Theorem 2.11 ([9], Theorem 3.1, p. 1003)**.** *Let $2 < p < \infty$, $1/(p-1) < \alpha < 1$, and let $\Omega_k \subset \Omega$, $k = 1,2,3,\ldots,m$, be a family of pairwise disjoint subdomains of the domain $\Omega \subset \mathbb{R}^N$, and let $0 < T_0 < \infty$. Furthermore, let $0 \leq q \in C(\overline{\Omega})$ and $\xi_k \in \Omega_k$ be such that $q(\xi_k) > 0$, $k = 1,2,3,\ldots,m$. Then there exists some $T \in (0,T_0]$ such that the initial-boundary value problem* (2.1) *possesses a nontrivial, nonnegative solution $u\colon (0,T) \times \Omega \to \mathbb{R}_+$ such that*

1. *$u(t,\xi_k) > 0$ for all $k = 1,2,\ldots,m$ and all $t \in (0,T)$;*
2. *$u(t,x) = 0$ for all $x \in \overline{\Omega} \setminus \cup_{k=1}^m \Omega_k$ and all $t \in [0,T]$.*

Finite speed of propagation of a solution of (2.1) appears for $p > 2$ since the diffusion is slow (weak or degenerate) in the case. In the case $1 < p < 2$, the diffusion is fast (strong, singular), see Figure 1.1. Hence there is a chance that a solution of (2.2) possesses *infinite speed of propagation* in the sense of following Definition 2.12.

Definition 2.12. A weak positive solution of (2.2) possesses an infinite speed of propagation if, for any fixed $t_0 \in (0,T)$, the solution $u(t_0,\cdot)$ is either positive on Ω or else identically zero on Ω.

Indeed this phenomenon was confirmed for any continuous, nonnegative solution of (2.2) in Khin and Su [32] for unbounded domain Ω and in [10] for bounded domain Ω. The continuity assumption is meaningful at least in the two special situations $b(s) \equiv s$ for all $s \in \mathbb{R}_+$ and $b(s) = s^\sigma$ for all $s \in \mathbb{R}_+$ with $\sigma \in \mathbb{R}$ is a constant $p - 1 \leq \sigma < +\infty$. Then

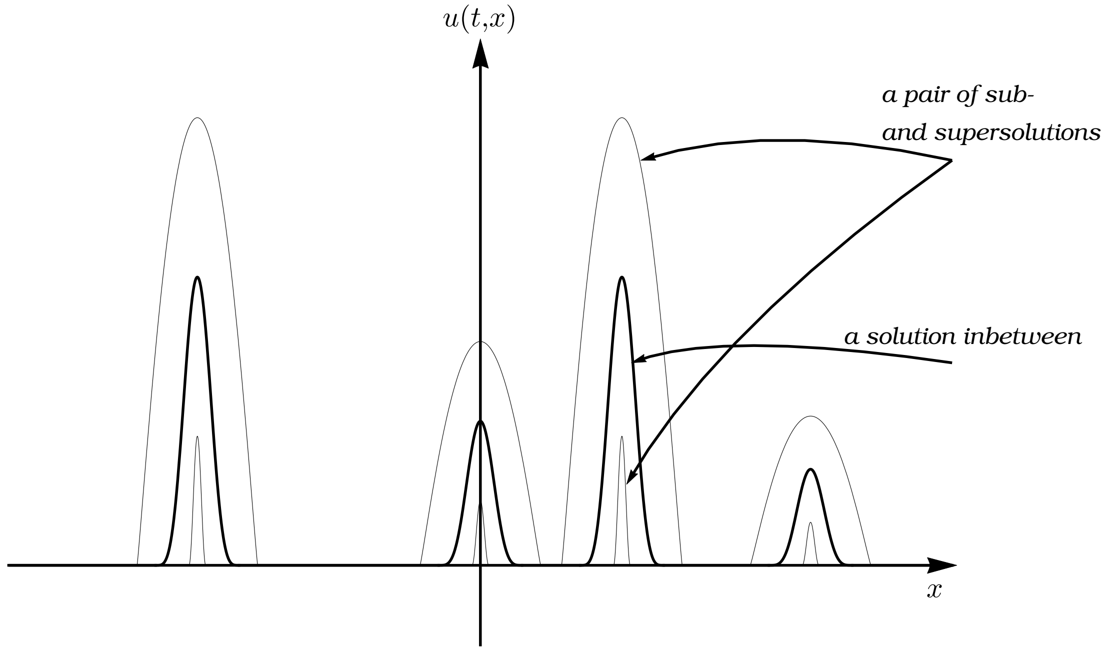

Figure 2.1: A multi-bump solution.

any solution is continuous due to the regularity result by Chen and DiBenedetto [19, Theorem 1, p. 320] or Ivanov [31, Proposition 3.1 and 3.2, p. 28] in the first case and [31, Eq. (1.7), p. 23] combined with [31, Proposition 3.1 and 3.2, p. 28] in the latter case. Let us formulate Theorem 2.13 following up *infinite speed of propagation* of solution of (2.2) where $\Omega \subset \mathbb{R}^N$ is not necessary bounded for a moment. We also assume that both, f and u_0, are continuous and nonnegative for simplicity.

Theorem 2.13 ([10], Theorem 1.1, p. 96)**.** *Let* $1 < p < 2$*,* $N \geq 1$ *and assume that* $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuously differentiable function in* $(0, +\infty)$ *with* $b' > 0$*,* $b(0) = 0$*, and such that*

$$\lim_{s \to 0+} \frac{s^{2-p}\, b'(s)}{|\log s|^{p-1}} = 0\,. \tag{2.8}$$

Finally assume that $u\colon [0,T) \times \overline{\Omega} \to \mathbb{R}_+$ *is a continuous, nonnegative, weak solution of* (2.2)*. Then, for any fixed* $t_0 \in (0,T)$*, the solution* $u(t_0,\cdot)$ *is either positive everywhere on* Ω *or else identically zero on* Ω*.*

In particular, if $u(0,\xi) = u_0(\xi) > 0$ *for some* $\xi \in \Omega$*, then there exists* $\tau \in (0,T]$ *such that* $u(t,x) > 0$ *for all* $(t,x) \in (0,\tau) \times \Omega$*, i.e. the strong maximum principle is valid in the* $(N+1)$*-dimensional space-time cylinder* $(0,\tau) \times \Omega$*. The number* $\tau \in (0,T)$ *can be estimated from below by*

$$\tau = \sup\{T' \in (0,T]\colon u(t,\xi) > 0 \text{ for all } t \in [0,T')\} > 0\,.$$

Sketch of proof: Let $t_0 \in (0,T)$ be fixed and denote $Z \equiv Z(t_0) = \{x \in \Omega \colon u(t_0,x) = 0\}$. Our aim is to show that Z is both, open and closed in Ω. The latter one easily follows from the continuity of u on connected set Ω. The proof is trivial if Z is empty set. Let $x_1 \in Z$ and we prove that also $B_{\frac{d}{2}}(x_1) \subset Z$ where $d = \operatorname{dist}(x_1,\partial\Omega)$. We assume by contradiction that there exists $x_2 \in B_{\frac{d}{2}}(x_1)$ and $x_2 \notin Z$. Due to the continuity of u, there exists $R \in (0,|x_1 - x_2|)$ and $\tau \in (0,t_0)$ such that

$$\eta \stackrel{\text{def}}{=} \inf_{(t,x)\in[t_0-\tau,t_0]\times \overline{B}_R(x_2)} u(t,x) > 0\,.$$

By triangle inequality, $|x_1 - x_2| < \operatorname{dist}(x_2,\partial\Omega)$ and hence there exists R^* such that $0 < R < |x_1 - x_2| < R^* < \operatorname{dist}(x_2,\partial\Omega)$. We will construct a subsolution $v \colon [t_0 - \tau,t_0] \times \big(\overline{B}_{R^*}(x_2) \setminus B_R(x_2)\big) \to \mathbb{R}_+$ of problem (2.2) satisfying

1. $v(t_0 - \tau,x) = 0$ for all $x \in \overline{B}_{R^*}(x_2) \setminus B_R(x_2)$;
2. $v(t_0,x_1) > 0$;
3. $v(t,x) = 0$ for all $x \in \partial B_{R^*}(x_2)$ and for all $t \in [t_0 - \tau,t_0]$;
4. $v(t,x) \le \eta$ for all $x \in \partial B_R(x_2)$ and all $t \in [t_0 - \tau,t_0]$.

Properties 1, 3, and 4 guarantee that $0 \le v(t,x) \le u(x,t)$ also on

$$\big((t_0 - \tau,t_0] \times \big(B_{R^*}(x_2) \setminus \overline{B}_R(x_2)\big)\big)$$

by the weak comparison principle. Property 4 provide a contradiction with $x_1 \in Z$. It follows that $Z(t_0) = \Omega$ since $x_1 \in Z$ by assumption. Assume that

$$v(t,x) = z(R + \omega(t - t_0 + \tau) - |x - x_2|)\,,$$

where $\omega = \dfrac{R^* - R}{\tau}$ and

$$z(\zeta) = \begin{cases} \exp\left[-(\varepsilon\zeta)^{-1/\varepsilon}\right] & \text{if } \zeta \in (0,+\infty)\,, \\ 0 & \text{if } \zeta \in (-\infty,0] \end{cases}$$

with $\varepsilon > 0$ sufficiently small. Then v satisfies the conditions 1. - 4. (see [10]). ∎

Let us note that the method of construction of the subsolution v is taken from [32].

2.3. Related Articles of the Author

2015 **Nonuniqueness of solutions of initial-value problems for parabolic p-Laplacian**; Benedikt, J., Bobkov, V. E., Girg, P., Kotrla, L., Takáč, P.; *Electron. J. Differential Equations*; [7].

Abstract:
We construct a positive solution to a quasilinear parabolic problem in a bounded spatial domain with the p-Laplacian and a nonsmooth reaction function. We obtain nonuniqueness for zero initial data. Our method is based on sub- and supersolutions and the weak comparison principle.

Using the method of sub- and supersolutions we construct a positive solution to a quasilinear parabolic problem with the p-Laplacian and a reaction function that is non-Lipschitz on a part of the spatial domain. Thereby we obtain nonuniqueness for zero initial data.

Reviewed by Haifeng Shang (MR3335768):
"In this paper, the authors study the problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = q(x)|u|^{\alpha-1}u & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $1 < p < \infty$, $0 < \alpha < 1$, $0 < T < \infty$ and the potential q satisfies

$$q \in C\left(\overline{\Omega}\right),\ q \geq 0, \text{ and } q(x_0) > 0 \text{ for some } x_0 \in \Omega. \tag{2}$$

Moreover $\Omega \subset \mathbb{R}^N$ is bounded domain with $C^{1+\mu}$-boundary $\delta\Omega$, where $0 < \mu < 1$.

Using the method of sub- and supersolution and a weak comparison principle, the authors prove the following nonuniqueness result:

Theorem 1. Assume that $0 < \alpha < \min\{1,p-1\}$ and condition (2) is satisfied. Then there exists $T > 0$ small enough, such that problem (1) possesses (besides the trivial solution $u \equiv 0$) a nontrivial, nonnegative weak solution

$$u \in C([0,T],L^2(\Omega)) \cap L^2((0,T),W^{1,p}(\Omega)),$$

which is bounded below by a subsolution $\underline{u} : \Omega \times (0,T) \to \mathbb{R}_+$ of form

$$\underline{u}(x,t) = \theta(t)\tilde{\varphi}^{\beta}_{1,R}(x) \geq 0 \quad \text{in } \Omega \times (0,T),$$

where $\theta : [0,T] \to \mathbb{R}_+$ is a strictly increasing, continuously differentiable function with $\theta(0) = 0$, and $\beta \in (1,\infty)$ is a suitable number."

Cited by:

1. Laister, R., Robinson, J. C., Sierzega, M. A necessary and sufficient condition for uniqueness of the trivial solution in semilinear parabolic equations. *J. Differential Equations 262*, 10 (2017), 4979–4987.

2016 **Nonuniqueness and multi-bump solutions in parabolic problems with the p-Laplacian**; Benedikt, J., Girg, P., Kotrla, L., Takáč, P.; *J. Differential Equations*; [9]

Abstract:
The validity of the weak and strong comparison principles for degenerate parabolic partial differential equations with the p-Laplace operator Δ_p is investigated for $p > 2$. This problem is reduced to the comparison of the trivial solution ($\equiv 0$, by hypothesis) with a nontrivial nonnegative solution $u(x,t)$. This problem is closely related also to the question of uniqueness of a nonnegative solution via the weak comparison principle. In this article, realistic counterexamples to the uniqueness of a nonnegative solution, the weak comparison principle, and the strong maximum principle are constructed with a nonsmooth reaction function that satisfies neither a Lipschitz nor an Osgood standard "uniqueness" condition. Nonnegative multi-bump solutions with spatially disconnected compact supports and zero initial data are constructed between sub- and supersolutions with supports of the same type.

Reviewed by Juha K. Kinnunen (MR3419719):
"Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^{1+\mu}$-boundary $\delta\Omega$, where $0 < \mu < 1$. Let $p > 2$, $\alpha \in (0,1)$, and $0 < T < \infty$.

This paper considers the nonlinear parabolic problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = q(x)u^\alpha & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = 0 & \text{for } x \in \Omega, \end{cases} \tag{1}$$

where potential q is assumed to satisfy $q \in C\left(\bar{\Omega}\right)$, $q \geq 0$ and $q(x_0) > 0$ for some $x_0 \in \Omega$.

The main goal of the paper is to construct nontrivial nonnegative solution to (1) with multiple positive bumps that have pairwise disjoint supports with respect to the space variable. This is closely related to the question of uniqueness of a nonnegative solution. In particular, this paper gives counterexamples to the uniqueness of a nonnegative solution, a weak comparison principle and a strong maximum principle with a nonsmooth reaction function $f(x,u) = q(x)u^{\alpha}$ that satisfies neither a Lipschitz nor an Osgood type uniqueness condition.

By the strong maximum principle for linear parabolic equations, this phenomenon is impossible for semilinear parabolic problems of the form (1) when $p = 2$. Each single bump solution is obtained by constructing a Barenblatt type supersolution and using it as an upper bound for a monotone iteration procedure, starting from a nontrivial nonnegative subsolution."

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2017 **The strong maximum principle in parabolic problems with the p-Laplacian in a domain**; Benedikt, J., Girg, P., Kotrla, L., Takáč, P. ; *Appl. Math. Lett.*; [10]

Abstract:
We establish a strong maximum principle for a nonnegative continuous solution $u : \overline{\Omega} \times [0,T) \to \mathbb{R}_+$ of a doubly nonlinear parabolic problem in a space-time cylinder $\Omega \times (0,\tau)$ with a domain $\Omega \subset \mathbb{R}^N$ and a sufficiently short time interval $(0,\tau) \subset (0,T)$. Our method takes advantage of a nonnegative subsolution derived from an expanding spherical wave.

2018 **Origin of the p-Laplacian and A. Missbach**; Benedikt, J., Girg, P., Kotrla, L., Takáč, P.; *Electron. J. Differential Equations*; [11]

Abstract:
We describe the historical process of derivation of the p-Laplace operator from a nonlinear Darcy law and the continuity equation. The story begins with nonlinear flows in channels and ditches. As the nonlinear Darcy law we use the power law discovered by O. Smreker and verified in experiments by A. Missbach for flows through porous media in one space dimension. These results were generalized by S. A. Christianovitch and L. S. Leibenson to porous media in higher space dimensions. We provide a brief description of Missbach's experiments.

Reviewed by Philip Broadbridge (MR3762803):
" When immersed in the spectral properties of exotic nonlinear elliptic operators, it is too easy to lose sight of their origins in practical mathematical modelling, and the mathematical insight that they provide. To this end, the authors make a valuable contribution in providing an objective historical account, to answer the question "How did the p-Laplacian Δ_p originate?" Here,

$$\Delta_p u = \nabla \cdot \left(|\nabla u|^{p-2} \nabla u \right).$$

This relatively short, mathematically relevant historical article draws on a wealth of background material. I learnt that the porous media models of the mid-19th century were grounded in channel flow and pipe flow models of 18th-century French engineers. Thereafter, a number of theoretical and experimental studies in both Western and Eastern Europe (with some groups ignorant of concurrent and antecedent work) questioned how the Darcy law should be modified to account for nonlinear dependence of hydraulic head gradient $|\nabla \Phi|$ on fluid speed v. The referenced literature from the early to mid-20th century makes it apparent that there remains some controversy on whether that relationship must reduce to the linear Darcy law at low Reynolds numbers. The simplest representatives of the opposing views are the Forchheimer equation (with $a \neq 0$) and the Missbach equation (with

$a = 0$) in $|\nabla\Phi| = a+bv^m$; $m > 0$. In relation to conventional groundwater flow theory, the classic 20th-century texts of Polubarinova-Kochina and of Bear are given here. I would also mention [E. C. Childs, An introduction to the physical basis of soil water phenomena, Wiley-Interscience, London, 1969]. Childs indicated how the Darcy law can be derived from the Navier-Stokes equations."

CHAPTER III

Generalized Trigonometric and Hyperbolic Functions

Chapter 3 is mainly devoted to generalized *sine* and *hyperbolic sine* functions. These functions are closely related to initial value problem

$$\left\{\begin{array}{rcl} -\left(|\widehat{u}\,'|^{p-2}\widehat{u}\,'\right)' & = & \widehat{\lambda}\,|\widehat{u}|^{p-2}\,\widehat{u}\,, \\ \widehat{u}(0) & = & 0\,, \\ \widehat{u}\,'(0) & = & \widehat{\alpha}\,, \end{array}\right. \tag{3.1}$$

where $\widehat{\lambda} \in \mathbb{R}$ and $\widehat{\alpha} > 0$. Problem (3.1) was studied in Elbert [25][1] in the particular case $\widehat{\lambda} = p-1$ and $\widehat{\alpha} = 1$ and/or in Del Pino, Elgueta and Manasevich [21] for $\widehat{\lambda} > 0$. Multiplying the equation in (3.1) by $\widehat{u}\,'$ and integrating from 0 to x, we obtain

$$\begin{aligned} &\int_0^x \left|\widehat{u}\,'(s;\widehat{\lambda},\widehat{\alpha})\right|^{p-2} \widehat{u}\,'(s;\widehat{\lambda},\widehat{\alpha})\widehat{u}\,''(s;\widehat{\lambda},\widehat{\alpha})\,\mathrm{d}s \\ &+ \frac{\widehat{\lambda}}{p-1}\int_0^x \left|\widehat{u}\,(s;\widehat{\lambda},\widehat{\alpha})\right|^{p-2} \widehat{u}(s;\widehat{\lambda},\widehat{\alpha})\widehat{u}\,'(s;\widehat{\lambda},\widehat{\alpha})\,\mathrm{d}s \\ &= 0\,. \end{aligned}$$

Hence we get

$$\left|\widehat{u}\,'(x;\widehat{\lambda},\widehat{\alpha})\right|^p + \frac{\widehat{\lambda}}{p-1}\left|\widehat{u}(x;\widehat{\lambda},\widehat{\alpha})\right|^p = \widehat{\alpha}^{\,p}\,, \qquad x \in \mathbb{R}\,. \tag{3.2}$$

[1] I would like to thank to Prof. Lomtatidze for pointing out that it was J. D. Mirzov [45] who treated the eigenvalue problem (1.2) already in 1979 (i.e. 2 years before Elbert's work was published). More precisely, he studied oscillatory properties of solutions of the Emden-Fowler type systems. The eigenvalue problem (1.2) is covered as a special case of his results. Unfortunately, I received this information in July 2018 when the Thesis was almost completed. Therefore, I do not include these interesting results into the Thesis in detail.

It follows that

$$\frac{\widehat{\lambda}}{p-1}\widehat{u}^{\,p} \le \widehat{\alpha}^{\,p} \quad \text{for } \widehat{\lambda} > 0\,, \qquad \text{and} \qquad \widehat{\alpha}^{\,p} - \frac{\widehat{\lambda}}{p-1}\widehat{u}^{\,p} > 0 \quad \text{for } \widehat{\lambda} < 0\,.$$

Since $\widehat{\alpha} > 0$, there is $\delta > 0$ such that $\widehat{u}' > 0$ on $(0,\delta)$. Therefore

$$\widehat{u}^{\,\prime} = \sqrt[p]{\widehat{\alpha}^{\,p} - \frac{\widehat{\lambda}}{p-1}\widehat{u}^{\,p}} \tag{3.3}$$

on $(0,\delta)$. Integrating (3.3) from 0 to x again, we have

$$\int_0^x \frac{\widehat{u}^{\,\prime}(s;\widehat{\lambda},\widehat{\alpha})}{\sqrt[p]{\widehat{\alpha}^{\,p} - \frac{\widehat{\lambda}}{p-1}\widehat{u}^{\,p}(s;\widehat{\lambda},\widehat{\alpha})}}\,\mathrm{d}s = x$$

on $(0,\delta)$. Substitute $\sigma = \widehat{u}(s;\widehat{\lambda},\widehat{\alpha})$, we obtain

$$\int_0^{\widehat{u}(x)} \frac{1}{\sqrt[p]{\widehat{\alpha}^{\,p} - \frac{\widehat{\lambda}}{p-1}\sigma^p}}\,\mathrm{d}\sigma = x\,. \tag{3.4}$$

Remark 3.1. Already in 1879, Lundberg [42] studied the properties of a solution $y(x)$ of the integral equation

$$x = \int_0^{y(x)} \frac{\mathrm{d}s}{(1-s^n)^{\frac{m}{n}}}\,,$$

where $m,n \in \mathbb{N}$, $n \ge m$ and $\dfrac{m}{n}$ is either irreducible fraction or 1.

Let $\widehat{\lambda} < 0$. Since $f(\sigma) = 1/\sqrt[p]{\widehat{\alpha}^{\,p} - \frac{\widehat{\lambda}}{p-1}\sigma^p}$ is continuous for any $\sigma \ge 0$, $\widehat{u}$ is continuous whenever $\widehat{u}^{\,\prime}$ is positive. Consequently, $\widehat{u}^{\,\prime}$ is continuous on the same interval. Hence from $\widehat{u}(0) = 0$, $\widehat{u}^{\,\prime}(0) = \widehat{\alpha}$, and (3.2) we obtain $\widehat{u}^{\,\prime}(x) > 0$ for any $x > 0$. It is easy to see that odd extension of $\widehat{u}$ satisfies (3.1) for any $x \in \mathbb{R}$. We denote by $\sinh_p$ the solution of (3.1) with $\widehat{\lambda} = -p+1$ and $\widehat{\alpha} = 1$. In Section 3.3, we will study extension of $\sinh_p$ to complex domain.

Henceforth let $\widehat{\lambda} > 0$. Since $f(\sigma) = 1/\sqrt[p]{\widehat{\alpha}^{\,p} - \frac{\widehat{\lambda}}{p-1}\sigma^p}$ is continuous provided $\widehat{\alpha}^{\,p} - \frac{\widehat{\lambda}}{p-1}\sigma^p \ge 0$, $\widehat{u}(x;\widehat{\lambda},\widehat{\alpha})$ is continuous on $[0,\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha})/2)$ and consequently $\widehat{u}^{\,\prime}(x;\widehat{\lambda},\widehat{\alpha})$ is continuous on the same interval by (3.3). Here

$$\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha}) \overset{\text{def}}{=} 2\int_0^{\frac{\widehat{\alpha}(p-1)^{\frac{1}{p}}}{\widehat{\lambda}^{\frac{1}{p}}}} \frac{1}{\sqrt[p]{\widehat{\alpha}^{\,p} - \frac{\widehat{\lambda}}{p-1}\sigma^p}}\,\mathrm{d}\sigma\,,$$

i.e. $\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha})/2$ is the point, where $\widehat{u}(x;\widehat{\lambda},\widehat{\alpha})$ achieves its maximum. Hence, $\delta = \widehat{\pi}_p/2$. Let us show that the value $\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha})$ is finite. Indeed,

$$\begin{aligned}
\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha}) &= 2\int_0^1 \frac{\widehat{\alpha}(p-1)}{\widehat{\alpha}p\widehat{\lambda}} \left(\frac{\widehat{\lambda}}{\widehat{\alpha}^p(p-1)t}\right)^{\frac{p-1}{p}} (1-t)^{-\frac{1}{p}}\,\mathrm{d}t \\
&= 2\frac{(p-1)^{\frac{1}{p}}\widehat{\alpha}^{\frac{1}{p}-1}}{p\widehat{\lambda}^{\frac{1}{p}}} \int_0^1 t^{\frac{1}{p}-1}(1-t)^{-\frac{1}{p}}\,\mathrm{d}t \\
&= 2\frac{(p-1)^{\frac{1}{p}}\widehat{\alpha}^{\frac{1}{p}-1}}{p\widehat{\lambda}^{\frac{1}{p}}} B\left(\frac{1}{p},1-\frac{1}{p}\right),
\end{aligned}$$

where B stands for *Beta function*. The value $\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha}) < +\infty$ since $B(a,b)$ is convergent for any $a > 0$ and any $b > 0$. Moreover,

$$\widehat{\pi}_p(\lambda,\widehat{\alpha}) = 2\frac{(p-1)^{\frac{1}{p}}\widehat{\alpha}^{\frac{1}{p}-1}}{p\widehat{\lambda}^{\frac{1}{p}}}\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right) = 2\frac{(p-1)^{\frac{1}{p}}\widehat{\alpha}^{\frac{1}{p}-1}}{p\widehat{\lambda}^{\frac{1}{p}}}\frac{\pi}{\sin(\pi/p)} \tag{3.5}$$

by Andrews et al. [4, Theorem 1.1.4, p. 5, and Theorem 1.2.1, p. 9]. *Gamma function* Γ and *Beta function* B are defined as usual (see, e.g. Chapter 1 of the handbook [4]).

Integral (3.4) defines the unique solution $\widehat{u}(x;\widehat{\lambda},\widehat{\alpha})$ on $[0,\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha})/2]$ such that $\widehat{u}\,'(\widehat{\pi}(\widehat{\lambda},\widehat{\alpha})/2;\widehat{\lambda},\widehat{\alpha}) = 0$ by (3.2). Moreover, $\widehat{u}(\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha}) - x;\widehat{\lambda},\widehat{\alpha}) = \widehat{u}(x;\widehat{\lambda},\widehat{\alpha})$ also satisfies (3.1) with $\widehat{u}\,'(0;\widehat{\lambda},\widehat{\alpha}) = -\widehat{\alpha}$ and, hence, the function

$$\widehat{S}_p(x;\widehat{\lambda},\widehat{\alpha}) \stackrel{\text{def}}{=} \begin{cases} \widehat{u}(x;\widehat{\lambda},\widehat{\alpha}) & x \in [0,\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha})/2)\,, \\ \widehat{u}(\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha}) - x;\widehat{\lambda},\widehat{\alpha}) & x \in (\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha})/2,\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha})]\,, \\ \widehat{u}(-x;\widehat{\lambda},\widehat{\alpha}) & x \in [-\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha}),0) \end{cases} \tag{3.6}$$

extended on $\mathbb{R}$ as $2\widehat{\pi}_p(\widehat{\lambda},\widehat{\alpha})$-periodic function is the unique continuous solution of (3.1) by (3.2).

Let us assume the particular case $\widehat{\lambda} = p-1$ and $\widehat{\alpha} = 1$. We defined π_p by (1.3) in Chapter 1. It follows from (3.5) that

$$\pi_p = \widehat{\pi}_p(p-1,1)\,.$$

It is easy to see that $\widehat{S}_p(x;p-1,1)$ is a solution of eigenvalue problem (1.2). Hence we may equivalently define the function $\sin_p$ as the unique solution of initial value problem

$$\begin{cases} -\left(|u'|^{p-2}u'\right)' &= (p-1)\,|u|^{p-2}\,u\,, \\ u(0) &= 0\,, \\ u'(0) &= 1\,. \end{cases} \tag{3.7}$$

Definition 3.2. Let $p > 1$. Function $\sin_p$ is defined as the solution of (3.7) and $\cos_p$ is its first derivative.

Remark 3.3 (Different definition of $\sin_p$). The notation $\sin_p$ was used originally for $\widehat{S}_p(x;1,1)$ in [21]. Denote $S_p(x) = \widehat{S}_p(x;1,1)$ for simplicity. Then

$$\left(S_p'\right)^p(x) + \frac{S_p^p(x)}{p-1} = 1 \tag{3.8}$$

by (3.2). LINDQVIST [40] equivalently define function S_p via the integral

$$\int_0^{S_p(x)} \frac{1}{\sqrt[p]{1 - \frac{1}{p-1}\sigma^p}}\,\mathrm{d}\sigma = x$$

and obtain the relation

$$\frac{S_p^p}{p-1} + \frac{C_{p'}^{p'}}{p'-1} = 1\,.$$

Here $p' > 1$ is a *conjugate exponent* to p, i.e. $1/p + 1/p' = 1$, and function $C_p : \left[0,\frac{\widehat{\pi}_p(1,1)}{2}\right] \to \left[0,(p-1)^{1/p}\right]$ is defined by

$$\int_{C_p(x)}^{(p-1)^{1/p}} \frac{\mathrm{d}s}{\left(1 - s^p/(p-1)\right)^{1/p}} = x\,.$$

Let us note that function C_p is not the first derivative of S_p, but it satisfies the relation

$$S_p(x) = C_p\left(\frac{\pi_p}{2} - x\right)$$

(see [40, Eq. (3.5), p. 277]). It is easy to verify that

$$\sin_p(x) = (p-1)^{-\frac{1}{p}} S_p((p-1)^{\frac{1}{p}}x)\,.$$

by (3.2). Definition 3.2 is used, e.g. in BINDING at al. [13], BOULTON and LORD [15], BUSHELL and EDMUNDS [18] and/or LANG and EDMUNDS [38].

By (3.4), $\sin_p$ satisfies

$$\int_0^{\sin_p(x)} \frac{1}{(1-\sigma^p)^{1/p}}\,\mathrm{d}\sigma = x$$

on $\left[0,\frac{\pi_p}{2}\right]$. Hence there exists inverse function

$$\arcsin_p(x) \stackrel{\text{def}}{=} \int_0^x \frac{1}{(1-\sigma^p)^{1/p}}\,\mathrm{d}\sigma \qquad \text{for } x \in [0,1]\,. \tag{3.9}$$

Let us note that $\arcsin_p$ can be extended to $[-1,1]$ as an odd function. Then we obtain

$$\sin_p\left(\arcsin_p(x)\right) = x \qquad \text{for } x \in [-1,1]\,.$$

We also introduce *Gauss' hypergeometric function* ${}_2F_1(a,b,c,z)$, where $a,b,c \in \mathbb{C}$ are parameters and $z \in \mathbb{C}$ is variable. The inquiring reader can found a general definition of ${}_2F_1$ and its properties, e.g. in ABRAMOWITZ and STEGUN [1, Chap. 15] and/or in [4, Chap. 2]. Let $x \in [0,1)$. Then the real integral

$$\int_0^x \frac{1}{(1-\sigma^p)^{1/p}}\,\mathrm{d}\sigma = x\ {}_2F_1\left(\frac{1}{p},\frac{1}{p},1+\frac{1}{p};x^p\right)$$

by [4, Theorem 2.2.1, p. 65] and

$$x\ {}_2F_1\left(\frac{1}{p},\frac{1}{p},1+\frac{1}{p};x^p\right) = \sum_{k=0}^{+\infty} \frac{\Gamma\left(k+\frac{1}{p}\right)}{(kp+1)k!\Gamma\left(\frac{1}{p}\right)} x^{kp+1}$$

by the definition of ${}_2F_1$. The series converge absolutely for $|x| < 1$ by [4, Theorem 2.1.1, p. 62]. Hence

$$\operatorname{arcsin}_p(x) = \sum_{k=0}^{+\infty} \frac{\Gamma\left(k+\frac{1}{p}\right)}{(kp+1)k!\Gamma\left(\frac{1}{p}\right)} x^{kp+1} \qquad \text{for } x \in [0,1)\,. \tag{3.10}$$

We apply the well-known procedure of inverting series (see, e.g. MORSE and FESHBACH [50, §4.5]) to the series (3.10) and we formally obtain

$$\sin_p(x) = x - \frac{1}{p(p+1)}x^{p+1} - \frac{(p^2-2p-1)}{2p^2(p+1)(2p+1)}x^{2p+1} + \dots\,. \tag{3.11}$$

The main goal of Chapter 3 is to find Maclaurin series of functions $\sin_p$. Let us provide two applications of $\sin_p$ which motivate our work. In Section 1.2, we mentioned the work of BOULTON and LORD [15] in order to provide an application of the functions $\sin_p$ in solving parabolic problems involving the p-Laplace operator (eq. (1.11)). In [15], the basis $\{\sin_p(k\pi_p\,x)\}_{k=1}^{+\infty}$ is used in numerical implementation of Galerkin method to find an approximate solution of the boundary-initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}\left(\left|\dfrac{\partial u}{\partial x}\right|^{p-2}\dfrac{\partial u}{\partial x}\right) = g\,, & \text{in } (0,+\infty)\times(0,1)\,, \\ u(t,0) = u(t,1) = 0\,, & \text{for } t > 0\,, \\ u(0,x) = 0\,, & \text{in } (0,1)\,, \end{cases} \tag{3.12}$$

where $g \in L^2(0,1)$. The choice of basis $\{\sin_p(k\pi_p\,x)\}_{k=1}^{+\infty}$ leads to a very good approximation of solution of (3.12) with only few terms of the basis in use provided t is sufficient large.

The other application is generalized Prüfer transformation where $\sin_p$ and $\cos_p$ are used instead of classical sine and cosine functions, respectively. Generalized Prüfer transformation is a very powerful theoretical tool in studying various initial and/or boundary value problems for the quasilinear equation of the type (or some of its generalization)

$$-\left(|u'|^{p-2}u'\right)' - q(x)|u|^{p-2}u = f(x)$$

(under various conditions on q and f), see, e.g. Elbert [25], Reichel and Walter [53], and/or Benedikt and Girg [8]. The numerical algorithm based on shooting method and generalized Prüfer transformation is introduced in Brown and Reichel [16] and it is used to determine the eigenvalues and the corresponding eigenfunctions of the radially symmetric p-Laplace operator. The usage of Prüfer transformation includes the evaluation of S_p (see Remark 3.3 for relation between S_p and $\sin_p$) and hence the efficiency of the algorithm relies also on the ability to find the values of S_p or $\sin_p$ fast. In [16], it is done by solving (3.8) numerically with $S_p(0) = 0$ and $S_p'(0) = 1$.

An explicit formula for coefficients of Maclaurin series of $\sin_p$ could help speed up the above stated methods. However it is very difficult to determine the coefficients in general and we are not able to deal with this problem for all $p > 1$. As a starting point for further research in this direction, we provide such formulas for any p being an integer greater than 2. Let us note that even this partial result can already be used in practical applications, since (3.12) with $p \to +\infty$ is considered as a model for sandpile growth (see [5] and [26] for more details).

3.1. Basic Properties of p-Trigonometric Functions

Function $\sin_p$ possesses many properties as classical *sine* has. Some of the most basic properties are listed in the following proposition.

Proposition 3.4 ([29], Lemma 4.1, p. 106)**.** *Let $p > 1$. Functions $\sin_p$ and $\cos_p$ have the following basic properties.*

1. $\sin_p(x) > 0$ *for* $x \in (0,\pi_p)$, $\sin_p(0) = 0$, $\sin_p(x) = \sin_p(\pi_p - x)$ *for* $x \in (\frac{\pi_p}{2},\pi_p)$, *and* $\sin_p(x) = -\sin_p(-x)$ *for* $x \in (-\pi_p,0)$. *The function* $\sin_p$ *extends to* $\mathbb{R}$ *as* $2\pi_p$*-periodic function.*
2. $\sin_p$ *is strictly increasing on* $(-\frac{\pi_p}{2},\frac{\pi_p}{2})$.
3. $\cos_p(x) > 0$ *for* $x \in (-\frac{\pi_p}{2},\frac{\pi_p}{2})$, $\cos_p(-\frac{\pi_p}{2}) = \cos_p(\frac{\pi_p}{2}) = 0$ *and* $\cos_p(x) < 0$ *for* $x \in \left[-\pi_p, -\frac{\pi_p}{2}\right) \cup \left(\frac{\pi_p}{2},\pi_p\right]$.
4. *For all* $n \in \mathbb{N}$, *if* $\sin_p^{(2n-1)}(\cdot)$ *exists on* $(-\frac{\pi_p}{2},\frac{\pi_p}{2})$, *then it is even function on* $(-\frac{\pi_p}{2},\frac{\pi_p}{2})$.

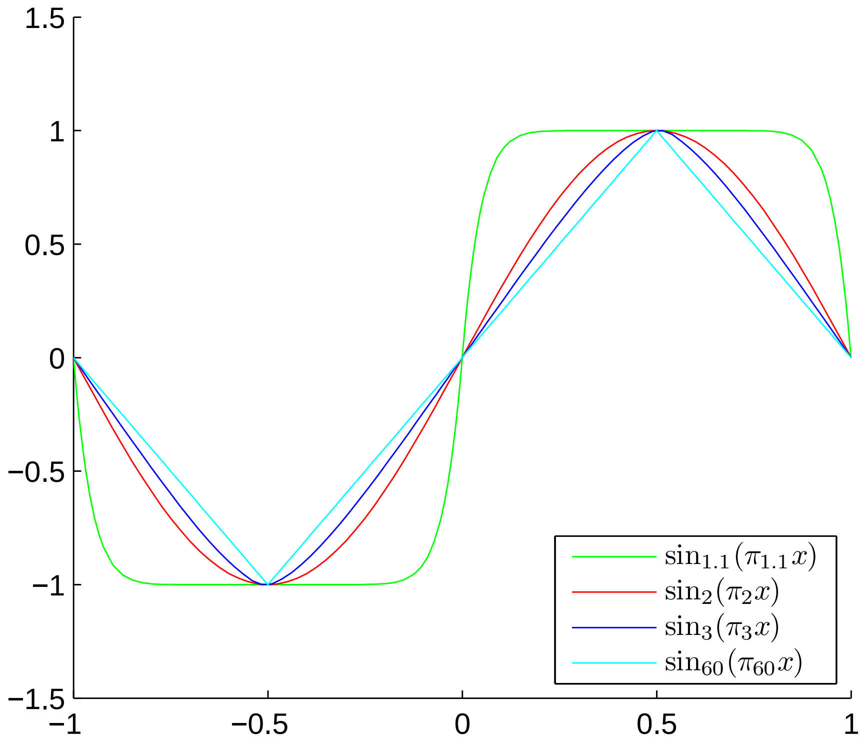

Figure 3.1: Comparison of $\sin_p$ functions.

5. *For all $n \in \mathbb{N}$, if $\sin_p^{(2n)}(\cdot)$ exists on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$, then it is odd function on $(-\frac{\pi_p}{2}, \frac{\pi_p}{2})$.*

Let as note that superscript (n) denotes n-th derivative. The comparison of functions $\sin_p$, $p = 1.1$, 3, 60 and classical *sine* function is visualized in Figure 3.1. Solution of (3.7) has to satisfy an analogy to the well-known Pythagorean identity (see (3.2) and/or Elbert [25]).

Proposition 3.5. *Functions $\sin_p$ and $\cos_p$ satisfy*

$$|\cos_p(x)|^p + |\sin_p(x)|^p = 1\,, \qquad x \in \mathbb{R}\,, \tag{3.13}$$

for any $p > 1$.

Remark 3.6. The curves which satisfy

$$\frac{x^n}{a^n} + \frac{y^n}{b^n} = 1 \quad a{,}b{,}n > 0$$

were studied by G. LAMÉ already in 1818 (see LAMÉ [37]). Their modern form

$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1 \tag{3.14}$$

is also known as superellipse. Obviously p trigonometric identity (3.13) is special case of (3.14) with $a = b = 1$. Hence $\sin_p$ and $\cos_p$ provide possible parametrization of some Lamé curves.

Following formulas for second derivative of $\sin_p$ ensue directly from (3.13).

Proposition 3.7 ([27], Lemma 4.2, p. 106)**.** *For all $p > 1$*

$$\sin_p''(x) = -\sin_p^{p-1}(x)\cos_p^{2-p}(x) \quad \textit{for } x \in \left(0,\frac{\pi_p}{2}\right), \tag{3.15}$$

and

$$\sin_p''(x) = \sin_p^{p-1}(-x)\cos_p^{2-p}(x) \quad \textit{for } x \in \left(-\frac{\pi_p}{2},0\right).$$

Relation (3.15) plays key role in our work since it enables us to express n-th derivative of $\sin_p(x)$ for $x \in (0,\pi_p/2)$ in the form

$$\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x)\cos_p^{1-q_{k,n}}(x), \tag{3.16}$$

for any $n \in \mathbb{N}$, $n \geq 2$. We assume for the rest of Section 3.1 that n denotes the order of derivative and it satisfies above stated restriction. The method, how to find an appropriate real numbers $a_{k,n}$ and $q_{k,n}$, will be shown later. We are able to prove following theorems concerning differentiability of $\sin_p$ using (3.16) and Propositions 3.4 and 3.7.

Proposition 3.8 ([27], Lemma 4.3, p. 107)**.** *Let $p \in \mathbb{R} \setminus \{2\}$ such that $p > 1$.*

1. *If $p > 2$, then the function $\sin_p(\cdot) \in C^1(\mathbb{R})$ and $\sin_p(\cdot) \notin C^2(\mathbb{R})$.*
2. *If $p \in (1,2)$, then the function $\sin_p(\cdot) \in C^2(\mathbb{R})$ and $\sin_p(\cdot) \notin C^3(\mathbb{R})$.*

Theorem 3.9 ([27], Theorem 3.1, p. 105)**.** *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then*

$$\sin_{2(m+1)}(\cdot) \in C^\infty\Big(-\frac{\pi_{2(m+1)}}{2},\frac{\pi_{2(m+1)}}{2}\Big).$$

Theorem 3.10 ([27], Theorem 3.2, p. 105)**.** *Let $p \in \mathbb{R} \setminus \{2m\}$, $m \in \mathbb{N}$, $p > 1$. Then*

$$\sin_p(\cdot) \in C^{\lceil p \rceil}(-\pi_p/2,\pi_p/2),$$

but

$$\sin_p(\cdot) \notin C^{\lceil p \rceil + 1}(-\pi_p/2,\pi_p/2).$$

Theorem 3.11. *Let* $p \in (1,2)$.

1. *If* $p' \notin \mathbb{N}$, *then* $\sin_p(\cdot) \in C^{\lceil p' \rceil - 1}(0,\pi_p)$, *but* $\sin_p(\cdot) \notin C^{\lceil p' \rceil}(0,\pi_p)$.
2. *If* p' *is odd, then* $\sin_p(\cdot) \in C^{p'-1}(0,\pi_p)$, *but* $\sin_p(\cdot) \notin C^{p'}(0,\pi_p)$.
3. *If* p' *is even, then* $\sin_p(\cdot) \in C^{\infty}(0,\pi_p)$.

Let us recall that $1/p + 1/p' = 1$. Above stated results are summarized in Table 3.1. All theorems are proved in [27] except Theorem 3.11. Nevertheless, the proof proceeds in the same steps as the proofs of Theorem 3.9 and Theorem 3.10. Let us provide a main ideas of the proof. At first, it is necessary to introduce some notions from formal languages.

p	**x in**	$(0,\frac{\pi_p}{2})$	$(-\frac{\pi_p}{2},\frac{\pi_p}{2})$	$\mathbb{R}$	$(0,\pi_p)$
$p = 2$		C^∞	C^∞	C^∞	C^∞
$p = 2k$	$k \in \mathbb{N} \setminus \{1\}$	C^∞	C^∞	C^1	C^1
$p = 2k+1$	$k \in \mathbb{N}$	C^∞	C^p	C^1	C^1
$p \in \mathbb{R} \setminus \mathbb{N}$	$p > 2$	C^∞	$C^{\lceil p \rceil}$	C^1	C^1
$p \in (1,2)$	$p' \notin \mathbb{N}$	C^∞	C^2	C^2	$C^{\lceil p' \rceil - 1}$
$p \in (1,2)$	p' odd	C^∞	C^2	C^2	$C^{p'-1}$
$p \in (1,2)$	p' even	C^∞	C^2	C^2	C^∞

Table 3.1: The order of differentiability of $\sin_p$.

Definition 3.12. (Salomaa and Soittola [54], I.2, p. 4, and/or Manna [43], p. 2–3, p. 47, and p. 78) An *alphabet* (denoted by V) is a finite nonempty set of letters. A *word* (denoted by w) over an alphabet V is a finite string of zero or more letters from the alphabet V. The word consisting of zero letters is called the *empty word*. The set of all words over an alphabet V is denoted by V^* and the set of all nonempty words over an alphabet V is denoted by V^+. For strings w_1 and w_2 over V, their juxtaposition $w_1 w_2$ is called *catenation* of w_1 and w_2, in operator notation $\mathrm{cat} : V^* \times V^* \to V^*$ and $\mathrm{cat}(w_1,w_2) = w_1 w_2$. We also define the length of the word w, in operator notation $\mathrm{len} : V^* \to \mathbb{N} \cup \{0\}$, which for a given word w yields the number of letters in w when each letter is counted as many times as it occurs in w. We also use *reverse function* $\mathrm{rev} : V^* \to V^*$ which reverses the order of the letters in any word w (see [43, p. 47, p. 78]).

We consider the alphabet $V = \{0,1\}$ and the set of all nonempty words V^+. Thus words in V^+ are, e.g.

$$\text{“}\mathtt{0}\text{”},\text{“}\mathtt{1}\text{”},\text{“}\mathtt{01}\text{”},\text{“}\mathtt{10}\text{”},\text{“}\mathtt{11}\text{”}\ldots.$$

For instance, cat(“1110”,“011”) = “1110011”, and

$$\begin{aligned} \operatorname{rev}(\text{“}\mathtt{010011000}\text{”}) &= \text{“}\mathtt{000110010}\text{”}\,, \\ \operatorname{len}(\text{“}\mathtt{010011000}\text{”}) &= 9\,. \end{aligned}$$

Symbol $(k)_{2,n-2}$ denotes the string of the length $n-2$ which represents binary expansion of any $k \in \mathbb{N} \cup \{0\}$ (it means, e.g. for $k = 5$ and $n = 6$ $(5)_{2,4} =$“0101”). We also define a set

$$T \stackrel{\text{def}}{=} \{a \sin_p^q(\cdot) \cos_p^{1-q}(\cdot) :\ a,q \in \mathbb{R}\}$$

of functions defined on $(0,\pi_p/2)$. One can easy see that $T \subset C^\infty(0,\pi_p/2)$ since $\sin_p(x) > 0$ and $\cos_p(x) > 0$ for any $x \in (0,\pi_p/2)$ by Proposition 3.4. It remains to introduce symbolic operators (rewriting rules) $\mathrm{D_s}\colon T \to T$ and $\mathrm{D_c}\colon T \to T$, as follows:

$$\mathrm{D_s}\, a \sin_p^q(\cdot) \cos_p^{1-q}(\cdot) = \begin{cases} aq \sin_p^{q-1}(\cdot) \cos_p^{1-(q-1)}(\cdot)\,, & q \neq 0 \text{ and } a \neq 0\,, \\ 0\,, & \text{otherwise}\,, \end{cases} \tag{3.17}$$

and

$$\mathrm{D_c}\, a \sin_p^q(\cdot) \cos_p^{1-q}(\cdot) = \begin{cases} -a(1-q) \sin_p^{q+p-1}(\cdot) \cos_p^{1-(q+p-1)}(\cdot)\,, & q \neq 1 \text{ and } a \neq 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.18}$$

Obviously

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \mathrm{D_s}\, f(x) + \mathrm{D_c}\, f(x)$$

for any $f \in T$. The application with $f = \sin_p^{q_0}(\cdot) \cos_p^{1-q_0}(\cdot) + \sin_p^{q_0+p}(\cdot) \cos_p^{1-q_0-p}(\cdot)$, $q_0 \in \mathbb{R} \setminus \{0,1\}$, is visualized on Figure 3.2. The special cases $q_0 = 1$ and $q_0 = 0$ are shown on Figures 3.3 and 3.4, respectively. Now we are ready to define a composition $\mathrm{D}_{k,n}$, $k \in \mathbb{N}$, $0 \leq k \leq 2^{n-2} - 1$, of $n-2$ symbolic operators (rewriting rules) $\mathrm{D_s}$ and $\mathrm{D_c}$. The procedure takes two steps:

Step 1 We create an ordered $n-2$-tuple $d_{k,n-2} \in \{\mathrm{D_s}\,, \mathrm{D_c}\}^{n-2}$ (cartesian product of sets $\{\mathrm{D_s}\,, \mathrm{D_c}\}$ of length $n-2$) from $\operatorname{rev}((k)_{2,n-2})$ such that for $1 \leq i \leq n-2$, $d_{k,n-2}$ contains $\mathrm{D_s}$ on the i-th position if $\operatorname{rev}((k)_{2,n-2})$ contains “0” on the i-th position, and $d_{k,n}$ contains $\mathrm{D_c}$ on the i-th position if $\operatorname{rev}((k)_{2,n-2})$ contains “1” on the i-th position (e.g. we obtain $d_{5,4} = (\mathrm{D_c}\,, \mathrm{D_s}\,, \mathrm{D_c}\,, \mathrm{D_s})$ for $k = 5$ and $n = 6$).

Step 2 We define $\mathrm{D}_{k,n}$ as the composition of operators $\mathrm{D_s}$, $\mathrm{D_c}$ in the order they appear in the ordered $n-2$-tuple $d_{k,n-2}$ (e.g. we obtain $\mathrm{D}_{5,6} = (\mathrm{D_c} \circ \mathrm{D_s} \circ \mathrm{D_c} \circ \mathrm{D_s})$ for $k=5$ and $n=6$).

Let us point out that it is possible to recover the index k from the positions of $\mathrm{D_c}$ in $\mathrm{D}_{k,n}$. We will denote by $j(k) \geq 0$ the number of $\mathrm{D_c}$ in $\mathrm{D}_{k,n}$ and, if $j(k) \neq 0$, we denote by $i_1, i_2, \ldots, i_{j(k)}$ its positions counted from back (i.e. in the order of application of $\mathrm{D_s}$ and/or $\mathrm{D_c}$). Then

$$k = 2^{n-2-(i_1-1)} + 2^{n-2-(i_2-1)} + \ldots + 2^{n-2-(i_{j(k)}-1)} . \tag{3.19}$$

If $j(k)=0$, $k=0$. We can prove by induction that

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} \mathrm{D}_{k,n} \sin_p''(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x) , \qquad x \in \left(0, \frac{\pi_p}{2}\right) ,$$

and it follows from (3.17) and (3.18) that

$$q_{k,n} = j(k)(p-1) + (n-2-j(k))(-1) + p - 1 . \tag{3.20}$$

Coefficient $a_{k,n}$ corresponding to $\mathrm{D}_{k,n}$ is obtained by recursion with base case $a_0 = -1$ and inductive clause

$$a_{i+1} = \begin{cases} q_i \cdot a_i & \text{if } \mathrm{D_s} \text{ is applied} , \\ -(1-q_i) a_i & \text{if } \mathrm{D_c} \text{ is applied} , \end{cases} \tag{3.21}$$

$i \in \mathbb{N}$, $0 \leq i \leq n-2$. The exponent q_i can be obtained from (3.20) with j equals to the number of $\mathrm{D_c}$ occurred on the last i positions in $\mathrm{D}_{k,n}$. The alternative way to obtain q_i is to use recursive formula with base case $q_0 = p-1$ and inductive clause

$$q_{i+1} = \begin{cases} q_i - 1 & \text{if } \mathrm{D_s} \text{ is applied} , \\ q_i + p - 1 & \text{if } \mathrm{D_c} \text{ is applied} . \end{cases} \tag{3.22}$$

For $\mathrm{D}_{5,6}$, we obtain $q_1 = p-2$, $q_2 = 2p-3$, $q_3 = 2p-4$, and $q_4 = 3p-5$ and hence

$$\begin{aligned} a_{5,6} &= (-1)(p-1)(-1)(1-(p-2))(2p-3)(-1)(1-(2p-4)) \\ &= (-1)^{j(k)+1}(p-1)(3-p)(2p-3)(5-2p) . \end{aligned}$$

Finally we get to the problem of differentiability. It easily follows from (3.16) that $\sin_p(x) \in C^\infty(0, \pi_p/2)$ since $\sin_p(x) > 0$ and $\cos_p(x) > 0$ for any $x \in (0, \pi_p/2)$ by Proposition 3.4. Due to Proposition 3.4, Part 1, the same statement holds for any interval $I \subset \mathbb{R}$, which has empty intersection with the set $\{k\pi_p/2,\ k \in \mathbb{Z}\}$. Continuity in the points $x = 0$ and $x = \pi_p/2$ depends only on terms in (3.16), where $q_{k,n} \leq 0$ and $1 - q_{k,n} \leq 0$, respectively. Otherwise

$$\lim_{x\to 0+} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x) = 0 \tag{3.23}$$

and

$$\lim_{x\to \pi_p/2-} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x) = 0\,. \tag{3.24}$$

Continuity at $x = 0$: Let $p > 1$ be an integer. Than it can be proved (see [27, Lemma 4.6, p. 113]) that $q_{k,n} \in \mathbb{N} \cup \{0\}$ or $a_{k,n} = 0$ for any $0 \leq k \leq 2^{n-2} - 1$. Hence the possible issue may occurs for

$$n = (j(k) + 1)p + 1 \tag{3.25}$$

by the condition $q_{k,n} = 0$. Otherwise

$$\lim_{x\to 0+} \sin_p^{(n)}(x) = 0$$

by (3.23),

$$\lim_{x\to 0-} \sin_p^{(n)}(x) = 0 = \lim_{x\to 0+} \sin_p^{(n)}(x)$$

by the oddness or evenness of $\sin_p^{(n)}(\cdot)$, and

$$\sin_p^{(n)}(0) = \lim_{x\to 0+} \sin_p^{(n)}(x)$$

by the definition of the first derivative and *L'Hôspital's rule.*

Let $n = p + 1$ for a moment. Then $q_{0,p+1} = 0$ by (3.20). Moreover it is the first order of derivative of $\sin_p$ where the discontinuity may occur by (3.25). If p is odd, function $\sin_p^{(p+1)}(x)$ has jump discontinuity at $x = 0$, since it is odd function and

$$\lim_{x\to 0+} a_{0,p+1} \sin_p^{q_{0,p+1}}(x) \cos_p^{1-q_{0,p+1}}(x) = a_{0,p+1} = -(p-1)! > 0\,.$$

It is the only nonzero limit by (3.20). Otherwise, $\sin_p^{(p+1)}(\cdot)$ is even function for p even. Hence $\sin_p^{(p+1)}(\cdot)$ is continuous since the only nonzero limit

$$\lim_{x\to 0+} a_{0,p+1} \sin_p^{q_{0,p+1}}(x) \cos_p^{1-q_{0,p+1}}(x) = a_{0,p+1} = -(p-1)! < +\infty$$

and

$$\sin_p^{(p+1)}(0) = \lim_{x\to 0+} \sin_p^{(n)}(x)$$

by the definition of the first derivative and *L'Hôspital's rule* again. Let p be even and $n = ip+1$, $i \in \mathbb{N} \setminus \{1\}$. Then n is odd and hence $\sin_p^{(ip+1)}(\cdot)$ is even function. Moreover

$$\lim_{x\to 0+} \sin_p^{(ip+1)}(x) = \sum_{k=0}^{2^{ip-1}-1} a_{k,ip+1} \lim_{x\to 0+} \sin_p^{q_{k,ip+1}}(x) \cos_p^{1-q_{k,ip+1}}(x)$$

$$= \sum_{\substack{k=0\\ q_{k,ip+1}=0}}^{2^{ip-1}-1} a_{k,ip+1} < +\infty\,.$$

Hence it can be prove by induction that $\sin_p^{(n)}(\cdot)$ is continuous for any $n \in \mathbb{N}$ provided p is an even integer.

Let us assume now that p is not an integer. Then the first possibly discontinuous derivative is $\sin_p^{(\lceil p\rceil+1)}(\cdot)$ by the condition $q_{k,n} \le 0$. Actually,

$$\lim_{x\to 0+} \sin_p^{(\lceil p\rceil+1)}(x) = \lim_{x\to 0+} a_{0,\lceil p\rceil+1} \sin_p^{q_{0,\lceil p\rceil+1}}(x) \cos_p^{1-q_{0,\lceil p\rceil+1}}(x) = -\infty$$

since $q_{0,\lceil p\rceil+1} < 0$ and $q_{k,\lceil p\rceil+1} > 0$ for $k \ne 0$ by (3.20).

Continuity at $x = \pi_p/2$: For $p > 2$,

$$\lim_{x\to \pi_p/2-} \sin_p''(x) = -\lim_{x\to \pi_p/2-} \sin_p^{p-1}(x) \cos_p^{2-p}(x) = -\infty$$

and, hence, it remains to deal with the case $p \in (1{,}2)$. It follows from (3.24) that $\sin_p^{(n)}(\cdot)$ is continuous provided $q_{k,n} < 1$ for all $0 \le k \le 2^{n-2}-1$. We reformulate (3.20) for $s(k) = n-2-j(k)$ which denotes the number of occurrences of $\mathrm{D_s}$ in $\mathrm{D}_{k,n}$. Then

$$q_{k,n} = (p-1)(n-2-s(k)) + (-1)s(k) + p - 1 \tag{3.26}$$

which follows that

$$q_{k,n} = \frac{1}{p'-1}(n-1-s(k)) - s(k)\,. \tag{3.27}$$

Then, the condition $q_{k,n} \ge 1$ is equivalent to

$$n \ge (s(k)+1)p' \ge p'\,.$$

Hence, $\sin_p(\cdot) \in C^{\lceil p'\rceil-1}(0,\pi_p)$. It is easy to see that

$$q_{2^{\lceil p'\rceil-2}-1,\lceil p'\rceil} = \frac{\lceil p'\rceil - 1}{p'-1} > 1$$

for $p > 1$ such that $p' \notin \mathbb{N}$ and, hence, $\sin_p^{(\lceil p'\rceil)}(\cdot) \notin C(0,\pi_p)$. We recall that $\mathrm{D}_{2^{\lceil p'\rceil-2}-1,\lceil p'\rceil}$ is composition of $\mathrm{D_c}$ only which implies that $s\left(2^{\lceil p'\rceil-2}-1\right) = 0$.

Let p be such that $p' > 2$ is an integer and $n = p'$. It follows that

$$q_{2^{p'-2}-1,p'} = \frac{1}{p'-1}(p'-1) = 1$$

by (3.27). Due to the reflection $\sin_p(x) = \sin_p(\pi_p - x)$ for $x \in (0,\pi_p/2)$, we obtain that $\sin_p^{(p')}(\cdot)$ is continuous for p' even and it has jump at $x = \pi_p/2$ for p' odd.

We prove that $q_{k,n} \leq 1$ or $a_{k,n} = 0$ for any $n \in \mathbb{N}$ and any $0 \leq k \leq 2^{n-2} - 1$. At first, let us show that

$$q_{k,n} \in \left\{ \frac{l}{p'-1}, l \in \mathbb{Z}, l \leq n-1 \right\} \tag{3.28}$$

for all $n \in \mathbb{N}$ and all $0 \leq k \leq 2^{n-2} - 1$. Indeed, let $l_k = n - 1 - s(k)p'$. Then $l_k \in \mathbb{Z}$ for all $0 \leq k \leq 2^{n-2} - 1$ and

$$q_{k,n} = \frac{1}{p'-1}(n-1-s(k)) - s(k) = \frac{n-1-s(k)p'}{p'-1} = \frac{l_k}{p'-1}$$

by (3.27). The proof continues by induction on n. The first exponent $q_{0,2} = p - 1 \leq 1$. Let $q_{k_0,n_0} \leq 1$ and apply $\mathrm{D_s}$ and $\mathrm{D_c}$ to obtain q_{2k_0,n_0+1} and q_{2k_0+1,n_0+1}, respectively. Then, $q_{2k,n+1} \leq 0$ by (3.17). The application of $\mathrm{D_c}$ should be divided into two cases. Let $q_{k_0,n_0} = 1$ at first. Then, $a_{2k_0+1,n_0+1} = 0$ by (3.18). If $q_{k_0,n_0} < 1$, then $q_{k_0,n_0} \leq (p'-2)/(p'-1)$ by (3.28). Hence,

$$q_{2k_0+1,n_0+1} = q_{k_0,n_0} + p - 1 = q_{k_0,n_0} + \frac{1}{p'-1} \leq 1\,.$$

It follows from (3.27) that $q_{k,n} = 1$ if and only if $n = ip'$, $i \in \mathbb{N}$. Henceforth, the proof takes similar steps as in the previous case. Let us emphasise that p and p' are even integers simultaneously only if $p = 2$.

3.2. Macluarin Series

Paredes and Uchiyama [52] proved that there exists analytic function $F(y)$ such that

$$\sin_p(x) = xF\left(|x|^p\right) = \sum_{n=0}^{+\infty} \alpha_n x|x|^{np} \tag{3.29}$$

on some neighbourhood of the origin, say on $(-\varepsilon,\varepsilon)$. We will assume that $p > 2$ is an integer. The goal of this section is to derive an explicit formula for the coefficients α_n and the radius of convergence of generalized Maclaurin series (3.29). It follows from (3.29) that there exist Maclaurin series

$$M_p(x) = \sum_{i=0}^{+\infty} \bar{\alpha}_i x^i$$

with $\bar{\alpha}_i = 0$ for $i \neq np+1$ for any $n \in \mathbb{N}\cup\{0\}$, which is convergent on some neighbourhood of origin such that

$$\sin_p(x) = M_p(x)$$

on $[0,\varepsilon)$. Hence

$$\bar{\alpha}_{np+1} = \frac{\lim_{x\to 0+} \sin_p^{(np+1)}(x)}{(np+1)!}$$

and, in particular, $\alpha_n = \bar{\alpha}_{np+1}$ due to the uniqueness of the coefficients of Maclaurin series and oddness of $\sin_p$. Maclaurin series of inverse function

$$\operatorname{arcsin}_p(x) = \sum_{j=0}^{+\infty} \beta_j x^j$$

with

$$\beta_j = \begin{cases} \frac{\Gamma\left(i+\frac{1}{p}\right)}{(ip+1)\Gamma\left(\frac{1}{p}\right)i!} & \text{if } j = ip+1 \text{ for some } i \in \mathbb{N}\cup\{0\}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

converge on $[0,1)$. Hence

$$\sin_p(\operatorname{arcsin}_p(x)) = \sum_{i=0}^{+\infty} c_i x^i\,, \tag{3.30}$$

on $[0,\delta)$ for some $\delta > 0$. Here

$$c_i = \sum_{\substack{k\in\mathbb{N}, j_1, j_2, j_3, \ldots, j_k \in \mathbb{N} \\ j_1+j_2+j_3+\ldots+j_k = i}} \alpha_k \cdot \beta_{j_1} \cdot \beta_{j_2} \cdot \beta_{j_3} \cdot \ldots \cdot \beta_{j_k}\,.$$

Moreover,

$$c_i = \begin{cases} 1\,, & i = 1\,, \\ 0\,, & \text{otherwise}\,, \end{cases}$$

since $\sin_p(\operatorname{arcsin}_p(x)) = x$ and the series on right hand side of (3.30) converges for some $x \neq 0$. It follows that the series in (3.30) converges for any $x \in \mathbb{R}$ and, in particular,

$$1 = \sum_{i=0}^{+\infty} c_i = \sum_{i=0}^{+\infty} \sum_{\substack{k\in\mathbb{N}, j_1, j_2, j_3, \ldots, j_k \in \mathbb{N} \\ j_1+j_2+j_3+\ldots+j_k = i}} \alpha_k \cdot \beta_{j_1} \cdot \beta_{j_2} \cdot \beta_{j_3} \cdot \ldots \cdot \beta_{j_k}\,,$$

where $\beta_j \geq 0$, $j \in \mathbb{N}$, $\alpha_1 = \lim_{x\to 0+} \cos_p(x) = 1$, and $\alpha_k \leq 0$, $k \geq 2$. The latter inequality follows from

$$\sin_p^{(n)}(x) \leq 0 \quad \text{on} \quad \left(0, \frac{\pi_p}{2}\right)$$

which is proved in [27, Lemma 4.7, p. 114] for any $n \in \mathbb{N}$, $n \geq 2$. Hence, all positive terms in

$$\sum_{i=0}^{+\infty} \sum_{\substack{k\in\mathbb{N}, j_1,j_2,j_3,\ldots,j_k\in\mathbb{N}\\ j_1+j_2+j_3+\ldots+j_k=i}} \alpha_k \cdot \beta_{j_1} \cdot \beta_{j_2} \cdot \beta_{j_3} \cdot \ldots \cdot \beta_{j_k}\,, \tag{3.31}$$

satisfy $k = 1$ and $j_1 = i$. In particular

$$\sum_{i=0}^{+\infty} \alpha_1 \beta_i = \operatorname{arcsin}_p(1) = \frac{\pi_p}{2}$$

and

$$\sum_{i=0}^{+\infty} \sum_{\substack{k\geq 2, j_1,j_2,j_3,\ldots,j_k\in\mathbb{N}\\ j_1+j_2+j_3+\ldots+j_k=i}} \alpha_k \cdot \beta_{j_1} \cdot \beta_{j_2} \cdot \beta_{j_3} \cdot \ldots \cdot \beta_{j_k} = 1 - \frac{\pi_p}{2}\,.$$

It follows that (3.31) converges absolutely and also any rearrangement of any subseries has to converge absolutely. It means that

$$\sum_{n=0}^{+\infty} \alpha_n (\sum_{j=0}^{M} \beta_j)^n$$

converges absolutely for any $M \in \mathbb{N}$ because it is rearranged subseries of (3.31). Since $\lim_{M\to+\infty} \sum_{j=0}^{M} \beta_j = \pi_p/2$ and b_j is positive for any $j \in \mathbb{N}$, series $M_p(x)$ converges absolutely on $(-\pi_p/2,\pi_p/2)$. That series converges toward $\sin_p$ by (3.29) provided p is even, but it does not converge toward $\sin_p$ for p odd. These results are summarized in following two theorems:

Theorem 3.13 ([27], Theorem 3.3, p. 106)**.** *Let $p = 2(m + 1)$ for $m \in \mathbb{N}$. Then the Maclaurin series $M_p(\cdot)$ of $\sin_p(\cdot)$ converges on $(-\pi_p/2,\pi_p/2)$.*

Theorem 3.14 ([27], Theorem 3.4, p. 106)**.** *Let $p = 2m + 1$, $m \in \mathbb{N}$. Then the formal Maclaurin series $M_p(\cdot)$ converges on $(-\pi_p/2,\pi_p/2)$. Moreover, the formal Maclaurin series $M_p(\cdot)$ converges towards $\sin_p(\cdot)$ on $[0,\pi_p/2)$, but does not converge towards $\sin_p(\cdot)$ on $(-\pi_p/2,0)$.*

We are also able to provide the explicit formula for the coefficient α_n. We omit terms with zero coefficients α_n in Theorem 3.15 and change the notation such that α_1 is the first nonzero coefficient, α_2 is the second nonzero coefficient, etc.

Theorem 3.15. *Let $p > 2$ be an integer and*

$$\sin_p(x) = \sum_{n=0}^{+\infty} \alpha_n x|x|^{np}\,, \quad x \in \left(-\frac{\pi_p}{2},\frac{\pi_p}{2}\right). \tag{3.32}$$

Then $\alpha_0 = 1$, $\alpha_1 = -\dfrac{1}{p(p+1)}$, *and for* $n \geq 2$,

$$\begin{aligned}\alpha_n = \frac{(-1)^n}{(np+1)!} &\sum_{\substack{i_1=1\\ i_1\neq p-1}}^{p} \sum_{\substack{i_2=i_1+1\\ i_2\neq 2p-1}}^{2p} \cdots \sum_{\substack{i_{n-1}=i_{n-2}+1\\ i_{n-1}\neq (n-1)p-1}}^{(n-1)p} \\ &\left[\prod_{m_1=1}^{i_1-1} (p-1-(m_1-1))\right] (1-(p-1-(i_1-1)) \\ &\cdot \left[\prod_{m_2=i_1+1}^{i_2-1} (2(p-1)-(m_2-2))\right] (1-(2(p-1)-(i_2-2))) \cdot \ldots \cdot \\ &\cdot \left[\prod_{m_{n-1}=i_{n-2}+1}^{i_{n-1}-1} ((n-1)(p-1)-(m_{n-1}-(n-1)))\right] \\ &\cdot (1-((n-1)(p-1)-(i_{n-1}-(n-1)))) \, [n(p-1)-(i_{n-1}-n+1)]! \, .\end{aligned} \tag{3.33}$$

The proof of Theorem 3.15 is very technical and we refer the curious reader to [34]. We only sketch the proof here. Since

$$\lim_{x\to 0+} \sin_p^{(np+1)}(x) = \sum_{\substack{k=0\\ q_{k,np+1}=0}}^{2^{np-1}-1} a_{k,np+1} \, ,$$

it is important to characterize k, $0 \leq k \leq 2^{np-1}-1$ such that $q_{k,np+1} = 0$. We use (3.19), i.e.

$$k = 2^{np-1-(i_1-1)} + 2^{np-1-(i_2-1)} + \ldots + 2^{np-1-(i_{n-1}-1)}.$$

The number of application of $\mathrm{D_c}$ in $\mathrm{D}_{k,np+1}$ is exactly $n-1$ which follows from the condition $q_{k,np+1} = 0$ (see (3.20)). Then we find all allowable composed operators $\mathrm{D}_{k,np+1}$ with exactly $n-1$ operators $\mathrm{D_c}$ (we exclude the composed operators when we should apply $\mathrm{D_c}$ on the term with $q = 1$, see (3.18)). Then we used recursive formulas (3.21) and (3.22) to obtain desired $a_{k,np+1}$ for any k such that $q_{k,np+1} = 0$.

Let us note that above stated procedure is not the only way how to get the coefficients for series (3.29). One can use inverse of Maclaurin series of $\arcsin_p(\cdot)$ as was shown at the beginning of Chapter 3. In our approach, the evaluation of the coefficients does not involve a numerical inversion which is usually expensive operation in terms of computational effort.

3.3. Extension to Complex Domain

This section is devoted to extension of $\sin_p$ to complex domain. The only attempt (even formal) known by us was done by Lindqvist [40] who proposed a definition of $\sin_p$ as the solution of

$$\frac{\mathrm{d}}{\mathrm{d}z}(w')^{p-1} + w^{p-1} = 0 \quad w(0) = 0\,, \quad w'(0) = 1$$

in complex domain for $p > 1$. But Lindqvist himself conjectured that real function $\sin_p(x)$ and complex function $w(z)$ of real variable could be different. Let us note that we use x for real variable and z for complex variable in this section. We define complex function $\sin_p(z)$ on $B_p \overset{\text{def}}{=} \{z \in \mathbb{C}\colon |z| < \pi_p/2\}$ naturally by its Maclaurin series provided $p \in \mathbb{N}$, $p > 2$.

Definition 3.16. Let $p \in \mathbb{N}$, $p > 2$, and $z \in B_p$. Then

$$\sin_p(z) = \sum_{n=0}^{+\infty} \alpha_n z^{np+1}\,, \tag{3.34}$$

where

$$\alpha_n = \lim_{x \to 0+} \frac{\sin_p^{(np+1)}(x)}{(np+1)!}\,.$$

We also define

$$\cos_p(z) = \frac{\mathrm{d}}{\mathrm{d}z}\sin_p(z)\,.$$

Then the following question arises: "Do above defined functions satisfy initial value problem (3.7) in the sense of differential equations in complex domain?" It was O. Došlý who motivated our work by asking the question. Let us consider initial value problem

$$\begin{cases} -(u')^{p-2}u'' - u^{p-1} &= 0\,, \\ u(0) &= 0\,, \\ u'(0) &= 1 \end{cases} \tag{3.35}$$

in complex domain.

Theorem 3.17 ([28],Theorem 2.1, p. 226)**.** *Let $p = 2(m+1)$, $m \in \mathbb{N}$, then the unique solution of the initial value problem* (3.35) *on B_p is the function* $\sin_p(z)$*.*

Theorem 3.18 ([28], Theorem 3.1, p. 229)**.** *Let $p = 2m+1$, $m \in \mathbb{N}$. Then the unique solution $u(z)$ of the complex initial value problem* (3.35) *differs from the solution* $\sin_p(x)$ *of the Cauchy problem* (3.7) *for $z = x \in (-\pi_p/2, 0)$.*

The proofs of Theorems 3.17 and 3.18 are based on the fact that (3.35) possesses the unique analytic solution at least on some small disk centered at the origin. The solution can be written as a power series which coincides with the series on the right hand side of (3.34) since, in real domain, (3.35) is equivalent to (3.7) for $x > 0$. Nevertheless, the real restriction of the series on the right hand side of (3.34) to real axis,

$$\sum_{n=0}^{+\infty} \alpha_n x^{np+1},$$

does not converge toward $\sin_p(x)$ even in real domain by Theorem 3.14 provided p is odd.

There are some interesting results concerning $\sin_p(z)$ in complex domain. At first let us mention that there is no hope for $\sin_p(z)$ to be entire function for $p \in \mathbb{N}$, $p > 2$, since it satisfies an analogy of p-trigonometric identity (3.13), i.e.

$$\cos_p^p(z) + \sin_p^p(z) = 1$$

on some disc D_r, $r > 0$, in complex domain centered at the origin (see [29, Lemma 13, p. 4]). Then we can use Proposition 3.19 which confirms our statement.

Proposition 3.19 ([17], Theorem 12.20 on p. 433)**.** *Let f and g be entire functions and for some positive integer n satisfy identity*

$$f^n + g^n = 1\,.$$

1. *If $n = 2$, then there is an entire function h such that $f = \cos \circ h$, $g = \sin \circ h$.*
2. *If $n > 2$, then f and g are each constant.*

Further, let us consider following initial value problem

$$\begin{cases} (u')^{p-2}u'' - u^{p-1} &= 0\,, \\ u(0) &= 0\,, \\ u'(0) &= 1\,. \end{cases} \tag{3.36}$$

Then there exists $r > 0$ such that (3.36) has the unique solution on complex disc $D_p \stackrel{\text{def}}{=} \{z \in \mathbb{C} \colon |z| < r\}$ by [29, Lemma 16, p. 6] and the following definition makes sense.

Definition 3.20. Let $p \in \mathbb{N}$,$p > 2$. The complex function $\sinh_p(z)$ is defined on D_p as the unique solution of the initial-value problem (3.36) and $\cosh_p(z) \stackrel{\text{def}}{=} \sinh_p'(z)$ for all $z \in D_p$.

Then we are able to prove an analogy to well known formula $\sin(z) = -\mathrm{i}\sin(\mathrm{i}z)$.

Theorem 3.21 ([29], Theorem 20, p. 6)**.** *Let $p = 4l + 2$, $l \in \mathbb{N}$. Then*

$$\begin{aligned} \sin_p(z) &= -\mathrm{i}\sinh_p(\mathrm{i}z)\,, &(3.37)\\ \cos_p(z) &= \cosh_p(\mathrm{i}z) &(3.38)\end{aligned}$$

for all $z \in B_p$. Moreover,

$$\sinh_p(z) = \sum_{k=0}^{\infty}(-1)^k\alpha_k z^{kp+1}\,. \tag{3.39}$$

Let us note that the coefficients of Maclaurin series of $\sinh_p$ alternates sign whereas the coefficients of Maclaurin series of $\sin_p$ are negative except the first provided $p = 4l + 2$, $l \in \mathbb{N}$. It follows from (3.39) and the fact that $\alpha_1 = 1$ and $\alpha_k \le 0$ for $k \ge 2$.

Theorem 3.22 ([29], Theorem 21, p. 6)**.** *Let $p = 4l$, $l \in \mathbb{N}$. Then*

$$\begin{aligned} \sin_p(z) &= -\mathrm{i}\sin_p(\mathrm{i}z)\,, &(3.40)\\ \cos_p(z) &= \cos_p(\mathrm{i}z) &(3.41)\end{aligned}$$

for all $z \in B_p$.

The statements of previous Theorems 3.21 and 3.22 are visualized on Figure 3.1.

Theorem 3.23 ([29], Theorem 22, p. 6)**.** *Let $p = 4l$, $l \in \mathbb{N}$. Then*

$$\begin{aligned} \sinh_p(z) &= -\mathrm{i}\sinh_p(\mathrm{i}z)\,, &(3.42)\\ \cosh_p(z) &= \cosh_p(\mathrm{i}z) &(3.43)\end{aligned}$$

for all $z \in D_p$.

3.4. Related Articles of the Author

2014 **Differentiability properties of p-trigonometric functions**; Girg, P., Kotrla, L.; *Electron. J. Differ. Equ. Conf.*; [27]

Abstract:

p-trigonometric functions are generalizations of the trigonometric functions. They appear in context of nonlinear differential equations and also in analytical geometry of the p-circle in the plain. The most important p-trigonometric function is $\sin_p(x)$. For $p > 1$, this function is defined as the unique solution of the initial-value problem

$$\left(|u'(x)|^{p-2}u'(x)\right)' = (p-1)|u(x)|^{p-2}u(x), \quad u(0) = 0,\, u'(0) = 1\,,$$

for any $x \in \mathbb{R}$. We prove that the n-th derivative of $\sin_p(x)$ can be expressed in the form

$$\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x)\,,$$

on $(0,\pi_p/2)$, where $\pi_p = \int_0^1 (1-s^p)^{-1/p}$,ds, and $\cos_p(x) = \sin_p'(x)$. Using this formula, we proved the order of differentiability of the function $\sin_p(x)$. The most surprising (least expected) result is that $\sin_p(x) \in C^\infty(-\pi_p/2,\pi_p/2)$ if p is an even integer. This result was essentially used in the proof of theorem, which says that the Maclaurin series of sinp(x) converges on $(-\pi_p/2,\pi_p/2)$ if p is an even integer. This completes previous results that were known e.g. by Lindqvist and Peetre where this convergence was conjectured.

2015 **Generalized trigonometric functions in complex domain**; Girg, P., Kotrla, L.; *Mathematica Bohemica*; [28]

Abstract:

In this paper we study extension of p-trigonometric functions $\sin_p$ and $\cos_p$ to complex domain. For $p = 4,6,8,\ldots$, the function $\sin_p$ satisfies initial value problem which is equivalent to

$$\begin{cases} -(u')^{p-2}u'' - u^{p-1} &= 0\,, \\ u(0) &= 0\,, \\ u'(0) &= 1 \end{cases} \qquad (*)$$

in $\mathbb{R}$. In our recent paper [27], we showed that $\sin_p(x)$ is a real analytic function for $p = 4,6,8,\ldots$ on $(-\pi_p/2,\pi_p/2)$, where $\pi_p/2 = \int_0^1 (1-s^p)^{-1/p}$. This allows us to extend $\sin_p$ to complex domain by its Maclaurin series convergent on disc $\{z \in \mathbb{C}\colon |z| < \pi_p/2\}$. The question is whether this extensions $\sin_p(z)$ satisfies (*) in the sense of differential equations in complex domain. This interesting question was posed by DOŠLÝ and we show that the answer is affirmative. We also discuss difficulties concerning extension of $\sin_p$ to complex domain for $p = 3,5,7\ldots$. Moreover, we show that the structure of the complex valued initial value problem (*) does not allow entire solutions for any $p \in \mathbb{N}$, $p > 2$. Finally, we provide some graphs of real and imaginary parts of $\sin_p(z)$ and suggest some new conjectures.

2016 **p-trigonometric and p-hyperbolic functions in complex domain**; Girg, P., Kotrla, L.; *Abstr. Appl. Anal.*; [29]

Abstract:
In this paper we study extension of p-trigonometric functions $\sin_p$ and $\cos_p$ and of p-hyperbolic functions $\sinh_p$ and $\cosh_p$ to complex domain. Our aim is to answer the question under what conditions on p these functions satisfy well known relations for usual trigonometric and hyperbolic functions, such as e.g. $\sin(z) = -\mathrm{i} \cdot \sinh(\mathrm{i} \cdot z)$. In particular, we prove in the paper that for $p = 6{,}10{,}14, \ldots$ the p-trigonometric and p-hyperbolic functions satisfy very analogous relations as their classical counterparts. Our methods are based on the theory of differential equations in the complex domain using the Maclaurin series for p-trigonometric and p-hyperbolic functions.

Reviewed by Mehdi Hassani (MR3498048):
"The authors generalize the properties of hyperbolic functions such as the well-known relations $\sin z = -i \sinh(iz)$, $\cos z = \cosh(iz)$, $\cos z = \sin' z$, $\cosh z = \sinh' z$, $\cos^2 z + \sin^2 z = 1$, and $\cosh^2 z - \sinh^2 z = 1$ with $z \in \mathbb{C}$ to have their counterparts for generalized p-trigonometric and p-hyperbolic functions. Also, they provide visualizations of the results obtained by introducing several graphs, including Laméc urves. "

2018 **Maclaurin series for** $\sin_p$ **with** $p > 2$ **be an integer**; Kotrla, L.; *Electron. J. Differential Equations*; [34]

Abstract: We find an explicit formula for the coefficients α_n, $n \in \mathbb{N}$, of the generalized Maclaurin series for $\sin_p$ provided $p > 2$ is an integer. Our method is based on an expression of the n-th derivative of $\sin_p$ in the form

$$\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{p-1}(x) \cos_p^{2-p}(x)\,, \quad x \in \left(0, \frac{\pi_p}{2}\right),$$

where $\cos_p$ stands for the first derivative of $\sin_p$. The formula allows us to compute the nonzero coefficients

$$\alpha_n = \frac{\lim_{x\to 0+} \sin_p^{(np+1)}(x)}{(np+1)!}\,.$$

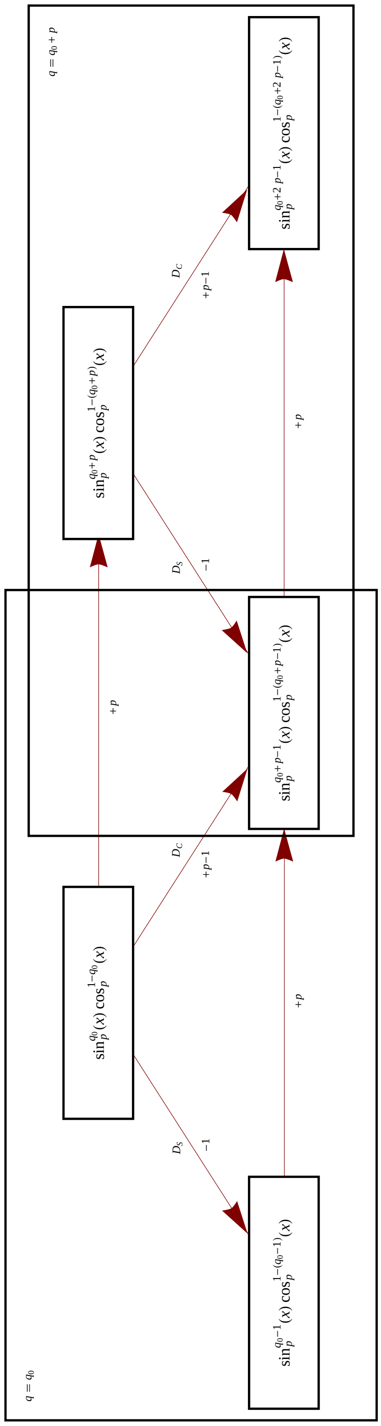

Figure 3.2: Rewriting diagram of the first derivative of $\sin_p^{q_0}(\cdot)\cos_p^{1-q_0}(\cdot) + \sin_p^{q_0+p}(\cdot)\cos_p^{1-q_0-p}(\cdot)$.

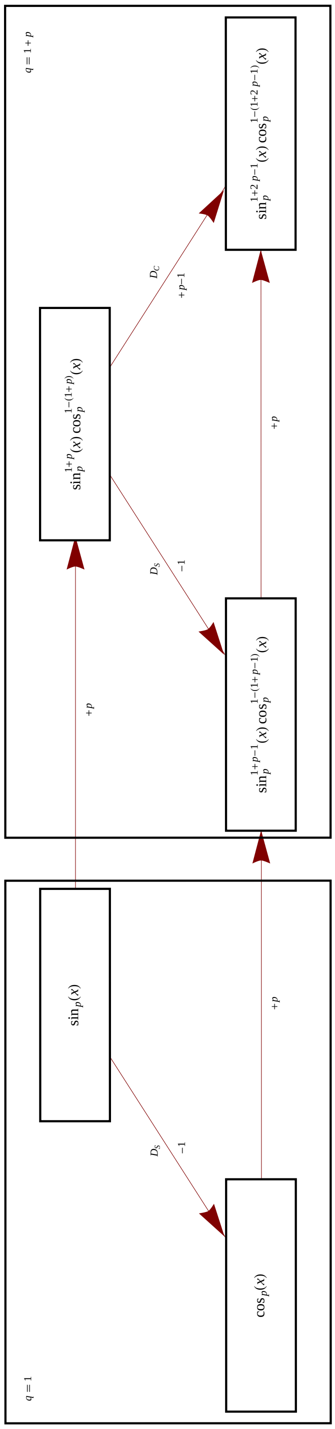

Figure 3.3: Rewriting diagram of the first derivative of $\sin_p(\cdot) + \sin_p^{1+p}(\cdot)\cos_p^{-p}(\cdot)$.

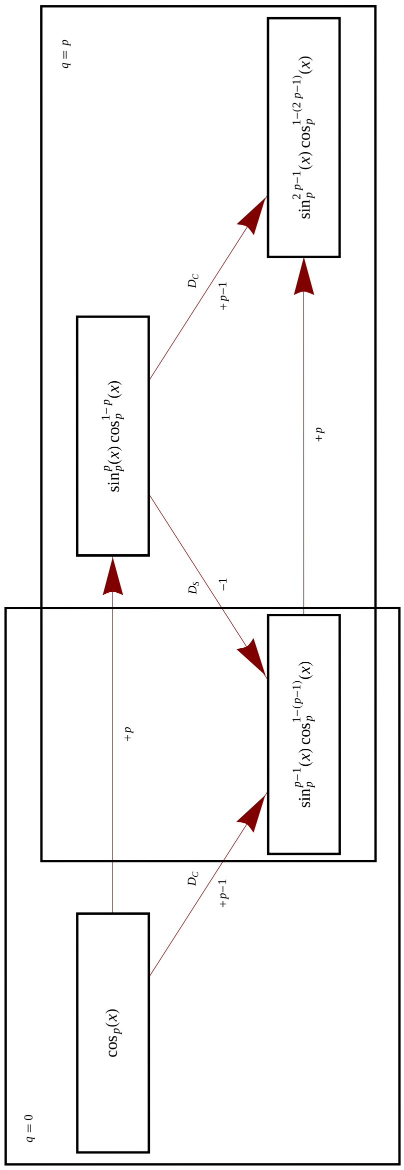

Figure 3.4: Rewriting diagram of the first derivative of $\cos_p(\cdot) + \sin_p^p(\cdot)\cos_p^{1-p}(\cdot)$.

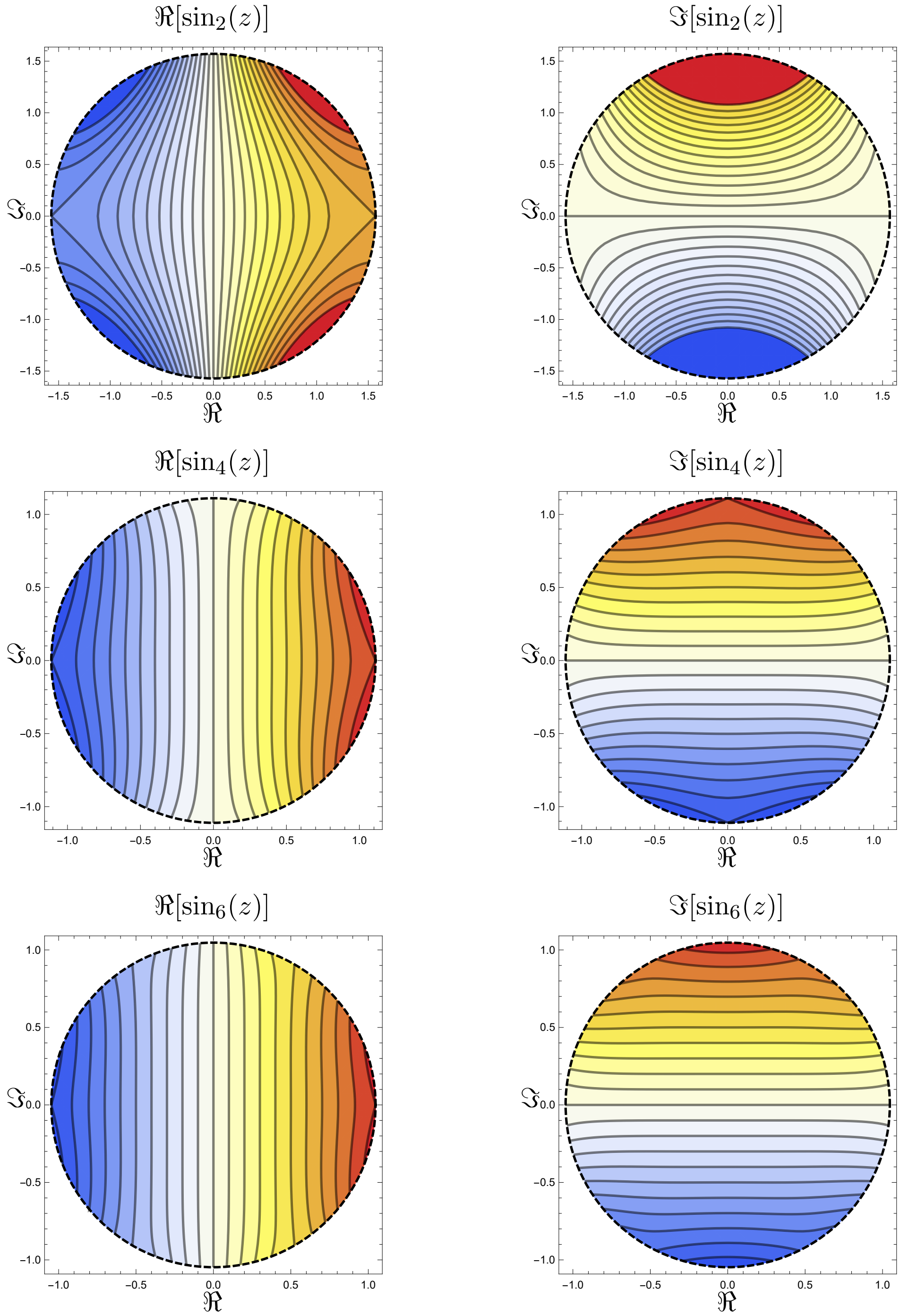

Figure 3.1: Contourlines of real and imaginary parts of $\sin_p(z)$ for $p = 2, 4, 6$.

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[2] The paper Girg and Kotrla [27] has been published before the doctoral study.

Author's active participations on conferences

Talks:

- Nonlinear Analysis Plzeň 2013, Pilsen, Czech Republic, August 24, 2013, *Differentiability properties of the p-trigonometric functions.*[3]
- Equadiff 13, Prague, Czech Republic, August 26–30, 2013, *Differentiability properties of the p-trigonometric functions.*[3]
- XXIX Seminar in Differential Equation, Monínec, Czech Republic, April 14–18, 2014, *One-dimensional p-Laplacian - the First eigenfunction and bifurcation.*[3]
- Studentská konference a Rektorysova soutěž, Czech Technical University in Prague, Prague, Czech Republic, December 3, 2014, *Funkce* $\sin_p$ *v reálném a komplexním oboru.*
- Meeting of Ph.D. Students of Mathematical Analysis and Differential Equations 2016, Mathematical Institute, Czech Academy of Sciences, Prague, Czech Republic, January 25–28, 2016, *The p-Trigonometric and p-Hyperbolic Functions.*
- Czech-Georgian Workshop on Boundray value problems 2016, Mathematical Institute, Czech Academy of Sciences, Brno, Czech Republic,February 8–11, 2016, *Differentiability of* $\sin_p$ *for p integer and its consequences.*
- XXX Seminar in Differential Equations, Tisá, Czech Republic, May 5–June 3, 2016, *Extension profile of the solution of the fast diffusion problem.*
- XXXI Seminar in Differential Equations, Velehrad, Czech Republic, May 21—25, 2018, *Bifurcation of Positive and Negative Continua for Quasilinear ODE Involving Nonlinearities Depending on Derivative.*

[3] I participated in the conference before the doctoral study.

- Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems, Flagstaff, Arizona, USA, June 5–8, 2018, *Multi-bump Solutions in Parabolic Problems with p-Laplacian.*

- 9. Czech-Israeli Workshop, Mathematical Institute, Czech Academy of Sciences, Brno, Czech Republic, July 9–13, 2018, *Nonuniqueness of solutions of initial-value problems for parabolic p-Laplacian and speed of propagation*

Poster:

- Emerging issues in nonlinear elliptic equations: singularities, singular perturbations and non local problems, Bedlewo, Poland, June 18–24 2017, *Bifurcation of Positive and Negative Continua for Quasilinear ODE Involving Nonlinearities Depending on Derivative*

Co-authors' statements

The co-authors' statement

The co-authors

- Vladimir E. Bobkov
- Jiří Benedikt
- Petr Girg
- Peter Takáč

of the paper "Nonuniqueness of Solutions of Initial-value Problems for Parabolic p-Laplacian" (Electron. J. Diff. Equ., Vol. 2015 (2015), No. 38, pp. 1-7). confirm that L. Kotrla significantly contributed to research related to this paper.

Indeed, the results were obtained during series of joint discussions during several exchange stays of the two groups (University of West Bohemia: Jiří Benedikt, Petr Girg, Lukáš Kotrla, and University of Rostock: Vladimir Bobkov, Peter Takáč). It is difficult to precisely describe contribution of each co-author separately, but we can say that the contribution of L. Kotrla is approximately 20% (as it is of every co-author). In particular, he focused on verification that given function is a subsolution.

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of the paper "Nonuniqueness and multi-bump solutions in parabolic problems with the *p*-Laplacian" (J. Differential Equations *260*, 2 (2016), pp. 991 - 1009) confirm that L. Kotrla significantly contributed to research related to this paper.

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of the paper "The strong maximum principle in parabolic problems with the p-Laplacian in a domain" (Appl. Math. Lett. *63* (2017), pp. 95 - 101) confirm that L. Kotrla significantly contributed to research related to this paper.

Indeed, the results were obtained during series of joint discussions during several exchange stays of the two groups (University of West Bohemia: Jiří Benedikt, Petr Girg, Lukáš Kotrla, and University of Rostock: Peter Takáč). It is difficult to precisely describe contribution of each co-author separately, but we can say that the contribution of L. Kotrla is approximately 25% (as it is of every co-author). In particular, he focused on verification that given spherically symmetric wave is a subsolution.

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of the paper "Origin of the *p*-Laplacian and A. Missbach" (Electron. J. Diff. Equ. *2018,* 16 (2018), pp. 1-17) confirm that L. Kotrla significantly contributed to research related to this paper.

Indeed, the results were obtained during series of joint discussions during several exchange stays of the two groups (University of West Bohemia: Jiří Benedikt, Petr Girg, Lukáš Kotrla, and University of Rostock: Peter Takáč). It is difficult to precisely describe contribution of each co-author separately, but we can say that the contribution of L. Kotrla is approximately 25% (as it is of every co-author). In particular, he focused on searching of information and obtaining the literature.

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of the paper "Differentiability properties of *p*-trigonometric functions " (In *Proceedings of the 2012 Conference on Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems* (2014), vol.~21 of Electron. J. Differ. Equ. Conf., Southwest Texas State Univ., San Marcos, TX, pp. 101--127) confirms that L. Kotrla significantly contributed to research related to this paper.

Indeed, the results were obtained during series of joint discussions. It is difficult to precisely describe contribution of each co-author separately, but we can say that the contribution of L. Kotrla is approximately 50%. In particular, he proved that differentiability of sin_p depends on the value of p. Most importantly, he showed that sin_p is infinitely many times differentiable on $(-\pi_p/2, \pi_p/2)$ for p > 2 even. Moreover, he proved a lemma concerning positivity/negativity of any derivative of sin_p for $p > 2$ even, which was used in the proof of convergence of Maclaurin series for sin_p.

Petr Girg

The co-authors' statement

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Indeed, the results were obtained during series of joint discussions. It is difficult to precisely describe contribution of each co-author separately, but we can say that the contribution of L. Kotrla is approximately 50%. In particular, he significantly contributed to the proof that Maclaurin series for sin_p with complex argument satisfies ODE in the sense of differential equations in complex domain.

Petr Girg

The co-authors' statement

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of the paper "p-Trigonometric and p-Hyperbolic Functions in Complex Domain " (Abstr. Appl. Anal (2016), Art. ID 3249439, 1 – 18.) confirms that L. Kotrla significantly contributed to research related to this paper.

Indeed, the results were obtained during series of joint discussions. It is difficult to precisely describe contribution of each co-author separately, but we can say that the contribution of L. Kotrla is approximately 50%. In particular, he focused on two equivalent ways how to obtain its coefficients of Maclaurin series for sin_p. He also significantly contributed to the derivation of the relationship $sin_p(z) = -i\ sinh_p(i\ z)$ ($p = 6, 10, 14, \ldots$) and $sin_p(z) = -i\ sin_p(i\ z)$ ($p = 4, 8, 12, \ldots$) which follows from the fact that Maclaurin series for sin_p solves certain ODE also in the sense of differential equations in complex domain.

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Petr Girg

Appendix A1

[7] Benedikt, J., Bobkov, V. E., Girg, P., Kotrla, L., and Takáč, P. Nonuniqueness of solutions of initial-value problems for parabolic p-Laplacian. *Electron. J. Differential Equations 2015* 38 (2015), 1–7.

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ftp ejde.math.txstate.edu

NONUNIQUENESS OF SOLUTIONS OF INITIAL-VALUE PROBLEMS FOR PARABOLIC p-LAPLACIAN

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Abstract. We construct a positive solution to a quasilinear parabolic problem in a bounded spatial domain with the p-Laplacian and a nonsmooth reaction function. We obtain nonuniqueness for zero initial data. Our method is based on sub- and supersolutions and the weak comparison principle.

Using the method of sub- and supersolutions we construct a positive solution to a quasilinear parabolic problem with the p-Laplacian and a reaction function that is non-Lipschitz on a part of the spatial domain. Thereby we obtain nonuniqueness for zero initial data.

1. Introduction

The problem of *uniqueness* and *nonuniqueness* of solutions to various types of initial (and boundary) value problems for quasilinear parabolic equations has been an interesting research topic for several decades (see, e.g., Fujita and Watanabe [3] and the references therein, Guedda [4], Ladyzhenskaya and Ural'tseva [6], and Oleinik and Kruzhkov [10]).

In this work we focus on the following problem with the p-Laplacian and a (partly) nonsmooth reaction function:

$$\begin{gathered}\frac{\partial u}{\partial t} - \Delta_p u = q(x)|u|^{\alpha-1}u \quad \text{for } (x,t)\in\Omega\times(0,T)\,;\\ u(x,t)=0 \quad \text{for } (x,t)\in\partial\Omega\times(0,T)\,,\\ u(x,0)=0 \quad \text{for } x\in\Omega\,.\end{gathered} \tag{1.1}$$

Here, $\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$ denotes the p-Laplacian for $1<p<\infty$, $\alpha\in(0,1)$ is a given number, $0<T<\infty$, and the potential q satisfies

(Q) $q\in C(\overline{\Omega})$, $q\ge 0$, and $q(x_0)>0$ for some $x_0\in\Omega$.

We assume that $\Omega\subset\mathbb{R}^N$ is a bounded domain with a $C^{1+\mu}$-boundary $\partial\Omega$ where $\mu\in(0,1)$.

In particular, we deal with degenerate (singular) diffusion if $2<p<\infty$ ($1<p<2$, respectively) and the reaction function $f(x,u):=q(x)|u|^{\alpha-1}u$. Notice that if $q(x_0)>0$ then the function $u\mapsto f(x_0,u)$ satisfies neither a local Lipschitz nor

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an Osgood (see [11]) condition near $u = 0$ provided $\alpha \in (0, 1)$. The case $p = 2$ (the Laplace operator) was treated in Fujita and Watanabe [3] by entirely different methods based on the Green's function for the heat equation. An important special case, $N = 1$, $1 < p < \infty$, and $q(x) \equiv \lambda > 0$ (a constant), was treated in Guedda [4] also by different methods. The main purpose of the present article is to fill in the gap left open for $1 < p < \infty$, $p \neq 2$, and $q \in C(\overline{\Omega})$, $q \geq 0$, where q is not necessarily positive everywhere in Ω. Because of this possibly nonuniform positivity of q over Ω, the method used in [4] cannot be applied here. We use a different approach based on sub- and supersolutions and the weak comparison principle. As a trivial consequence of the fact that problem (1.1) possesses a nontrivial non-negative solution (see our main result, Theorem 1), we conlude that the weak comparison principle does not hold for problem (1.1) considered with nontrivial initial conditions, say, in $W_0^{1,p}(\Omega)$.

Observe that our assumption (Q) implies that there exists $R > 0$ such that $q(x) \geq q_0 \equiv \text{const} > 0$ for all $x \in B_R(x_0)$ where

$$B_R(x_0) := \{x \in \mathbb{R}^N : |x - x_0| < R\} \subset \Omega.$$

Let $(\lambda_1, \varphi_{1,R})$ denote the first eigenpair for the operator $-\Delta_p \colon W_0^{1,p}(B_R(x_0)) \to W^{-1,p'}(B_R(x_0))$; that is,

$$\begin{gathered} -\Delta_p \varphi_{1,R} = \lambda_{1,R}\, \varphi_{1,R}^{p-1} \quad \text{in } B_R(x_0)\,; \\ \varphi_{1,R} = 0 \quad \text{on } \partial B_R(x_0)\,, \end{gathered} \tag{1.2}$$

and $\varphi_{1,R} \in W_0^{1,p}(B_R(x_0))$ is normalized by $\varphi_{1,R}(x_0) = 1$. Note that this normalization yields $0 < \varphi_{1,R}(x) \leq 1$ for all $x \in B_R(x_0)$. Moreover, we denote by

$$\widetilde{\varphi}_{1,R}(x) := \begin{cases} \varphi_{1,R}(x) & \text{for } x \in B_R(x_0)\,; \\ 0 & \text{for } x \in \overline{\Omega} \setminus B_R(x_0)\,, \end{cases} \tag{1.3}$$

the natural zero extension of $\varphi_{1,R}$ from $B_R(x_0)$ to the whole of $\overline{\Omega}$. Our main theorem is the following nonuniqueness result.

Theorem 1.1. *Assume that $0 < \alpha < \min\{1, p-1\}$ and (Q) are satisfied. Then there exists $T > 0$ small enough, such that problem (1.1) possesses (besides the trivial solution $u \equiv 0$) a nontrivial, nonnegative weak solution*

$$u \in C\big([0,T] \to L^2(\Omega)\big) \cap L^p\big((0,T) \to W^{1,p}(\Omega)\big)$$

which is bounded below by a subsolution $\underline{u} : \Omega \times (0,T) \to \mathbb{R}_+$ of type

$$\underline{u}(x,t) = \theta(t)\widetilde{\varphi}_{1,R}(x)^{\beta} \geq 0 \quad \text{in } \Omega \times (0,T)\,,$$

where $\theta \colon [0,T] \to \mathbb{R}_+$ is a strictly increasing, continuously differentiable function with $\theta(0) = 0$, and $\beta \in (1, \infty)$ is a suitable number.

In contrast with this nonuniqueness result, several uniqueness results have been established in [2].

Remark 1.2. Assume that $q \in L^\infty(\Omega)$ satisfies $0 \leq q(x) \leq \lambda_1$ a.e. in Ω, where λ_1 stands for the principal eigenvalue of $-\Delta_p$ with zero Dirichlet boundary conditions on Ω. Then the condition $\alpha < p - 1$ is essential for obtaining our nonuniqueness result. Namely, if $\alpha = p-1$ then $u \equiv 0$ is the unique weak solution of (1.1). The uniqueness follows directly from the following standard energy estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u(x,t)|^2\,\mathrm{d}x + \int_\Omega |\nabla u|^p\,\mathrm{d}x = \int_\Omega q(x)|u|^p\,\mathrm{d}x \le \lambda_1 \int_\Omega |u|^p\,\mathrm{d}x\,.$$

By the variational characterization of λ_1 (Poincaré's inequality in Lindqvist [8]), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u(x,t)|^2\,\mathrm{d}x \le -\int_\Omega |\nabla u|^p\,\mathrm{d}x + \lambda_1\int_\Omega |u|^p\,\mathrm{d}x \le 0\,,$$

which implies $u(x,t) \equiv 0$ in $\Omega \times (0,T)$, thanks to $u(x,0) \equiv 0$ in Ω.

A weaker result than our Theorem 1.1 has recently been published in Merchán, Montoro, and Peral [9, Theorem 2.2, p. 248]. There, a very strong uniform positivity condition on the potential q is assumed, $q_0 = \inf_\Omega q > 0$. This means that it suffices to treat the constant case $q(x) \equiv q_0 = \text{const} > 0$ and then use the resulting solution as a subsolution for the general case $q(x) \ge q_0 = \text{const} > 0$. In contrast, our Theorem 1.1 above does not assume $q_0 > 0$; we assume only $q \ge 0$ and $q \not\equiv 0$ in Ω. Nevertheless, our proof of this result, especially our construction of a nonzero subsolution, is simpler than in [9].

2. Proof of Theorem 1.1

Note that $\widetilde{\varphi}_{1,R}$ defined in (1.3) is continuous on $\overline{\Omega}$ and $\widetilde{\varphi}_{1,R}^{\beta}$ is continuously differentiable for any constant $\beta > 1$. We need to establish a few additional properties of $\varphi_{1,R}(x) \equiv \varphi_{1,R}(|x-x_0|) = \varphi_{1,R}(r)$, with $r = |x-x_0|$ and the usual harmless abuse of notation.

Lemma 2.1. *If $\beta \in (0,\infty)$ then*

$$-\Delta_p\Big(\varphi_{1,R}^{\beta}\Big) = \beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)-1}\big[\lambda_{1,R}\varphi_{1,R}^{p} - (p-1)(\beta-1)|\nabla\varphi_{1,R}|^p\big] \tag{2.1}$$

holds pointwise a.e. in $B_R(x_0)$. In particular, for $\beta \ge 1$ we have

$$\frac{-\Delta_p(\varphi_{1,R}^{\beta})}{\varphi_{1,R}^{\beta}} \le C \equiv \text{const} < \infty \quad \textit{pointwise a.e. in } B_R(x_0)\,. \tag{2.2}$$

Proof. Any function $u\colon B_R(x_0) \to \mathbb{R}$ that is radially symmetric around x_0 can be written as $u(x) = u(r)$ where $r = |x-x_0|$. Using this notation we obtain, by formal differentiation,

$$\begin{aligned}\Delta_p u(|x-x_0|) &= \operatorname{div}\Big(|u'(r)|^{p-2}u'(r)\frac{x-x_0}{r}\Big)\\ &= \Big(|u'(r)|^{p-2}u'(r)\Big)' + \frac{N-1}{r}|u'(r)|^{p-2}u'(r)\,.\end{aligned} \tag{2.3}$$

It is well-known that the first eigenfunction $\varphi_{1,R}$ is radially symmetric around x_0, positive, and C^2 in $\overline{B_R}(x_0)\setminus\{x_0\}$, see e.g. [1]. Therefore, we get a.e. in $B_R(x_0)$,

$$\begin{aligned}&\Delta_p\Big(\varphi_{1,R}^{\beta}(r)\Big)\\ &= \Big(\beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)}|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\Big)'\\ &\quad + \frac{N-1}{r}\beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)}|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\end{aligned}$$

$$
\begin{aligned}
&= \beta^{p-1}\Big\{(p-1)(\beta-1)\varphi_{1,R}^{(p-1)(\beta-1)-1}|\varphi_{1,R}'|^p \\
&\quad + \varphi_{1,R}^{(p-1)(\beta-1)}\Big(|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\Big)' + \frac{N-1}{r}\varphi_{1,R}^{(p-1)(\beta-1)}|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\Big\} \\
&= \beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)-1}\Big\{(p-1)(\beta-1)|\varphi_{1,R}'|^p - \lambda_{1,R}\varphi_{1,R}^p\Big\} \\
&= \beta^{p-1}\varphi_{1,R}^{(p-1)\beta}\Big\{(p-1)(\beta-1)\,\frac{|\varphi_{1,R}'|^p}{\varphi_{1,R}^p} - \lambda_{1,R}\Big\}\,.
\end{aligned}
$$

Hence,

$$
-\Delta_p\big(\varphi_{1,R}^\beta\big) \le \beta^{p-1}\lambda_{1,R}\varphi_{1,R}^{(p-1)\beta}
$$

for $\beta \ge 1$. For $p \ge 2$ this yields

$$
\frac{-\Delta_p\big(\varphi_{1,R}^\beta\big)}{\varphi_{1,R}^\beta} \le \beta^{p-1}\lambda_{1,R}\varphi_{1,R}^{(p-2)\beta} \le \beta^{p-1}\lambda_{1,R}\,,
$$

thanks to our normalization $0 < \varphi_{1,R} \le 1$. On the other hand, for $1 < p < 2$,

$$
\frac{-\Delta_p\big(\varphi_{1,R}^\beta\big)}{\varphi_{1,R}^\beta} = \beta^{p-1}\varphi_{1,R}^{(p-2)\beta}\big\{\lambda_{1,R} - (p-1)(\beta-1)\varphi_{1,R}^{-p}|\varphi_{1,R}'|^p\big\}. \tag{2.4}
$$

Since $\varphi_{1,R}$ is radially decreasing and satisfies the Hopf maximum principle on the boundary of $B_R(x_0)$, we can choose $\varepsilon > 0$ such that $\varphi_{1,R}'(r) < \varphi_{1,R}'(R)/2 < 0$ for all $r \in (R-\varepsilon, R)$.

Hence, (2.4) implies (2.2) for $R-\varepsilon \le r < R$ provided $\varepsilon > 0$ is small enough, such that

$$
\lambda_{1,R} - (p-1)(\beta-1)\varphi_{1,R}^{-p}|\varphi_{1,R}'|^p \le 0 \quad \text{for } R-\varepsilon \le r < R\,.
$$

At the same time, the ratio $-\Delta_p\big(\varphi_{1,R}^\beta\big)/\varphi_{1,R}^\beta$ is bounded for $0 < r \le R-\varepsilon$. Thus, estimate (2.2) holds a.e. in $B_R(x_0)$. ☐

Proposition 2.2. *Assume that* $0 < \alpha < \min\{1, p-1\}$ *and* (Q) *are satisfied. Given any fixed number* $S \in (0,\infty)$, *we define*

$$
\underline{u}(x,t) := \theta(t)\widetilde{\varphi}_{1,R}(x)^\beta \quad \text{for } (x,t) \in \Omega \times [0,S]\,,
$$

where $\beta > 1$, $\widetilde{\varphi}_{1,R}$ *is given by* (1.3), *and* $\theta : [0,S] \to \mathbb{R}_+$ *is the positive solution of the Cauchy problem*

$$
\frac{\mathrm{d}\theta}{\mathrm{d}t}(t) = \frac{q_0}{2}\theta^\alpha(t) \quad \text{for } t \in (0,S)\,; \quad \theta(0) = 0\,, \tag{2.5}
$$

such that $0 < \theta(t) < \infty$ *for every* $t \in (0,S)$. *Then* $\underline{u} : \Omega \times (0,S) \to \mathbb{R}_+$ *is a subsolution of problem* (1.1) *in a smaller domain* $\Omega \times (0,\underline{\sigma})$, *i.e., for* $t \in (0,\underline{\sigma})$ *only, where* $\underline{\sigma} \in (0,S)$ *is small enough.*

Proof. We will show that the following inequality holds

$$
\frac{\partial \underline{u}}{\partial t} - \Delta_p \underline{u} \le q(x)|\underline{u}|^{\alpha-1}\underline{u}.
$$

Using $0 < \alpha < \min\{1, p-1\}$, equation (2.5), and the continuity of $\theta\colon [0,S) \to \mathbb{R}_+$, we get

$$
\frac{\mathrm{d}\theta}{\mathrm{d}t} \le\; -C\theta(t)^{p-1} + q_0\theta(t)^\alpha \quad \text{for all } t \in [0,\underline{\sigma}]\,, \tag{2.6}
$$

where $\underline{\sigma} \in (0, S)$ is small enough, such that $\theta(t)^{p-1-\alpha} \le q_0/(2C)$ holds for all $t \in [0, \underline{\sigma}]$.

Inserting the inequality

$$\varphi_{1,R}^{-\beta} \Delta_p(\varphi_{1,R}^{\beta}) \ge -C \equiv \text{const}$$

in Ω from Lemma 2.1, inequality (2.2), into (2.6), we obtain

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}t} &\le \varphi_{1,R}^{-\beta} \Delta_p(\varphi_{1,R}^{\beta}) \theta(t)^{p-1} + q_0 \theta(t)^{\alpha} \\ &\le \varphi_{1,R}^{-\beta} \Delta_p(\varphi_{1,R}^{\beta}) \theta(t)^{p-1} + q_0 \varphi_{1,R}^{(\alpha-1)\beta} \theta(t)^{\alpha}, \end{aligned}$$

thanks to the normalization $0 < \varphi_{1,R} \le 1$ in $B_R(x_0)$ combined with $(\alpha - 1)\beta < 0$. Finally, multiplying by $\varphi_{1,R}^{\beta}$, we arrive at

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}t} \varphi_{1,R}^{\beta} &\le \Delta_p(\varphi_{1,R}^{\beta}) \theta(t)^{p-1} + q_0 \theta(t)^{\alpha} \varphi_{1,R}^{\alpha} \\ &\le \Delta_p(\varphi_{1,R}^{\beta}) \theta(t)^{p-1} + q(x) \theta(t)^{\alpha} \varphi_{1,R}^{\alpha} \,. \end{aligned}$$

This inequality, combined with our definition of the function $\widetilde{\varphi}_{1,R}$, guarantees that $\underline{u}(x,t) = \theta(t)\widetilde{\varphi}_{1,R}(x)$ is a subsolution to problem (1.1). ☐

Proof of Theorem 1.1. First, let us observe that $\overline{u}(x,t) = \|q\|_{\infty}^{\frac{1}{1-\alpha}} t$ is a supersolution of (1.1) for $0 < t \le 1$. Indeed, a straightforward calculation shows that

$$\frac{\partial \overline{u}}{\partial t} - \Delta_p \overline{u} = \|q\|_{\infty}^{\frac{1}{1-\alpha}} \ge q(x) \Big(\|q\|_{\infty}^{\frac{1}{1-\alpha}} t \Big)^{\alpha} = q(x) |\overline{u}|^{\alpha-1} \overline{u}$$

holds for $0 < t \le 1$, since $q \in C(\overline{\Omega})$, $q \ge 0$, and $\|q\|_{\infty} = \sup_{x\in\Omega} q(x)$.

Second, we show now that $\underline{u} \le \overline{u}$ for all $x \in \Omega$ and all $t > 0$ sufficiently small, say, $0 < t \le \overline{\sigma}$. Evidently,

$$\underline{u}(x,t) = \theta(t)\widetilde{\varphi}_1(x)^{\beta} = c_1 t^{\frac{1}{1-\alpha}} \widetilde{\varphi}_1(x)^{\beta} \le c_1 t^{\frac{1}{1-\alpha}} \le \overline{u}(x,t) = \|q\|_{\infty}^{\frac{1}{1-\alpha}} t$$

for $0 < t \le \overline{\sigma}$, where $\overline{\sigma}$ satisfies

$$\overline{\sigma}^{\alpha} \le \|q\|_{\infty} / c_1^{1-\alpha} \,.$$

Now it remains to show the existence of weak solution u for (1.1), such that

$$\underline{u} \le u \le \overline{u} \quad \text{in } \Omega \times (0,T) \,, \quad \text{where } T := \min\{\underline{\sigma}, \overline{\sigma}\} > 0.$$

Let us define a sequence of functions $u_n \colon \Omega \times (0,T) \to \mathbb{R}$ recursively for $n = 1, 2, 3, \dots$, such that u_n is the unique weak solution of

$$\begin{gathered} \frac{\partial u_n}{\partial t} - \Delta_p u_n = q(x) |u_{n-1}|^{\alpha-1} u_{n-1}, \quad (x,t) \in \Omega \times (0,T), \\ u_n(x,0) = 0, \quad x \in \Omega, \\ u_n(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \end{gathered} \tag{2.7}$$

with $u_0 = \underline{u}$. By a weak solution of (2.7), we mean a Lebesgue-measurable function $u_n \colon \Omega \times (0,T) \to \mathbb{R}$ that satisfies

$$u_n \in C([0,T] \to L^2(\Omega)) \cap L^p\big((0,T) \to W_0^{1,p}(\Omega)\big)$$

and the equation

$$\begin{aligned}
&\int_\Omega u_n(x,t)\phi(x,t)\,\mathrm{d}x - \int_0^t \int_\Omega u_n(x,s)\frac{\partial \phi}{\partial t}(x,s)\,\mathrm{d}x\,\mathrm{d}s \\
&+ \int_0^t \int_\Omega |\nabla u_n(x,s)|^{p-2}\langle \nabla u_n(x,s), \nabla\phi(x,s)\rangle\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^t \int_\Omega q(x)|u_{n-1}(x,s)|^{\alpha-1}u_{n-1}(x,s)\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{2.8}$$

for every $t \in (0,T)$ and every test function

$$\phi \in C\left([0,T] \to L^2(\Omega)\right) \cap L^p\left((0,T) \to W_0^{1,p}(\Omega)\right) \cap W^{1,p'}\left((0,T) \to W^{-1,p'}(\Omega)\right).$$

The questions of existence and uniqueness of weak solutions of problems of type (2.7) obtained by monotone iterations have been discussed in [12, Appendix A, §A.1]. Let us deduce from the fact that $u_0 = \underline{u}$ is a subsolution of (1.1) the inequalities $u_{n-1} \le u_n$ in $\Omega \times (0,T)$ for every $n = 1, 2, 3, \dots$. The proof is by induction on n. The first inequality, $u_0 \le u_1$ in $\Omega \times (0,T)$, holds by the Weak Comparison Principle (see [12, Lemma 4.9, p. 618]) and the fact that $u_0 = \underline{u}$ is a subsolution of (1.1). Now assume that $u_{n-1} \le u_n$ in $\Omega \times (0,T)$ for some $n \in \mathbb{N}$. Then we have

$$\frac{\partial u_n}{\partial t} - \Delta_p u_n = |u_{n-1}|^{\alpha-1}u_{n-1} \le |u_n|^{\alpha-1}u_n = \frac{\partial u_{n+1}}{\partial t} - \Delta_p u_{n+1}$$

in $\Omega \times (0,T)$ and consequently $u_n \le u_{n+1}$ in $\Omega \times (0,T)$ again, by [12, Lemma 4.9, p. 618]. Therefore, monotonicity holds: $\underline{u} = u_0 \le u_1 \le u_2 \le \dots \le \overline{u}$ in $\Omega \times (0,T)$. The comparison with the supersolution $\overline{u}$ is deduced again from the Weak Comparison Principle. Hence, u_n is uniformly bounded in $\Omega \times (0,T)$ by $\underline{u} \le u \le \overline{u}$. A global regularity result from [7, Theorem 0.1, p. 552] (cf. [12, Lemma 4.6, p. 617]) guarantees $u_n \in C^{1+\gamma, \frac{1+\gamma}{2}}(\overline{\Omega} \times [0,T])$ uniformly for $n \in \mathbb{N}$, where $\gamma \in (0,1)$ is independent of n. We follow the notations and definitions of Hölder spaces of functions on $\Omega \times [0,T]$ from [5, Chpt. 1, p. 7]. Thus, by the Arzelà-Ascoli theorem, $\{u_n\}$ is relatively compact in $C^{1,0}(\overline{\Omega} \times [0,T])$. Hence, the sequence $\{u_n\}$ possesses a subsequence which converges to $u \in C^{1,0}(\overline{\Omega} \times [0,T])$. Therefore, in the weak formulation of (2.8) we may pass to the limit as $n \to \infty$, thus verifying that the limit function u is a weak solution of (1.1) in $\Omega \times (0,T)$, such that $\underline{u} \le u \le \overline{u}$. □

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Appendix A2

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Nonuniqueness and multi-bump solutions in parabolic problems with the p-Laplacian

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Abstract

The validity of the weak and strong comparison principles for degenerate parabolic partial differential equations with the p-Laplace operator Δ_p is investigated for $p > 2$. This problem is reduced to the comparison of the trivial solution ($\equiv 0$, by hypothesis) with a nontrivial nonnegative solution $u(x,t)$. The problem is closely related also to the question of uniqueness of a nonnegative solution via the weak comparison principle. In this article, realistic counterexamples to the uniqueness of a nonnegative solution, the weak comparison principle, and the strong maximum principle are constructed with a nonsmooth reaction function that satisfies neither a Lipschitz nor an Osgood standard "uniqueness" condition. Nonnegative multi-bump solutions with spatially disconnected compact supports and zero initial data are constructed between sub- and supersolutions that have supports of the same type.

MSC: 35B05; 35B30; 35K15; 35K55; 35K65

Keywords: Quasilinear parabolic equations with p-Laplacian; Nonuniqueness for initial-boundary value problem; Sub- and supersolutions; Comparison principle; Barenblatt-type solutions

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1. Introduction

The main purpose of this article is to investigate the validity of the weak and strong comparison principles for degenerate parabolic partial differential equations with the p-Laplace operator Δ_p for $p > 2$. In its simplest form, this problem is reduced to the comparison of the trivial solution ($\equiv 0$, by hypothesis) with a nontrivial nonnegative solution $u(x,t)$. In this special setting, the problem is closely related also to the question of uniqueness of a nonnegative solution. Typically, the validity of the weak comparison principle implies the uniqueness of a solution. Conditions on existence, uniqueness, and regularity have been studied, e.g., in Ladyzhenskaya et al. [12], Ladyzhenskaya and Ural'tseva [13], and Oleĭnik and Kružkov [16]. On the other hand, important examples of nonuniqueness for standard parabolic problems, even with the regular Laplace operator Δ (i.e., $p = 2$) and a nonsmooth reaction function, have been constructed first in Fujita and Watanabe [10], then in Redheffer and Walter [18], and for the p-Laplace operator in DiBenedetto et al. [8] and Guedda [11], and more recently in Bobkov and Takáč [4], Merchán et al. [15], and Benedikt et al. [2]. Most of these nonuniqueness examples show the nonuniqueness of the trivial solution ($\equiv 0$) to a given parabolic initial-boundary value problem with the trivial initial and boundary conditions. More specifically, the following parabolic problem,

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta_p u &= q(x)u^{\alpha} && \text{for } (x,t) \in \Omega \times (0,T)\,; \\ u(x,t) &= 0 && \text{for } (x,t) \in \partial\Omega \times (0,T)\,, \\ u(x,0) &= 0 && \text{for } x \in \Omega\,, \end{aligned} \tag{1.1}$$

is considered in these examples. Here, $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ denotes the p-Laplacian for $1 < p < \infty$, $\alpha \in (0,1)$ is a given number, $0 < T < \infty$, and the potential q satisfies the following condition:

$$\text{(Q)} \qquad q \in C(\overline{\Omega}),\ q \geq 0, \quad \text{and} \quad q(x_0) > 0 \quad \text{for some } x_0 \in \Omega.$$

We extend q to the whole of $\mathbb{R}^N$ by $q \equiv 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$ if needed. Although we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a $C^{1+\mu}$-boundary $\partial\Omega$ where $\mu \in (0,1)$, the validity of our parabolic problems extends from the bounded domain $\Omega \times (0,T)$ to all of $\mathbb{R}^N \times (0,T)$; i.e., to the case of the whole space $\mathbb{R}^N$ in place of Ω. This is a trivial consequence of the fact that all our weak solutions to a problem of type (1.1), and sub- and supersolutions as well, will be spatially supported in a compact subset of Ω. Hence, when working with such a (weak) solution $u: \Omega \times (0,T) \to \mathbb{R}$, we extend it automatically to an "entire" solution $\widetilde{u}: \mathbb{R}^N \times (0,T) \to \mathbb{R}$ defined by

$$\widetilde{u}(x,t) \stackrel{\text{def}}{=} \begin{cases} u(x,t) & \text{for } (x,t) \in \Omega \times (0,T)\,; \\ 0 & \text{for } (x,t) \in \left(\mathbb{R}^N \setminus \Omega\right) \times (0,T)\,. \end{cases}$$

A nontrivial nonnegative solution to (1.1) is often constructed by nondecreasing monotone iterations starting from a nontrivial nonnegative subsolution, cf. Bobkov and Takáč [4], Merchán et al. [15], and Benedikt et al. [2]. In this approach, originating in Sattinger [20], which we use also in our present work, a suitable supersolution has to be constructed that provides an *upper bound* for the monotone iterations. Besides a spatially constant function, a simple example of such a supersolution is the well-known Barenblatt solution (and its modifications), cf. DiBenedetto et

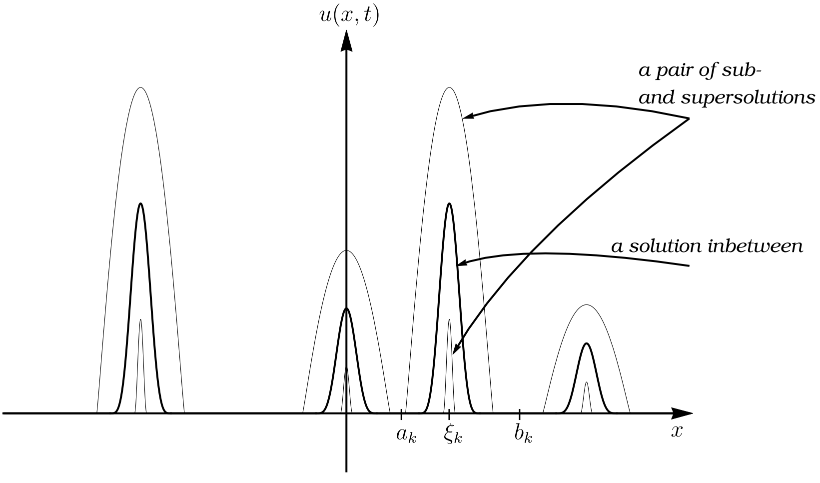

Fig. 1. A multi-bump solution.

al. [8] and Samarskii et al. [19]. An important advantage of a Barenblatt-type supersolution is its *compact support* with respect to the space variable $x \in \Omega$. This property allows us to use also linear combinations of several supersolutions that have pairwise disjoint supports in order to construct more complicated and, perhaps, also more surprising examples.

Our main results are formulated for problem (1.1) and can be outlined as follows. Of course, the zero function $u \equiv 0$ is a solution. In particular, we focus on the *degenerate* diffusion $2 < p < \infty$ and the *nonsmooth* reaction function $f(x,u) \stackrel{\text{def}}{=} q(x)u^\alpha$ for $(x,u) \in \Omega \times \mathbb{R}_+$, $0 < \alpha < 1$, where $\mathbb{R}_+ \stackrel{\text{def}}{=} [0,+\infty)$. Notice that if $q(x_0) > 0$ then the function $u \mapsto f(x_0,u)$ satisfies neither a local Lipschitz nor an Osgood condition near $u = 0$ provided $\alpha \in (0,1)$, see Osgood [17]. The case $p = 2$ was treated in Fujita and Watanabe [10] by different methods based on the Green's function for the heat equation. An important special case, $N = 1$, $1 < p < \infty$, and $q(x) \equiv \lambda > 0$ (a constant), was treated in Guedda [11] also by different methods. The main purpose of the present article is to obtain nontrivial nonnegative solutions with *multiple* positive "bumps" that have pairwise disjoint *compact* supports with respect to the space variable $x \in \Omega$. The bump supports, contained in Ω, do not overlap during a given time interval $[0,T]$; this is impossible if $p = 2$, by the strong maximum principle (Hopf's lemma) for linear parabolic problems (Friedman [9]), cf. Aguirre and Escobedo [1, Cor. 2.6, p. 190]. Each single bump solution is obtained by constructing a *Barenblatt-type* supersolution and using it as an upper bound for nondecreasing monotone iterations, starting from a nontrivial nonnegative subsolution (cf. [2]), that converge to the desired single bump solution described in Theorem 1.3. (See Fig. 1.)

Definition 1.1. A weak *subsolution* (*supersolution*, and *solution*, respectively) to problem (1.1) has been defined in DiBenedetto et al. [8, §3.1, p. 23]. We say that a weak subsolution (supersolution, or solution)

$$u \in C\left([0,T] \to L^2_{\text{loc}}(\Omega)\right) \cap L^p\left((0,T) \to W^{1,p}_{\text{loc}}(\Omega)\right)$$

to problem (1.1) is an *m-bump* subsolution (supersolution, or solution) if it has the following properties:

(a) $u : \Omega \times [0, T] \to \mathbb{R}$ is continuous, $u \geq 0$ in $\Omega \times [0, T]$, and $u \not\equiv 0$;
(b) u has a compact support

$$\operatorname{supp}(u) \overset{\text{def}}{=} \text{closure}\,\{(x,t) \in \Omega \times [0,T] \colon u(x,t) > 0\} \quad \text{ in } \mathbb{R}^N \times [0,T],$$

$\operatorname{supp}(u) \subset \Omega \times [0, T]$;
(c) there exist precisely m ($m \in \mathbb{N} = \{1, 2, 3, \dots\}$) pairwise disjoint, connected compact subsets K_k ($k = 1, 2, \dots, m$) of Ω such that

$$\operatorname{supp}(u) \subset \left(\cup_{k=1}^{m} K_k\right) \times [0, T];$$

(d) for all $t \in (0, T]$ and $k = 1, 2, \dots, m$,

$$\{x \in K_k \colon u(x,t) > 0\}$$

is a nonempty connected open subset of Ω.

Notice that an m-bump subsolution (supersolution, or solution) may have identically zero initial values (for $t = 0$). We use an analogous definition if the interval $[0, T]$ is replaced by $[t_0, T]$ with $0 \leq t_0 < T$.

Theorem 1.2. *Let $2 < p < \infty$, $1/(p-1) < \alpha < 1$, and let $I_k = [a_k, b_k]$; $k = 1, 2, 3, \dots, m$, be a family of pairwise disjoint compact intervals in $\mathbb{R}$, $-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < +\infty$, and let $0 < T_0 < \infty$. Furthermore, let $\xi_k \in (a_k, b_k)$ be an arbitrary point; $k = 1, 2, 3, \dots, m$. Then there exists some $T \in (0, T_0]$ such that the initial-boundary value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = u(x,t)^{\alpha} & \text{for } x \in (a_1, b_m)\,, 0 < t < T; \\ u(a_1, t) = u(b_m, t) = 0 & \text{for } 0 < t < T; \\ u(x, 0) = 0 & \text{for } x \in (a_1, b_m)\,, \end{cases} \tag{1.2}$$

possesses a nontrivial nonnegative solution $u \colon (a_1, b_m) \times (0, T) \to \mathbb{R}_+$ such that

(i) *$u(\xi_k, t) > 0$ for all $k = 1, 2, \dots, m$ and all $t \in (0, T)$;*
(ii) *$u(x, t) = 0$ for all $x \in \mathbb{R} \setminus \cup_{k=1}^{m} (a_k, b_k)$ and all $t \in (0, T)$.*

Hence, u is a multi-bump solution with at least m bumps. This theorem is derived from the special case of a one-bump solution that we formulate below in a domain $\Omega \subset \mathbb{R}^N$. A careful inspection of our proof of Theorem 1.3 below reveals that the solution $u \colon (a_1, b_m) \times (0, T) \to \mathbb{R}_+$ from Theorem 1.2 has precisely m bumps. This claim follows from the fact that each iterate u_{n+1} defined in Eq. (2.17) has precisely one bump, by the uniqueness of the weak solution u_{n+1} to problem (2.17) in our proof of Theorem 1.3. (See Fig. 2.)

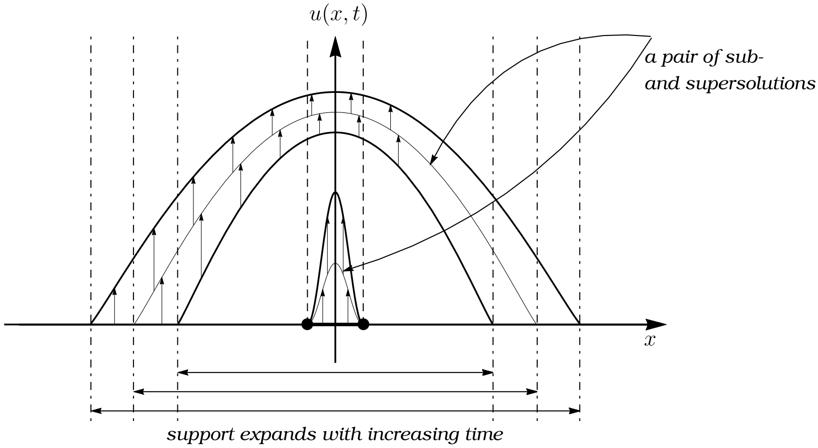

Fig. 2. Our choice of sub- and supersolutions; the graphs grow in time in the direction of the arrows.

Theorem 1.3. *Let $2 < p < \infty$, $1/(p-1) < \alpha < 1$, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Assume that q satisfies hypothesis* (Q), *$\xi \in \Omega$ is such that $q(\xi) > 0$, and*

$$\overline{B}_r(\xi) \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N : |x - \xi| \leq r\} \subset \Omega$$

and $0 < T_0 < \infty$. Then there exists some $T \in (0, T_0]$ such that the initial-boundary value problem (1.1) *possesses a nontrivial nonnegative solution $u: \Omega \times (0, T) \to \mathbb{R}_+$ such that*

(i) $u(\xi, t) > 0$ *for all $t \in (0, T)$;*
(ii) $u(x, t) = 0$ *for all $x \in \Omega \setminus B_r(\xi)$ and all $t \in (0, T)$.*

In addition, if $\Omega = B_R(\xi)$ is a ball with radius R centered at ξ, $0 < r < R < \infty$, and q is radially symmetric about ξ, i.e., $q(x) \equiv q(|x - \xi|)$ for $x \in \Omega$, then the nontrivial solution u above can be constructed radially symmetric about ξ in the space variable $x \in \Omega$, i.e., $u(x, t) \equiv u(|x - \xi|, t)$.

Both theorems will be derived from more general results in Section 3 that are too "technical" to state at this point.

As remarked above, Theorem 1.2 (Theorem 1.3, respectively) may be reformulated for solutions $u(x, t)$ extended by zero values for all $x \in \mathbb{R}$ ($x \in \mathbb{R}^N$, respectively). Such extended solutions are typical for the slow diffusion case ($p > 2$) and provide a new type of solutions, in addition to the spatially constant solutions obtained in Aguirre and Escobedo [1, Cor. 2.6, p. 190] for the regular diffusion ($p = 2$).

This article is organized as follows. In Section 2 we construct Barenblatt-type supersolutions which are nonnegative, nontrivial, and compactly supported in the spatial domain. We apply these supersolutions to prove Theorem 1.3 which provides the most important tool for proving more interesting results stated in Section 3. These supersolutions play also a decisive role in obtaining the solutions in Theorem 1.2. We employ Theorem 1.3 to establish our main results with multiple bumps in Section 3, Theorem 3.1 with m-bumps, and Theorem 3.2 with a variable number of bumps increasing in time. Alternatively, we use Barenblatt-type supersolutions to prove analogous results for radially symmetric solutions in Theorem 3.4.

2. A Barenblatt-type supersolution

Recall that a *subsolution*, *supersolution*, and *solution*

$$u \in C\left([0,T] \to L^2_{\mathrm{loc}}(\Omega)\right) \cap L^p\left((0,T) \to W^{1,p}_{\mathrm{loc}}(\Omega)\right)$$

to problem (1.1) in the weak sense are formally defined in the same way as in [8, §3.1, p. 23].

We look for a supersolution with compact support and properties similar to solutions in Theorems 1.2 and 1.3 (Properties (i) and (ii)), cf. Definition 1.1. The form of this supersolution is motivated by the well-known Barenblatt solution (see [8], eq. (3.3) on p. 63). As usual, the symbol $a_+ = \max\{a, 0\}$ stands for the nonnegative part of $a \in \mathbb{R}$.

Theorem 2.1. *Let $p > 2$, $1/(p-1) < \alpha < 1$, $0 \le q \in L^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^N$ is an arbitrary domain, $K = \|q\|_{L^\infty(\Omega)} > 0$, $0 < T < 1/(K\alpha)$, and let $\varepsilon > 0$ be such that $B_\varepsilon(0) \subset \Omega$. Define*

$$\sigma \overset{\text{def}}{=} \frac{K}{1-KT\alpha} \quad \textit{and} \quad \varrho \overset{\text{def}}{=} \frac{1}{2}\sigma\varepsilon(1-\alpha) + \frac{2^{p-1}(1+\sigma T)^{p-1}\alpha(p-1)}{(1-\alpha)^{p-1}\varepsilon^{p-1}}. \tag{2.1}$$

Then the radially symmetric, nonnegative nontrivial function

$$\bar{u}(x,t) \equiv \bar{u}(|x|,t) \overset{\text{def}}{=} (1+\sigma t)\left[1 - \left(\frac{|x|}{\varepsilon + \varrho t}\right)^2\right]_+^{\frac{1}{1-\alpha}} \tag{2.2}$$

is a supersolution of problem (1.1) in $\mathbb{R}^N \times (0,T)$. Furthermore, we have

$$\frac{\partial \bar{u}}{\partial t}, \; \frac{\partial \bar{u}}{\partial x_i}, \; \Delta_p \bar{u} \in C^\gamma(\mathbb{R}^N \times [0,T]) \tag{2.3}$$

with some $\gamma \in (0,1)$ depending only on p and α.

Proof. We abbreviate $r = |x|$ for $x \in \mathbb{R}^N$. Since we are interested in a radially symmetric supersolution $\bar{u}(x,t) \equiv \bar{u}(r,t)$ only in the interior of its support, the following calculations are performed only for $0 < r < \varepsilon + \varrho t$ where $0 \le t \le T$. The boundary points $r = 0$ and $r = \varepsilon + \varrho t$ are treated separately, thanks to the regularity in (2.3). If we wish that $\operatorname{supp}(u(\cdot,t)) \subset \Omega$ for all $t \in [0,T]$, then we need to take $T > 0$ small enough, such that, in addition to $T < 1/(K\alpha)$, also $B_{\varepsilon+\varrho T}(0) \subset \Omega$ holds. However, as we consider problem (1.1) in $\mathbb{R}^N \times (0,T)$ below, it will not be necessary to require that $B_{\varepsilon+\varrho T}(0) \subset \Omega$.

Abbreviating

$$[\ldots] \equiv \left[1 - \left(\frac{r}{\varepsilon+\varrho t}\right)^2\right] \in (0,1) \quad \text{for } 0 < r < \varepsilon + \varrho t,$$

we calculate

$$\begin{aligned}\frac{\partial \bar u}{\partial t}(r,t) &= \sigma\,[\ldots]^{\frac{1}{1-\alpha}} + \frac{1+\sigma t}{1-\alpha}\,[\ldots]^{\frac{\alpha}{1-\alpha}}\,\frac{2\varrho r^2}{(\varepsilon+\varrho t)^3}\\ &= [\ldots]^{\frac{\alpha}{1-\alpha}}\left(\sigma + r^2\frac{2\varrho-\sigma\varepsilon(1-\alpha)+\sigma t\varrho(1+\alpha)}{(1-\alpha)(\varepsilon+\varrho t)^3}\right)\\ &\ge [\ldots]^{\frac{\alpha}{1-\alpha}}\left(\sigma + r^2\frac{2\varrho-\sigma\varepsilon(1-\alpha)}{(1-\alpha)(\varepsilon+\varrho t)^3}\right),\end{aligned}$$

then

$$\frac{\partial \bar u}{\partial r}(r,t) = \frac{1+\sigma t}{1-\alpha}\,[\ldots]^{\frac{\alpha}{1-\alpha}}\,\frac{-2r}{(\varepsilon+\varrho t)^2},$$

$$\left|\frac{\partial \bar u}{\partial r}\right|^{p-2}\frac{\partial \bar u}{\partial r} = -\left(\frac{2}{1-\alpha}\right)^{p-1}\left(\frac{1+\sigma t}{(\varepsilon+\varrho t)^2}\right)^{p-1} r^{p-1}\,[\ldots]^{\frac{\alpha(p-1)}{1-\alpha}}, \tag{2.4}$$

and finally

$$\begin{aligned}-\frac{\partial}{\partial r}\left(\left|\frac{\partial \bar u}{\partial r}\right|^{p-2}\frac{\partial \bar u}{\partial r}\right) &= \left(\frac{2}{1-\alpha}\right)^{p-1}\left(\frac{1+\sigma t}{(\varepsilon+\varrho t)^2}\right)^{p-1}\\ &\quad\times\left\{(p-1)r^{p-2}[\ldots]^{\frac{\alpha(p-1)}{1-\alpha}} + r^{p-1}\frac{\alpha(p-1)}{1-\alpha}[\ldots]^{\frac{\alpha p-1}{1-\alpha}}\frac{-2r}{(\varepsilon+\varrho t)^2}\right\}\\ &= \left(\frac{2}{1-\alpha}\right)^{p-1}\left(\frac{1+\sigma t}{(\varepsilon+\varrho t)^2}\right)^{p-1}(p-1)r^{p-2}\\ &\quad\times[\ldots]^{\frac{\alpha p-1}{1-\alpha}}\left[1-\left(\frac{r}{\varepsilon+\varrho t}\right)^2 - \frac{2r^2\alpha}{(\varepsilon+\varrho t)^2(1-\alpha)}\right]\\ &\ge -\left(\frac{2}{1-\alpha}\right)^{p-1}\left(\frac{1+\sigma t}{(\varepsilon+\varrho t)^2}\right)^{p-1}(p-1)r^{p-2}\frac{2r^2\alpha}{(\varepsilon+\varrho t)^2(1-\alpha)}\\ &\quad\times[\ldots]^{\frac{\alpha p-1}{1-\alpha}}\\ &\ge -[\ldots]^{\frac{\alpha}{1-\alpha}}\frac{r^2}{(1-\alpha)(\varepsilon+\varrho t)^3}\,\frac{2^p(1+\sigma t)^{p-1}\alpha(p-1)}{(1-\alpha)^{p-1}(\varepsilon+\varrho t)^{p-1}}[\ldots]^{\frac{\alpha(p-1)-1}{1-\alpha}}\\ &\ge -[\ldots]^{\frac{\alpha}{1-\alpha}}\frac{r^2}{(1-\alpha)(\varepsilon+\varrho t)^3}\,\frac{2^p(1+\sigma T)^{p-1}\alpha(p-1)}{(1-\alpha)^{p-1}\varepsilon^{p-1}}\end{aligned} \tag{2.5}$$

for $0 < r < \varepsilon + \varrho t$. Recall that $[\ldots] \in (0,1)$ and $\alpha(p-1) > 1$.

We apply these estimates to the p-Laplacian of a radially symmetric function with $r = |x|$,

$$\begin{aligned}\Delta_p \bar u(x,t) &= \frac{\partial}{\partial r}\left(\left|\frac{\partial \bar u}{\partial r}(r,t)\right|^{p-2}\frac{\partial \bar u}{\partial r}(r,t)\right) + \frac{N-1}{r}\left|\frac{\partial \bar u}{\partial r}(r,t)\right|^{p-2}\frac{\partial \bar u}{\partial r}(r,t)\\ &\le \frac{\partial}{\partial r}\left(\left|\frac{\partial \bar u}{\partial r}(r,t)\right|^{p-2}\frac{\partial \bar u}{\partial r}(r,t)\right),\end{aligned} \tag{2.6}$$

thanks to $\frac{\partial \bar u}{\partial r}(r,t) \le 0$, by our choice of $\bar u(r,t)$ in (2.2) combined with (2.4).

These inequalities yield the following lower bound for the left-hand side of the equation in (1.1):

$$\begin{aligned}\text{L.H.S.} &\equiv \frac{\partial \bar{u}}{\partial t} - \Delta_p \bar{u} \\ &\geq [\dots]^{\frac{\alpha}{1-\alpha}} \left[\sigma + \frac{r^2}{(1-\alpha)(\varepsilon+\varrho t)^3} \left(2\varrho - \sigma\varepsilon(1-\alpha) - \frac{2^p(1+\sigma T)^{p-1}\alpha(p-1)}{(1-\alpha)^{p-1}\varepsilon^{p-1}} \right) \right] \\ &= \sigma\, [\dots]^{\frac{\alpha}{1-\alpha}} \,,\end{aligned}$$

thanks to our choice of ϱ in (2.1). Now we estimate from above the right-hand side of the equation in (1.1):

$$\begin{aligned}\text{R.H.S.} \equiv q(x)\bar{u}^\alpha \leq K\bar{u}^\alpha(r,t) &= K(1+\sigma t)^\alpha\, [\dots]^{\frac{\alpha}{1-\alpha}} \\ &\leq K(1+\sigma t\alpha)\, [\dots]^{\frac{\alpha}{1-\alpha}} \,.\end{aligned}$$

Comparing the last two estimates and using (2.1), we arrive at

$$\text{L.H.S.} \geq \text{R.H.S.},$$

i.e.,

$$\frac{\partial}{\partial t}\bar{u}(x,t) - \Delta_p \bar{u}(x,t) \geq q(x)\bar{u}^\alpha(x,t) \quad \text{for} \quad 0 < |x| < \varepsilon + \varrho t\,,\ 0 \leq t \leq T\,. \tag{2.7}$$

From the formulas above for $\partial\bar{u}/\partial t$, $\partial\bar{u}/\partial r$, and $\Delta_p \bar{u}$ we deduce that all partial derivatives $\partial\bar{u}/\partial t$, $\partial\bar{u}/\partial x_i$, and $\Delta_p\bar{u}$ exist in the classical sense and belong to $C^\gamma(\mathbb{R}^N \times [0,T])$ with some $\gamma \in (0,1)$. Consequently, ineq. (2.7) holds pointwise in the entire space–time domain $\mathbb{R}^N \times (0,T)$. It is now easy to conclude that our function $\bar{u}(x,t)$ is a supersolution to problem (1.1) in the sense of [8, §3.1, p. 23]. □

The Barenblatt-type supersolution provided by formula (2.2) in Theorem 2.1 will now be used to construct the one-bump solution in the proof of Theorem 1.3 below. A suitable subsolution has been constructed in [2, Prop. 2.2, p. 4] as follows:

Assuming $q(x) \geq q_0 \equiv \text{const.} > 0$ on an open ball $B_R(x_0) \subset \Omega$, we construct a nontrivial nonnegative subsolution $\underline{u}(x,t)$ to the auxiliary problem

$$\frac{\partial \underline{u}}{\partial t} - \Delta_p \underline{u} \leq q_0 \underline{u}^\alpha \quad \text{in } \mathbb{R}^N \times (0,T) \tag{2.8}$$

that is supported in a compact subset of $B_R(x_0) \times [0,T]$. Consequently, $\underline{u}$ is also a subsolution to the original problem (1.1).

Let $(\lambda_{1,R}, \varphi_{1,R})$ denote the first eigenpair for the operator

$$-\Delta_p \colon W_0^{1,p}(B_R(x_0)) \to W^{-1,p'}(B_R(x_0))\,,$$

that is,

$$-\Delta_p \varphi_{1,R} = \lambda_{1,R}\, \varphi_{1,R}^{p-1} \quad \text{in } B_R(x_0)\,; \qquad \varphi_{1,R} = 0 \quad \text{on } \partial B_R(x_0)\,. \tag{2.9}$$

Here $\varphi_{1,R} \in W_0^{1,p}(B_R(x_0))$ is normalized by $\varphi_{1,R}(x_0) = 1$; this normalization yields $0 < \varphi_{1,R}(x) \le 1$ for all $x \in B_R(x_0)$. Moreover, we denote by

$$\widetilde{\varphi}_{1,R}(x) := \begin{cases} \varphi_{1,R}(x) & \text{for } x \in B_R(x_0)\,; \\ 0 & \text{for } x \in \mathbb{R}^N \setminus B_R(x_0)\,, \end{cases} \tag{2.10}$$

the natural zero extension of $\varphi_{1,R}$ from $B_R(x_0)$ to the whole of $\mathbb{R}^N$. Clearly, $\widetilde{\varphi}_{1,R} \in W^{1,p}(\mathbb{R}^N)$.

The following lemma is a version of [2, Lemma 2.1] with complete hypotheses and a correct proof.

Lemma 2.2. *If $\beta \in (0, \infty)$ then*

$$-\Delta_p\left(\varphi_{1,R}^{\beta}\right) = \beta^{p-1} \varphi_{1,R}^{(p-1)(\beta-1)-1}\left[\lambda_{1,R}\varphi_{1,R}^{p} - (p-1)(\beta-1)|\nabla\varphi_{1,R}|^{p}\right] \tag{2.11}$$

holds pointwise a.e. in $B_R(x_0)$. In particular, if $p \ge 2$ and $\beta \ge 1$, or else $1 < p < 2$ and $\beta > 1$, then we have

$$\frac{-\Delta_p(\varphi_{1,R}^{\beta})}{\varphi_{1,R}^{\beta}} \le C \equiv \text{const} < \infty \quad \textit{pointwise a.e. in } B_R(x_0)\,. \tag{2.12}$$

Proof. Any function $u\colon B_R(x_0) \to \mathbb{R}$ that is radially symmetric around x_0 can be written as $u(x) = u(r)$ where $r = |x - x_0|$, by harmless abuse of notation. Using this notation we obtain, by formal differentiation,

$$\begin{aligned} \Delta_p u(|x - x_0|) &= \operatorname{div}\left(|u'(r)|^{p-2}u'(r)\frac{x - x_0}{r}\right) \\ &= \left(|u'(r)|^{p-2}u'(r)\right)' + \frac{N-1}{r}|u'(r)|^{p-2}u'(r)\,. \end{aligned} \tag{2.13}$$

It is well-known that the first eigenfunction $\varphi_{1,R}$ is radially symmetric around x_0, positive, and C^2 in $\overline{B_R}(x_0) \setminus \{x_0\}$, see e.g. [3]. Therefore, we get a.e. in $B_R(x_0)$,

$$\begin{aligned} &\Delta_p\left(\varphi_{1,R}^{\beta}(r)\right) \\ &= \left(\beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)}|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\right)' + \frac{N-1}{r}\beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)}|\varphi_{1,R}'|^{p-2}\varphi_{1,R}' \\ &= \beta^{p-1}\Big\{(p-1)(\beta-1)\varphi_{1,R}^{(p-1)(\beta-1)-1}|\varphi_{1,R}'|^{p} \\ &\quad +\varphi_{1,R}^{(p-1)(\beta-1)}\left(|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\right)' + \frac{N-1}{r}\varphi_{1,R}^{(p-1)(\beta-1)}|\varphi_{1,R}'|^{p-2}\varphi_{1,R}'\Big\} \\ &= \beta^{p-1}\varphi_{1,R}^{(p-1)(\beta-1)-1}\left\{(p-1)(\beta-1)|\varphi_{1,R}'|^{p} - \lambda_{1,R}\varphi_{1,R}^{p}\right\} \\ &= \beta^{p-1}\varphi_{1,R}^{(p-1)\beta}\left\{(p-1)(\beta-1)\,\frac{|\varphi_{1,R}'|^{p}}{\varphi_{1,R}^{p}} - \lambda_{1,R}\right\}. \end{aligned}$$

Hence,

$$-\Delta_p\big(\varphi_{1,R}^{\beta}\big) \le \beta^{p-1}\lambda_{1,R}\varphi_{1,R}^{(p-1)\beta}$$

for $\beta \ge 1$. For $p \ge 2$, the last inequality yields

$$\frac{-\Delta_p\big(\varphi_{1,R}^{\beta}\big)}{\varphi_{1,R}^{\beta}} \le \beta^{p-1}\lambda_{1,R}\varphi_{1,R}^{(p-2)\beta} \le \beta^{p-1}\lambda_{1,R}\,,$$

thanks to our normalization $0 < \varphi_{1,R} \le 1$. On the other hand, for $1 < p < 2$,

$$\frac{-\Delta_p\big(\varphi_{1,R}^{\beta}\big)}{\varphi_{1,R}^{\beta}} = \beta^{p-1}\varphi_{1,R}^{(p-2)\beta}\big\{\lambda_{1,R} - (p-1)(\beta-1)\varphi_{1,R}^{-p}|\varphi_{1,R}'|^p\big\}. \tag{2.14}$$

Since $\varphi_{1,R}$ is radially decreasing and satisfies the Hopf maximum principle on the boundary of $B_R(x_0)$, we can choose $\varepsilon > 0$ such that $\varphi_{1,R}'(r) < \varphi_{1,R}'(R)/2 < 0$ for all $r \in [R-\varepsilon, R)$.

Hence, assuming $\beta > 1$, (2.14) implies (2.12) for $R - \varepsilon \le r < R$ provided $\varepsilon > 0$ is small enough, such that

$$\lambda_{1,R} - (p-1)(\beta-1)\varphi_{1,R}^{-p}|\varphi_{1,R}'|^p \le 0 \quad \text{for } R-\varepsilon \le r < R\,.$$

At the same time, the ratio $-\Delta_p\big(\varphi_{1,R}^{\beta}\big)/\varphi_{1,R}^{\beta}$ is bounded for $0 < r \le R - \varepsilon$. Thus, estimate (2.12) holds a.e. in $B_R(x_0)$. □

The following proposition is a special case of [2, Prop. 2.2, p. 4]. Recall that $q(x) \ge q_0 \equiv$ const. > 0 on an open ball $B_R(x_0) \subset \Omega$ specified before Lemma 2.2.

Proposition 2.3. *Assume that $2 < p < \infty$, $0 < \alpha < 1$, and* (Q) *are satisfied. Given a ball $B_R(x_0) \subset \Omega$ and a fixed number $T_0 \in (0, \infty)$, we define*

$$\underline{u}(x,t) := \theta(t)\widetilde{\varphi}_{1,R}(x)^{\beta} \quad \textit{for } (x,t) \in \mathbb{R}^N \times [0, T_0]\,,$$

where $\beta > 1$, $\widetilde{\varphi}_{1,R}$ is given by (2.10)*, and $\theta : [0, T_0] \to \mathbb{R}_+$ is the nonnegative solution of the Cauchy problem*

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}(t) = \frac{q_0}{2}\theta^{\alpha}(t) \quad \textit{for } t \in (0, T_0)\,; \quad \theta(0) = 0\,, \tag{2.15}$$

such that $0 < \theta(t) < \infty$ for every $t \in (0, T_0)$. Then $\underline{u} : \mathbb{R}^N \times (0, T_0) \to \mathbb{R}_+$ is a subsolution of problem (1.1) *in a smaller domain $\Omega \times (0, \underline{T})$, i.e., for $x \in \Omega$ and $t \in (0, \underline{T})$ only, where $\underline{T} \in (0, T_0)$ is small enough.*

Proof. Since the proof of [2, Prop. 2.2, p. 4] contains minor misprints, we provide the corrected proof below, under the more general hypotheses $1 < p < \infty$ and $0 < \alpha < \min\{1, p-1\}$ used in [2, Prop. 2.2, p. 4].

We will show that the following inequality holds

$$\frac{\partial \underline{u}}{\partial t} - \Delta_p \underline{u} \le q(x)\underline{u}^{\alpha}\,.$$

Using $0 < \alpha < \min\{1, p-1\}$, equation (2.15), and the continuity of $\theta\colon [0, T_0) \to \mathbb{R}_+$, we get

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \le -C\theta(t)^{p-1} + q_0\theta(t)^{\alpha} \quad \text{for all } t \in [0, \underline{T}]\,, \tag{2.16}$$

where $\underline{T} \in (0, T_0)$ is small enough, such that $\theta(t)^{p-1-\alpha} \le q_0/(2C)$ holds for all $t \in [0, \underline{T}]$, and the constant $C > 0$ has been specified in Ineq. (2.12).

Inserting the inequality

$$\varphi_{1,R}^{-\beta}\Delta_p(\varphi_{1,R}^{\beta}) \ge -C \equiv \text{const}$$

in Ω from Lemma 2.2, inequality (2.12), into (2.16), we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \le \varphi_{1,R}^{-\beta}\Delta_p(\varphi_{1,R}^{\beta})\theta(t)^{p-1} + q_0\theta(t)^{\alpha} \le \varphi_{1,R}^{-\beta}\Delta_p(\varphi_{1,R}^{\beta})\theta(t)^{p-1} + q_0\varphi_{1,R}^{(\alpha-1)\beta}\theta(t)^{\alpha},$$

thanks to the normalization $0 < \varphi_{1,R} \le 1$ in $B_R(x_0)$ combined with $(\alpha-1)\beta < 0$. Finally, multiplying by $\varphi_{1,R}^{\beta}$, we arrive at

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t}\varphi_{1,R}^{\beta} &\le \Delta_p(\varphi_{1,R}^{\beta})\theta(t)^{p-1} + q_0\theta(t)^{\alpha}\varphi_{1,R}^{\alpha\beta} \\ &\le \Delta_p(\varphi_{1,R}^{\beta})\theta(t)^{p-1} + q(x)\theta(t)^{\alpha}\varphi_{1,R}^{\alpha\beta}\,.\end{aligned}$$

This inequality, combined with our definition of the function $\widetilde{\varphi}_{1,R}$, guarantees that $\underline{u}(x,t) = \theta(t)\widetilde{\varphi}_{1,R}(x)^{\beta}$ is a subsolution to problem (1.1). □

Proof of Theorem 1.3. We apply the well-known monotone iteration method from Sattinger [20] as adapted to parabolic problems with the p-Laplacian in Derlet and Takáč [7]. In fact, for ordinary differential equations, the method of monotone iterations dates back to the work of Osgood [17].

An ordered pair of sub- and supersolutions in Proposition 2.3 and Theorem 2.1, respectively, is chosen as follows. Let $\xi \in \Omega$ and $0 < r < \infty$ be such that $\overline{B}_r(\xi) \subset \Omega$, $q(\xi) > 0$, and let $K = \|q\|_{L^\infty(\Omega)}$ and $0 < T_0 < \infty$. We choose $\varepsilon = \frac{r}{4} > 0$ and $0 < T < \min\{\frac{1}{K\alpha}, T_0\}$ in Theorem 2.1. This choice of T determines the positive constants σ and ϱ in eq. (2.1). Let us denote $T' = \min\{\frac{r}{4\varrho}, T\}$. Hence,

$$\varepsilon + \varrho t \le \frac{r}{2} \quad \text{and} \quad \left[1 - \left(\frac{|x-\xi|}{\varepsilon+\varrho t}\right)^2\right]_+ = 0 \quad \text{whenever } |x-\xi| \ge \frac{r}{2} \text{ and } t \in [0, T']\,.$$

Let $\bar{u}$ denote the (radially symmetric) supersolution constructed in Theorem 2.1 centered at a given point $\xi \in \Omega$ instead of $\xi = 0 \in \mathbb{R}^N$. We will not need the radial symmetry of $\bar{u}$ about ξ in the remaining part of the proof. Our choice of the constants ε and T' above guarantees

$\bar{u}(\xi, t) = 1 + \sigma t \geq 1$ and $\bar{u}(x, t) = 0$ for all $x \in \mathbb{R}^N \setminus B_{r/2}(\xi)$, both for all $t \in [0, T']$. We choose the radius $R = r\sqrt{2}/8$ in Proposition 2.3; hence,

$$1 - \left(\frac{|x - \xi|}{\varepsilon + \varrho t} \right)^2 \geq \frac{1}{2} \quad \text{for all} \quad x \in B_R(\xi) \quad \text{and} \quad t \in [0, T'] .$$

Consequently, we have

$$\bar{u}(x, t) \geq (1 + \sigma t) 2^{-\frac{1}{1-\alpha}} \geq 2^{-\frac{1}{1-\alpha}} .$$

We may choose $\underline{T} \in (0, \infty)$ in Proposition 2.3 small enough such that

$$\underline{u}(x, t) \leq 2^{-\frac{1}{1-\alpha}} \quad \text{holds for all} \quad x \in B_r(\xi) \quad \text{and} \quad t \in [0, \underline{T}] ,$$

thanks to $0 < R < r$. Setting $T'' = \min\{T', \underline{T}\} > 0$ and recalling our extensions of $\underline{u}$ and $\bar{u}$ to all of $\Omega \times [0, T'']$, we observe that the inequality $\underline{u} \leq \bar{u}$ is valid in all of $\Omega \times [0, T'']$. In addition, we recall that $\underline{u}$ and $\bar{u}$ is a pair of sub- and supersolutions to problem (1.1) in $\Omega \times (0, T'')$. Since the constant T from Theorem 2.1 does not appear explicitly any more, we relabel T'' ($\leq T$) by T in order to keep our notation compatible with the statement of Theorem 1.3.

We start our monotone iteration procedure from the subsolution $u_1(x, t) = \underline{u}(x, t)$ described in Proposition 2.3. For each $n = 1, 2, 3, \dots$ we define $u_{n+1}(x, t)$ recursively to be the unique weak solution to following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u_{n+1}}{\partial t} - \Delta_p u_{n+1} = q(x) u_n^\alpha & \text{for } (x, t) \in \Omega \times (0, T); \\ u_{n+1}(x, t) = 0 & \text{for } (x, t) \in \partial\Omega \times (0, T); \\ u_{n+1}(x, 0) = 0 & \text{for } x \in \Omega . \end{cases} \tag{2.17}$$

Since $\underline{u}(x, t) = u_1(x, t) \leq \bar{u}(x, t)$ for $(x, t) \in \Omega \times (0, T)$, it follows from (2.17) that $\underline{u} = u_1 \leq u_2 \leq \dots \leq u_n \leq \bar{u}$ implies also $u_n \leq u_{n+1} \leq \bar{u}$, by induction on $n = 1, 2, 3, \dots$. Thus, we have constructed a monotone increasing sequence of subsolutions $\underline{u} = u_1 \leq u_2 \leq \dots \leq u_n \leq \dots$ to problem (1.1) bounded above by the supersolution $\bar{u}$. Standard regularity and compactness arguments from [7] now guarantee that the sequence $\{u_n\}_{n=1}^\infty$ converges uniformly in $\overline{\Omega} \times [\tau, T]$, for any $\tau \in (0, T)$, to a continuous function $u: \overline{\Omega} \times (0, T] \to \mathbb{R}$. Since also $u_n(x, 0) \equiv 0$ for each $n = 1, 2, 3, \dots$, we obtain also $u(x, 0) \equiv 0$. By such a regularity result proved in Lieberman [14, Thm. 0.1, p. 552], one obtains also the convergence in $C^{1+\gamma, (1+\gamma)/2}(\overline{\Omega} \times [\tau, T])$ for any $\tau \in (0, T)$. It is proved in [7] that u is a weak solution to problem (1.1) in the sense of [8, §3.1, p. 23] (cf. Definition 1.1) and it satisfies $\underline{u} \leq u \leq \bar{u}$. Finally, the properties of $\underline{u}$ and $\bar{u}$ yield the conclusion of Theorem 1.3.

If $\Omega = B_R(\xi)$, $0 < r < R < \infty$, and q is radially symmetric about ξ, then also all subsolutions $\underline{u} = u_1 \leq u_2 \leq \cdots u_n \leq \cdots$ constructed above are radially symmetric about ξ, by the uniqueness of the weak solution $u_{n+1}(x, t)$ to the initial-boundary value problem (2.17). The starting subsolution, $u_1 = \underline{u}$, is radially symmetric about ξ, by construction in Proposition 2.3. Consequently, also the limit $u: \overline{\Omega} \times [0, T] \to \mathbb{R}$ of the monotone increasing sequence $u_n \nearrow u$ $(n \to \infty)$ is radially symmetric about ξ, as desired, $u \leq \bar{u}$. □

3. Main results

Theorem 1.2 is a special one-dimensional case of the following more general result.

Theorem 3.1. *Let $2 < p < \infty$, $1/(p-1) < \alpha < 1$, and let $\Omega_k \subset \Omega$; $k = 1, 2, 3, \dots, m$, be a family of pairwise disjoint subdomains of the domain $\Omega \subset \mathbb{R}^N$, and let $0 < T_0 < \infty$. Furthermore, let $0 \le q \in C(\overline{\Omega})$ and $\xi_k \in \Omega_k$ be such that $q(\xi_k) > 0$; $k = 1, 2, 3, \dots, m$. Then there exists some $T \in (0, T_0]$ such that the initial-boundary value problem (1.1) possesses a nontrivial nonnegative solution $u : \Omega \times (0, T) \to \mathbb{R}_+$ such that*

(i) $u(\xi_k, t) > 0$ *for all $k = 1, 2, \dots, m$ and all $t \in (0, T)$;*
(ii) $u(x, t) = 0$ *for all $x \in \overline{\Omega} \setminus \cup_{k=1}^{m} \Omega_k$ and all $t \in [0, T]$.*

Proof. We will apply Theorem 1.3 in each subdomain Ω_k; $k = 1, 2, \dots, m$, separately. In fact, by Theorem 1.3 we may replace each subdomain Ω_k by an open ball $B_{r_k}(\xi_k) \subset \Omega$ with a sufficiently small radius $r_k > 0$. Furthermore, we may replace each r_k by any $r \in \mathbb{R}$ satisfying $0 < r < \min\{r_1, r_2, \dots, r_m\}$. In particular, the balls $B_r(\xi_k) \subset \Omega$; $k = 1, 2, \dots, m$, have pairwise disjoint closures $\overline{B}_r(\xi_k)$. Now, by Theorem 1.3, there exists a nontrivial nonnegative solution $u_k : \Omega \times (0, T) \to \mathbb{R}_+$ such that

(i) $u_k(\xi_k, t) > 0$ for all $t \in (0, T)$;
(ii) $u_k(x, t) = 0$ for all $x \in \Omega \setminus B_r(\xi_k)$ and all $t \in [0, T]$.

In order to keep our construction more explicit, we may assume that each solution u_k is radially symmetric about ξ_k.

The supports of functions u_k being pairwise disjoint, we conclude that also the sum $u = \sum_{k=1}^{m} u_k$ is a weak solution to problem (1.1) satisfying $0 \le u_k \le u$ in $\Omega \times (0, T)$ and $u = u_k$ in $\Omega_k \times [0, T]$ for each $k = 1, 2, \dots, m$. This function, u, satisfies conditions (i) and (ii); the theorem is proved. □

Inspecting the proof of Theorem 3.1, one may generalize this theorem as follows.

Theorem 3.2. *Let $2 < p < \infty$, $1/(p-1) < \alpha < 1$, and let $\Omega_k \subset \Omega$; $k = 1, 2, 3, \dots, m$, be a family of pairwise disjoint subdomains of the domain $\Omega \subset \mathbb{R}^N$. Assume that $0 \le q \in C(\overline{\Omega})$ and $\xi_k \in \Omega_k$ is such that $q(\xi_k) > 0$; $k = 1, 2, 3, \dots, m$. Finally, let $T \in (0, T_0]$ be the time constant obtained in Theorem 3.1 and let $0 \le t_1 \le t_2 \le \cdots \le t_m < T$. Then the initial-boundary value problem (1.1) possesses a nontrivial nonnegative solution $u : \Omega \times (0, T) \to \mathbb{R}_+$ such that*

(i_0) $u(x, t) = 0$ *for all $(x, t) \in \overline{\Omega}_k \times [0, t_k]$; $k = 1, 2, \dots, m$;*
(i_+) $u(\xi_k, t) > 0$ *for all $t \in (t_k, T)$; $k = 1, 2, \dots, m$;*
(ii) $u(x, t) = 0$ *for all $x \in \overline{\Omega} \setminus \cup_{k=1}^{m} \Omega_k$ and all $t \in [0, T]$.*

Remark 3.3. It is obvious from condition (ii) that u is a weak solution to (1.1) also in the entire space–time domain $\mathbb{R}^N \times (0, T)$, i.e., we may replace Ω by the entire space $\mathbb{R}^N$.

Proof of Theorem 3.2. In the proof of Theorem 3.1 above, we replace the function u_k by the time-shifted function

$$\hat{u}_k(x,t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{for } x \in \Omega,\ t \in [0, t_k]; \\ u_k(x, t-t_k) & \text{for } x \in \Omega,\ t \in (t_k, T); \end{cases} \tag{3.1}$$

$k = 1, 2, \dots, m$. The remaining part of the proof is identical. □

For the porous medium equation, related results have been obtained in de Pablo and Vázquez [5,6].

In many technical constructions in the proofs of our results and those in [2,4–6,10,15], spatially radially symmetric sub- and/or supersolutions to various (auxiliary) parabolic problems have played a crucial role. This is the main reason why we formulate the following analogue of Theorems 1.2 and 3.1 for radially symmetric solutions in a ball $\Omega = B_R(0)$ centered at the origin, $0 < R < \infty$. Recall that we use the standard notation for radially symmetric functions: $q(x) \equiv q(|x|) = q(r)$ and $u(x,t) \equiv u(|x|,t) = u(r,t)$ with the radial variable $r = |x|$ for $x \in \mathbb{R}^N$.

Theorem 3.4. *Let $2 < p < \infty$, $1/(p-1) < \alpha < 1$, $0 < R < \infty$, and let $I_k = [a_k, b_k]$; $k = 1, 2, 3, \dots, m$, be a family of pairwise disjoint compact intervals in $(0, R]$, $0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m \le R$, and let $0 < T_0 < \infty$. Assume that $q\colon [0, R] \to \mathbb{R}_+$ is a continuous function and $\xi_k \in (a_k, b_k)$ such that $q(\xi_k) > 0$; $k = 1, 2, 3, \dots, m$. Then there exists some $T \in (0, T_0]$ such that the initial-boundary value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = q(|x|)u(x,t)^\alpha & \text{for } |x| < R\,, 0 < t < T; \\ \quad u(x,t) = 0 & \text{for } |x| = R\,, 0 < t < T; \\ \quad u(x,0) = 0 & \text{for } |x| < R\,, \end{cases} \tag{3.2}$$

possesses a nontrivial nonnegative solution $u(x,t) \equiv u(|x|,t)$, $u\colon [0, R) \times (0, T) \to \mathbb{R}_+$ such that

(i) *$u(\xi_k, t) > 0$ for all $k = 1, 2, \dots, m$ and all $t \in (0, T)$;*
(ii) *$u(r,t) = 0$ for all $r \in [0, R] \setminus \cup_{k=1}^m (a_k, b_k)$ and all $t \in (0, T)$.*

Remark 3.5. We may combine Theorems 1.3 and 3.4 in order to generalize the latter (Theorem 3.4) to the case when the family of intervals I_k; $k = 0, 1, 2, \dots, m$, contains also an interval $[0, b_0]$ with $0 < b_0 < a_1$. Clearly, if $q(0) > 0$, we may apply Theorem 1.3 in the ball $B_{b_0}(0)$ in order to obtain a nontrivial nonnegative solution u_0 to problem (3.2), $u_0\colon [0, R] \times (0, T) \to \mathbb{R}_+$, such that $u_0(0, t) > 0$ for all $t \in (0, T)$ and $u_0(r, t) = 0$ for all $r \in [b_0, R]$ and all $t \in [0, T]$. The remaining functions u_k for $k = 1, 2, \dots, m$ can be taken from Theorem 3.4 (see our construction of $\underline{u}$ and $\bar{u}$ in its proof below).

Proof of Theorem 3.4. We abbreviate $\Omega_k = B_{b_k}(0) \setminus \overline{B}_{a_k}(0)$, an open spherical shell centered at the origin. The closures $\overline{\Omega}_k$ of Ω_k; $k = 1, 2, \dots, m$, are pairwise disjoint, concentric spherical shells. The function q being continuous on $[a_k, b_k]$ with $q(\xi_k) > 0$ for some $\xi_k \in (a_k, b_k)$, there is a number $\delta_k \in \mathbb{R}$ such that

$$0 < \delta_k < \min\{\xi_k - a_k, b_k - \xi_k\}$$

and

$$q(|x|) \geq q_k \stackrel{\text{def}}{=} \frac{1}{2} q(\xi_k) > 0 \qquad \text{holds for all}$$

$$x \in \overline{\Sigma}_k \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^N : \xi_k - \delta_k \leq |x| \leq \xi_k + \delta_k \right\} \subset \Omega_k .$$

Abbreviating by Σ_k the interior domain of the compact spherical shell $\overline{\Sigma}_k$, we denote by $\lambda_{1,k}$ the first eigenvalue of the operator $-\Delta_p: W_0^{1,p}(\Sigma_k) \to W_0^{-1,p'}(\Sigma_k)$, $1/p + 1/p' = 1$. This eigenvalue is simple with an eigenfunction $\varphi_{1,k} \in C^1\left(\overline{\Sigma}_k\right)$, which is radially symmetric and can be normalized by $\varphi_{1,k}(|x|) > 0$ for all $x \in \Sigma_k$ and $\sup_{x\in\Sigma_k} \varphi_{1,k}(|x|) = 1$.

By (2.11) in Lemma 2.2, we have

$$-\Delta_p \left(\varphi_{1,k}^{\beta}(r) \right) = \beta^{p-1} \left[\varphi_{1,k}^{\beta}(r) \right]^{p-1} \left\{ \lambda_{1,k} - (p-1)(\beta-1) \frac{|\varphi_{1,k}'(r)|^p}{\varphi_{1,k}^p(r)} \right\} .$$

Hence,

$$-\Delta_p \left(\varphi_{1,k}^{\beta}(r) \right) \leq \beta^{p-1} \lambda_{1,k} \varphi_{1,k}^{(p-1)\beta}(r) ,$$

for $\beta \geq 1$. Since $p \geq 2$, for all $r \in (\xi_k - \delta_k, \xi_k + \delta_k)$ this yields

$$\frac{-\Delta_p(\varphi_{1,k}^{\beta})}{\varphi_{1,k}^{\beta}} \leq \beta^{p-1} \lambda_{1,k} \varphi_{1,k}^{(p-2)\beta} \leq \beta^{p-1} \lambda_{1,k} = \text{const} \equiv C ,$$

thanks to our normalization $0 < \varphi_{1,k} \leq 1$. Let us denote

$$\tilde{\varphi}_{1,k}(r) = \begin{cases} \varphi_{1,k}(r) & \text{for } \xi_k - \delta_k < r < \xi_k + \delta_k ; \\ 0 & \text{for } r \in \mathbb{R}_+ \setminus (\xi_k - \delta_k, \xi_k + \delta_k) , \end{cases}$$

and

$$\underline{u}_k(x,t) := \theta(t) \widetilde{\varphi}_{1,k}(x)^{\beta} \quad \text{ for } (x,t) \in \mathbb{R}^N \times [0, T_0] ,$$

where function $\theta(t)$ is defined by (2.15), with $q_0 = \min\{q_1, q_2, \ldots, q_m\}$. Calculations similar to those in the proof of Proposition 2.3 given in [2, Prop. 2.2, pp. 4–5] yield that

$$\underline{u}_k : \Sigma_k \times \left(0, \underline{T}\right) \mapsto \mathbb{R}_+$$

is a subsolution to problem (3.2) for some $\underline{T}$ sufficiently small, $0 < \underline{T} \leq T_0$. In fact, since $\beta > 1$, $\underline{u}_k$, extended by zero to $\left(\mathbb{R}^N \setminus \Sigma_k\right) \times \left(0, \underline{T}\right)$, is a subsolution to problem (3.2) in all of $\mathbb{R}^N \times \left(0, \underline{T}\right)$. Since θ is independent of k, so is $\underline{T}$. Consequently, also the sum $\underline{u} = \sum_{k=1}^m \underline{u}_k$ yields a subsolution to problem (3.2) in $\mathbb{R}^N \times \left(0, \underline{T}\right)$ which is supported in $\cup_{k=1}^m \overline{\Sigma}_k$.

A supersolution $\bar{u}$ to problem (3.2) in $\mathbb{R}^N \times \left(0, \overline{T}\right)$ is constructed in an analogous way, $\bar{u} = \sum_{k=1}^m \bar{u}_k$. Let $0 < \tau < \frac{1}{K\alpha}$ where $K = \|q\|_{L^\infty(B_R)}$. Denote

$$\sigma = \frac{K}{1-K\tau\alpha}\,, \quad \varepsilon_k = \frac{1}{2}(\delta_k + \min\{\xi_k - a_k, b_k - \xi_k\})\,,$$

$$\varrho_k = \frac{p-2}{p-1}\sigma(1-\alpha)\varepsilon_k + \frac{(1+\sigma\tau)^{p-1}(p-1)^p\alpha}{(1-\alpha)^{p-1}(p-2)^{p-1}\varepsilon_k^{p-1}} + \frac{(N-1)(1+\sigma\tau)^{p-1}(p-1)^{p-2}}{a_k(1-\alpha)^{p-2}(p-2)^{p-2}\varepsilon_k^{p-2}}\,,$$

$$\overline{T} = \min\left\{T_0, \tau,\ \frac{1}{2\varrho_1}\,(\min\{\xi_1 - a_1, b_1 - \xi_1\} - \delta_1)\,, \dots, \frac{1}{2\varrho_m}\,(\min\{\xi_m - a_m, b_m - \xi_m\} - \delta_m)\right\}.$$

Then $\bar{u}(x,t) = \Sigma_{k=1}^m \bar{u}_k(x,t)$ is a supersolution on $B_R \times [0, \overline{T}]$, where

$$\bar{u}_k(x,t) = (1+\sigma t)\left[1 - \left(\frac{||x| - \xi_k|}{\varepsilon_k + \varrho_k t}\right)^{\frac{p-1}{p-2}}\right]_+^{\frac{1}{1-\alpha}}.$$

Since the supports of $\bar{u}_k$ are pairwise disjoint, it suffices to show that each $\bar{u}_k$ is a supersolution on $\Omega_k \times [0, \overline{T}]$; more precisely, we have

$$\frac{\partial \bar{u}_k}{\partial t} - \Delta_p \bar{u}_k \geq q(x)\bar{u}_k^\alpha \quad \text{for all } t \in [0, \overline{T}]\,,\ 0 < ||x| - \xi_k| < \varepsilon_k + \varrho_k t\,.$$

In order to simplify lengthy expressions, we abbreviate the bracket

$$[\dots]_k \equiv \left[1 - \left(\frac{|r - \xi_k|}{\varepsilon_k + \varrho_k t}\right)^{\frac{p-1}{p-2}}\right] \quad \text{for } r = |x| \in (\xi_k - \varepsilon_k - \varrho_k t, \xi_k + \varepsilon_k + \varrho_k t)\,.$$

Indeed, we estimate

$$\begin{aligned}
\frac{\partial \bar{u}_k}{\partial t}(r,t) &= \sigma\,[\dots]_k^{\frac{1}{1-\alpha}} + \frac{1+\sigma t}{1-\alpha}\,[\dots]_k^{\frac{\alpha}{1-\alpha}}\ \frac{p-1}{p-2}\,\frac{\varrho_k |r-\xi_k|^{\frac{p-1}{p-2}}}{(\varepsilon_k + \varrho_k t)^{\frac{p-1}{p-2}+1}} \\
&= [\dots]_k^{\frac{\alpha}{1-\alpha}}\left(\sigma + |r-\xi_k|^{\frac{p-1}{p-2}}\,\frac{-\sigma(1-\alpha)(p-2)(\varepsilon_k + \varrho_k t) + (1+\sigma t)(p-1)\varrho_k}{(1-\alpha)(p-2)(\varepsilon_k + \varrho_k t)^{\frac{p-1}{p-2}+1}}\right) \\
&= [\dots]_k^{\frac{\alpha}{1-\alpha}}\left(\sigma + |r-\xi_k|^{\frac{p-1}{p-2}}\,\frac{(p-1)\varrho_k - \sigma(1-\alpha)(p-2)\varepsilon_k + \sigma\varrho_k t(1 + \alpha(p-2))}{(1-\alpha)(p-2)(\varepsilon_k + \varrho_k t)^{\frac{p-1}{p-2}+1}}\right) \\
&\geq [\dots]_k^{\frac{\alpha}{1-\alpha}}\left(\sigma + \frac{|r-\xi_k|^{\frac{p-1}{p-2}}}{(1-\alpha)(p-2)(\varepsilon_k + \varrho_k t)^{\frac{p-1}{p-2}+1}}\big((p-1)\varrho_k - \sigma(1-\alpha)(p-2)\varepsilon_k\big)\right),
\end{aligned}$$

by $t \geq 0$ and $\varrho_k > 0$. We further estimate the radial p-Laplacian in several steps:

$$\frac{\partial \bar{u}_k}{\partial r}(r,t) = -\frac{1+\sigma t}{1-\alpha}\ [\dots]_k^{\frac{\alpha}{1-\alpha}}\ \frac{p-1}{p-2}\,\frac{|r-\xi_k|^{\frac{1}{p-2}}}{(\varepsilon_k + \varrho_k t)^{\frac{p-1}{p-2}}}\,\mathrm{sgn}(r - \xi_k)\,;$$

$$\left|\frac{\partial \bar{u}_k}{\partial r}\right|^{p-2}\frac{\partial \bar{u}_k}{\partial r} = \left(\frac{(1+\sigma t)(p-1)}{(1-\alpha)(p-2)}\right)^{p-1} [\ldots]_k^{\frac{\alpha(p-1)}{1-\alpha}} \ \frac{|r-\xi_k|^{\frac{p-1}{p-2}}}{(\varepsilon_k+\varrho_k t)^{\frac{(p-1)^2}{p-2}}} \ \operatorname{sgn}(r-\xi_k)\,,$$

and

$$\begin{aligned}
-\frac{\partial}{\partial r}\left(\left|\frac{\partial \bar{u}_k}{\partial r}\right|^{p-2}\frac{\partial \bar{u}_k}{\partial r}\right) &= \left(\frac{(1+\sigma t)(p-1)}{(1-\alpha)(p-2)}\right)^{p-1} \frac{\operatorname{sgn}(r-\xi_k)}{(\varepsilon_k+\varrho_k t)^{\frac{(p-1)^2}{p-2}}} \\
&\quad \times \left\{-\frac{\alpha(p-1)}{1-\alpha}[\ldots]^{\frac{\alpha p-1}{1-\alpha}} \ \frac{p-1}{p-2} \ \frac{|r-\xi_k|^{\frac{p}{p-2}}}{(\varepsilon_k+\varrho_k t)^{\frac{p-1}{p-2}}} \operatorname{sgn}(r-\xi_k)\right. \\
&\qquad \left. +[\ldots]^{\frac{\alpha(p-1)}{1-\alpha}} \ \frac{p-1}{p-2}\,|r-\xi_k|^{\frac{1}{p-2}} \operatorname{sgn}(r-\xi_k)\right\} \\
&= [\ldots]_k^{\frac{\alpha p-1}{1-\alpha}} \left(\frac{1+\sigma t}{1-\alpha}\right)^{p-1} \left(\frac{p-1}{p-2}\right)^{p} \frac{|r-\xi_k|^{\frac{1}{p-2}}}{(\varepsilon_k+\varrho_k t)^{\frac{(p-1)^2}{p-2}}} \\
&\quad \times \left\{-\frac{\alpha(p-1)}{1-\alpha} \ \frac{|r-\xi_k|^{\frac{p-1}{p-2}}}{(\varepsilon_k+\varrho_k t)^{\frac{p-1}{p-2}}} + [\ldots]_k\right\} \\
&\geq -[\ldots]_k^{\frac{\alpha}{1-\alpha}} \left(\frac{1+\sigma t}{1-\alpha}\right)^{p-1} \left(\frac{p-1}{p-2}\right)^{p} \ \frac{|r-\xi_k|^{\frac{1}{p-2}}}{(\varepsilon_k+\varrho_k t)^{\frac{(p-1)^2}{p-2}}} \\
&\quad \times \frac{\alpha(p-1)}{1-\alpha} \ \frac{|r-\xi_k|^{\frac{p-1}{p-2}}}{(\varepsilon_k+\varrho_k t)^{\frac{p-1}{p-2}}} \\
&= -[\ldots]_k^{\frac{\alpha}{1-\alpha}} \ \frac{|r-\xi_k|^{\frac{p-1}{p-2}}}{(1-\alpha)(p-2)(\varepsilon_k+\varrho_k t)^{\frac{p-1}{p-2}+1}} \\
&\quad \times \frac{(1+\sigma t)^{p-1}(p-1)^{p+1}\alpha}{(1-\alpha)^{p-1}(p-2)^{p-1}(\varepsilon_k+\varrho_k t)^{p-1}} \left(\frac{|r-\xi_k|}{\varepsilon_k+\varrho_k t}\right)^{\frac{1}{p-2}} \\
&\geq -[\ldots]_k^{\frac{\alpha}{1-\alpha}} \ \frac{|r-\xi_k|^{\frac{p-1}{p-2}}}{(1-\alpha)(p-2)(\varepsilon_k+\varrho_k t)^{\frac{p-1}{p-2}+1}} \\
&\quad \times \frac{(1+\sigma \tau)^{p-1}(p-1)^{p+1}\alpha}{(1-\alpha)^{p-1}(p-2)^{p-1}\varepsilon_k^{p-1}}\,,
\end{aligned}$$

since $[\ldots]_k \in (0,1)$, $\alpha p - 1 > \alpha$, $\frac{|r-\xi_k|}{\varepsilon_k+\varrho_k t} \in (0,1)$, and $t \in [0,\tau]$.

Furthermore, we have

$$\frac{N-1}{r}\left|\frac{\partial \bar{u}_k}{\partial r}\right|^{p-2}\frac{\partial \bar{u}_k}{\partial r} = -[\ldots]_k^{\frac{\alpha(p-1)}{1-\alpha}} \ \frac{|r-\xi_k|^{\frac{p-1}{p-2}}}{(1-\alpha)(p-2)(\varepsilon_k+\varrho_k t)^{\frac{p-1}{p-2}+1}}$$

$$\times \frac{(N-1)(1+\sigma t)^{p-1}(p-1)^{p-1}}{r(1-\alpha)^{p-2}(p-2)^{p-2}(\varepsilon_k+\varrho_k t)^{p-2}}$$

$$\geq -[\dots]_k^{\frac{\alpha}{1-\alpha}} \frac{|r-\xi_k|^{\frac{p-1}{p-2}}}{(1-\alpha)(p-2)(\varepsilon_k+\varrho_k t)^{\frac{p-1}{p-2}+1}}$$

$$\times \frac{(N-1)(1+\sigma\tau)^{p-1}(p-1)^{p-1}}{a_k(1-\alpha)^{p-2}(p-2)^{p-2}\varepsilon_k^{p-2}},$$

due to $[\dots]_k \in (0,1)$, $p-1>1$, $r>a_k$, and $t \in [0,\tau]$.

Finally,

$$\begin{aligned}
&\frac{\partial \bar{u}_k}{\partial t}(x,t) - \Delta_p \bar{u}_k(x,t) \\
&= \frac{\partial \bar{u}_k}{\partial t}(r,t) - \frac{\partial}{\partial r}\left(\left|\frac{\partial \bar{u}_k}{\partial r}(r,t)\right|^{p-2}\frac{\partial \bar{u}_k}{\partial r}(r,t)\right) - \frac{N-1}{r}\left|\frac{\partial \bar{u}_k}{\partial r}(r,t)\right|^{p-2}\frac{\partial \bar{u}_k}{\partial r}(r,t) \\
&\geq [\dots]_k^{\frac{\alpha}{1-\alpha}}\Bigg\{\sigma + \frac{|r-\xi_k|^{\frac{p-1}{p-2}}}{(1-\alpha)(p-2)(\varepsilon_k+\varrho_k t)^{\frac{p-1}{p-2}+1}} \\
&\qquad \times \Bigg((p-1)\varrho_k - \sigma(1-\alpha)(p-2)\varepsilon_k - \frac{(1+\sigma\tau)^{p-1}(p-1)^{p+1}\alpha}{(1-\alpha)^{p-1}(p-2)^{p-1}\varepsilon_k^{p-1}} \\
&\qquad - \frac{(N-1)(1+\sigma\tau)^{p-1}(p-1)^{p-1}}{a_k(1-\alpha)^{p-2}(p-2)^{p-2}\varepsilon_k^{p-2}}\Bigg)\Bigg\} \\
&= \sigma[\dots]_k^{\frac{\alpha}{1-\alpha}} = K(1+\sigma\tau\alpha)[\dots]_k^{\frac{\alpha}{1-\alpha}} \geq K(1+\sigma t)^{\alpha}[\dots]_k^{\frac{\alpha}{1-\alpha}} \\
&= K\bar{u}^{\alpha}(r,t) \geq q(x)\bar{u}^{\alpha}(x,t),
\end{aligned}$$

thanks to our choice of ϱ_k.

By a simple time-continuity argument, there exists some $T \in \mathbb{R}$,

$$0 < T \leq \min\{T_0, \underline{T}, \overline{T}\},$$

such that $\underline{u} \leq \bar{u}$ in $\mathbb{R}^N \times [0,T]$. The desired solution $u(x,t) \equiv u(|x|,t)$ in Theorem 3.4 is now obtained by monotone iterations as in the proof of Theorem 1.3, cf. (2.17). □

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Appendix A3

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The strong maximum principle in parabolic problems with the p-Laplacian in a domain

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ABSTRACT

We establish a strong maximum principle for a nonnegative continuous solution $u\colon \overline{\Omega} \times [0,T) \to \mathbb{R}_+$ of a doubly nonlinear parabolic problem in a space–time cylinder $\Omega \times (0,\tau)$ with a domain $\Omega \subset \mathbb{R}^N$ and a sufficiently short time interval $(0,\tau) \subset (0,T)$. Our method takes advantage of a nonnegative subsolution derived from an expanding spherical wave.

1. Introduction

The strong maximum principle in parabolic problems in a spatial domain $\Omega \subset \mathbb{R}^N$ describes a propagation property for the mass or energy in a variety of mathematical reaction–diffusion models, including quasilinear models with convection and absorption studied in [1–5]. These kinds of models exhibit the following two phenomena, among others:

Finite speed of propagation. In our recent work [1] we have constructed nontrivial nonnegative solutions with compact support in the space variable $x \in \mathbb{R}^N$ and zero initial data for a quasilinear parabolic problem with the p-Laplacian for $p > 2$ (weak, degenerate diffusion) and a nonsmooth reaction function (only Hölder-continuous). The positivity of the solution appears thanks to a nonsmooth reaction function, whereas the compact support expands with finite speed, thanks to $p > 2$.

In contrast, *infinite speed of propagation* has been suggested in the work of Khin and Su [4]. In our present work we establish this phenomenon in the "complementary" case of $1 < p < 2$ (strong, singular diffusion).

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More precisely, we prove a strong maximum principle for the following quasilinear parabolic problem:

$$\begin{cases} \dfrac{\partial}{\partial t} b\left(u(x,t)\right) - \Delta_p u(x,t) = f(x,t) & \text{for } (x,t) \in \Omega \times (0,T)\,; \\ u(x,0) = u_0(x) & \text{for } x \in \Omega; \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T)\,. \end{cases} \tag{1.1}$$

As usual, we abbreviate $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Our most important hypothesis is $1 < p < 2$. In addition, we assume the following standard hypotheses: $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, $b(0) = 0$, and $b \in C^1(0,+\infty)$ with $b' > 0$ in $(0,+\infty)$. For simplicity, we assume that both, $f\colon \Omega \times (0,T) \to \mathbb{R}$ and $u_0\colon \Omega \to \mathbb{R}$, are continuous and nonnegative. The following kind of strong maximum principle describes the propagation with infinite speed throughout the domain Ω.

Theorem 1.1. *Let $1 < p < 2$, $N \geq 1$ and assume that $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ is as above and satisfies also*

$$\lim_{s\to 0+} \frac{s^{2-p}\, b'(s)}{|\log s|^{p-1}} = 0. \tag{1.2}$$

Finally, assume that $u\colon \overline{\Omega} \times [0,T) \to \mathbb{R}_+$ is a continuous, nonnegative, weak solution of (1.1). Then, for any fixed $t_0 \in (0,T)$, the solution $u(\cdot,t_0)$ is either positive everywhere on Ω or else identically zero on Ω.

In particular, if $u(\xi,0) = u_0(\xi) > 0$ for some $\xi \in \Omega$, then there exists $\tau \in (0,T]$ such that $u(x,t) > 0$ for all $(x,t) \in \Omega \times (0,\tau)$, i.e., the strong maximum principle is valid in the $(N+1)$-dimensional space–time cylinder $\Omega \times (0,\tau)$. The number $\tau \in (0,T)$ can be estimated from below by

$$\tau = \sup\left\{T' \in (0,T]\colon u(\xi,t) > 0 \text{ for all } t \in [0,T')\right\} > 0. \tag{1.3}$$

A related, spatially localized result around $\xi \in \mathbb{R}$ is proved in [6]. The initial positivity hypothesis $u_0(\xi) > 0$ is not required. It is obtained for any $t > 0$ small enough, i.e., $u(\xi,t) > 0$, from a positive subsolution that can be constructed if $f = f(u)$ is only Hölder continuous as $u \searrow 0$. Since the subsolution is positive only on a small ball $B_R(x_0) \subset \Omega$, also the positivity of u may be spatially localized. In fact, we have constructed such a solution for $p > 2$ in our work [1, Theorem 1.3]. Our present result, Theorem 1.1, is of somewhat different nature. We do not work with a nonsmooth reaction function $f = f(u)$, that would produce a nontrivial, nonnegative solution $u(x,t)$ for every $t > 0$ small enough, in spite of $u_0 \equiv 0$ in $\mathbb{R}^N$. Rather, we assume the positivity of the initial data $u_0(\xi) > 0$ at some point $\xi \in \Omega$ and derive from it that $u > 0$ throughout $\Omega \times (0,\tau)$, where $\tau \in (0,T)$. By the finite time extinction property proved in Chen and DiBenedetto [7, p. 323] and DiBenedetto [2, Chapt. VII, Section 3, Prop. 3.1], if $1 < p < 2$ and $f(x,t) \equiv 0$ then one has $u(x,t) \equiv 0$ for all $t \geq T^*$, provided $T^* > 0$ is sufficiently large. Their proof makes use of a spatially localized result related to our Theorem 1.1, see Chen and DiBenedetto [7, Theorem 2, p. 323]. The same results are established also in A.V. Ivanov [8, p. 32] Theorems 6.2 and 6.3, in a more general setting.

2. Proof of the main result

Our continuity hypothesis on the solution $u\colon \overline{\Omega} \times [0,T) \to \mathbb{R}$ in Theorem 1.1 has been verified in the following two important situations:

(a) For $b(s) \equiv s$ for all $s \in \mathbb{R}_+$, see Chen and DiBenedetto [7, Theorem 1, p. 320] and A.V. Ivanov [8, p. 28], Propositions 3.1 and 3.2.

(b) For $b(s) = s^\sigma$ for all $s \in \mathbb{R}_+$, where $\sigma \in \mathbb{R}$ is a constant, $(0 <)\ p - 1 \leq \sigma < +\infty$, see A.V. Ivanov [8, Eq. (1.7), p. 23] combined with [8, p. 28], Propositions 3.1 and 3.2.

Proof of Theorem 1.1. Let us fix any $t_0 \in (0,T)$ and denote $Z \equiv Z(t_0) = \{x \in \Omega : u(x,t_0) = 0\}$. Our aim is to prove that either $Z = \Omega$ or else $Z = \emptyset$. It is sufficient to prove that Z is an open subset of Ω since it is closed thanks to the continuity of u and the connected set Ω possesses only two open and closed subsets, Ω and $\emptyset$.

Let $x_1 \in Z$. We prove that $B_{d/2}(x_1) \subset Z$ where $d = \operatorname{dist}(x_1, \partial\Omega)$. Note that if $\Omega = \mathbb{R}^N$ then $d = \infty$. We assume, by contradiction, that $x_2 \notin Z$ for some $x_2 \in B_{d/2}(x_1)$. In other words, we have $u(x_2,t_0) > u(x_1,t_0) = 0$ and $|x_1 - x_2| < \frac{1}{2}d = \frac{1}{2}\operatorname{dist}(x_1,\partial\Omega)$. Consequently, since $\operatorname{dist}(x_2,\partial\Omega) > \frac{1}{2}d$ holds by the triangle inequality, we have $|x_1 - x_2| < \operatorname{dist}(x_2,\partial\Omega)$. Due to the continuity of u, there exist $R \in (0, |x_1 - x_2|)$ and $\tau \in (0, t_0)$ such that

$$\eta \stackrel{\text{def}}{=} \inf_{(x,t)\in \overline{B}_R(x_2)\times[t_0-\tau,t_0]} u(x,t) > 0. \tag{2.1}$$

Notice that $\overline{B}_R(x_2) \cap Z(t) = \emptyset$ for all $t \in [t_0 - \tau, t_0]$.

Let us fix a constant $R^* \in (|x_1 - x_2|, \operatorname{dist}(x_2,\partial\Omega))$. Hence, $0 < R < |x_1 - x_2| < R^* < \operatorname{dist}(x_2,\partial\Omega)$. We will construct a subsolution $v\colon \left(\overline{B}_{R^*}(x_2) \setminus B_R(x_2)\right) \times [t_0 - \tau, t_0] \to \mathbb{R}_+$ of problem (1.1) satisfying

(a) $v(x, t_0 - \tau) = 0$ for all $x \in \overline{B}_{R^*}(x_2) \setminus B_R(x_2)$;
(b) $v(x_1, t_0) > 0$;
(c) $v(x,t) = 0$ for all $x \in \partial B_{R^*}(x_2)$ and for all $t \in [t_0 - \tau, t_0]$;
(d) $v(x,t) \le \eta$ for all $x \in \partial B_R(x_2)$ and all $t \in [t_0 - \tau, t_0]$.

The properties (a) and (c) guarantee $v(x,t) = 0 \le u(x,t)$ for all

$$(x,t) \in \left(\overline{B}_{R^*}(x_2) \setminus B_R(x_2) \times \{t_0 - \tau\}\right) \cup \left(\partial B_{R^*}(x_2) \times [t_0 - \tau, t_0]\right).$$

Furthermore our choice of η in Eq. (2.1) entails $u(x,t) \ge \eta$ for all $x \in \partial B_R(x_2)$ and for all $t \in [t_0 - \tau, t_0]$. Combining this inequality with (d), we obtain $0 \le v(x,t) \le u(x,t)$ for all $x \in \partial B_R(x_2)$ and for all $t \in [t_0 - \tau, t_0]$.

Then the weak comparison principle in the space–time domain

$$Q \stackrel{\text{def}}{=} \left(B_{R^*}(x_2) \setminus \overline{B}_R(x_2)\right) \times (t_0 - \tau, t_0]$$

guarantees that $0 \le v(x,t) \le u(x,t)$ holds for all $(x,t) \in Q$. The reader is referred to Alt and Luckhaus [9, Thm. 2.2, p. 325], Díaz [10, Thm. 3, p. 313], and Otto [11, Thm. on p. 25] for the appropriate version of the weak comparison principle. In particular, thanks to $|x_1 - x_2| < R^*$ in property (c), we obtain $(x_1,t_0) \in Q$ and $u(x_1,t_0) \ge v(x_1,t_0) > 0$ which is the desired contradiction with our choice of $x_1 \in Z$. We have proved that the set $Z(t_0)$ is open in Ω. Since $Z(t_0) \ni x_1$ is also relatively closed in Ω and Ω is connected, we conclude that $Z(t_0) = \Omega$ as claimed. This concludes the proof of the main part of the theorem.

To construct the desired subsolution v, in analogy with [4, pp. 599–600], proof of Theorem 4.1, we construct it in the form of a spherically symmetric wave

$$v(x,t) = z\left(R + \omega(t - t_0 + \tau) - |x - x_2|\right), \quad (x,t) \in \overline{B}_{R^*}(x_2) \times [t_0 - \tau, t_0], \tag{2.2}$$

with the velocity $\omega \in (0, +\infty)$ to be determined later and the function $z\colon \mathbb{R} \to \mathbb{R}_+$ satisfying the differential equation

$$\frac{\mathrm{d}z}{\mathrm{d}\xi} = f_\varepsilon(z(\xi)) \quad \text{for } \xi \in \mathbb{R};\ z(0) = 0. \tag{2.3}$$

The crucial point of our construction is a suitable choice of the nonnegative function $f_\varepsilon\colon \mathbb{R}_+ \to \mathbb{R}_+$ which depends on a small parameter $\varepsilon \in (0,1)$. We choose

$$f_\varepsilon(s) \stackrel{\text{def}}{=} \begin{cases} s\,|\log s|^{1+\varepsilon} & \text{if } s \in (0,+\infty)\,, \\ 0 & \text{if } s \in (-\infty,0]\,. \end{cases} \tag{2.4}$$

Clearly, f_ε is a continuous function on $\mathbb{R}$ with the derivative

$$f'_\varepsilon(s) = \begin{cases} (|\log s| - 1 - \varepsilon)\,|\log s|^{\varepsilon} & \text{if } 0 < s < +\infty, \\ 0 & \text{if } -\infty < s < 0. \end{cases}$$

Furthermore, given any fixed $\eta_0 \in (0,1)$, the integral

$$\zeta_0 \equiv \zeta_0(\varepsilon) \stackrel{\text{def}}{=} \int_0^{\eta_0} \frac{ds}{f_\varepsilon(s)}$$

converges and satisfies $\zeta_0(\varepsilon) \nearrow +\infty$ as $\varepsilon \searrow 0$. Finally, the classical method of separation of variables applied to the initial value problem (2.3) yields the following formula for the unknown function $z \equiv z_\varepsilon\colon \mathbb{R} \to \mathbb{R}_+$:

$$z(\zeta) = \begin{cases} \exp\left[-(\varepsilon\cdot\zeta)^{-1/\varepsilon}\right] & \text{if } \zeta \in (0,+\infty)\,, \\ 0 & \text{if } \zeta \in (-\infty,0]\,. \end{cases} \tag{2.5}$$

We remark that, for any fixed $\zeta \in (0,+\infty)$, we have $z_\varepsilon(\zeta) \searrow 0$ as $\varepsilon \searrow 0$.

In order to complete the proof, let us now verify that the function $v(x,t)$ defined in (2.2) has all properties (a), (b), (c), and (d) stated above.

Property (a): For every $x \in \overline{B}_{R^*}(x_2)$ with $|x - x_2| \geq R$ we have $v(x, t_0 - \tau) = z(R - |x - x_2|) = z(0) = 0$.

Property (b): The desired inequality

$$v(x_1, t_0) = z(R + \omega\tau - |x_1 - x_2|) > 0$$

will be satisfied whenever $R + \omega\tau - |x_1 - x_2| > 0$, i.e., if the number $\omega \in (0,+\infty)$ is chosen such that

$$\omega > \underline{\omega}(x_1,t_0) \stackrel{\text{def}}{=} \frac{|x_1 - x_2| - R}{\tau} > 0; \quad \text{we take } \omega \stackrel{\text{def}}{=} \frac{R^* - R}{\tau}.$$

Property (c): Similarly as above, $v(x,t) = 0$ for all $(x,t) \in \partial B_{R^*}(x_2) \times [t_0 - \tau, t_0]$ if and only if $R + \omega(t - t_0 + \tau) - R^* \leq 0$ for all $t \in [t_0 - \tau, t_0]$, which is satisfied by our choice of $\omega = (R^* - R)/\tau > \underline{\omega}(x_1,t_0)$ above.

Property (d): For every $x \in \partial B_R(x_2)$ and for all $t \in [t_0 - \tau, t_0]$, we have

$$\begin{aligned} v(x,t) &= z(R + \omega(t - t_0 + \tau) - |x - x_2|) = z(\omega(t - t_0 + \tau)) \\ &\leq z(\omega\tau) \leq z(R^*) \equiv z_\varepsilon(R^*) \leq \eta, \end{aligned}$$

provided $\varepsilon > 0$ is small enough, say, $0 < \varepsilon \leq \varepsilon_1$, by Eq. (2.5). This proves Property (d).

Let us denote the spherical shell $A \stackrel{\text{def}}{=} B_{R^*}(x_2) \setminus \overline{B}_R(x_2) \subset \Omega$; hence $Q = A \times (t_0 - \tau, t_0]$. It remains to verify the differential inequality

$$\frac{\partial}{\partial t} b\,(v(x,t)) - \Delta_p v(x,t) \leq 0 \quad \text{in } A \times (t_0 - \tau, t_0] \tag{2.6}$$

in the sense of distributions. Let us fix $t \in (t_0 - \tau, t_0]$. Take any nonnegative test function $\psi \in C^1_c(A)$, $\psi \geq 0$. Recalling our definition (2.2) of $v(x,t) = z(R + \omega(t - t_0 + \tau) - |x - x_2|)$, $v \geq 0$, and setting

$$A_+ \stackrel{\text{def}}{=} \{x \in A\colon v > 0\} = \left\{x \in \mathbb{R}^N\colon R < |x - x_2| < R + \omega(t - t_0 + \tau)\right\},$$

we calculate

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_A b(v)\,\psi\,\mathrm{d}x + \int_A |\nabla v|^{p-2}\,\nabla v\cdot\nabla\psi\mathrm{d}x\\
&= \omega\int_A b'(z)z'\psi\mathrm{d}x - \int_A (z')^{p-1}\,\frac{x-x_2}{|x-x_2|}\cdot\nabla\psi\mathrm{d}x\\
&= \omega\int_{A_+} b'(z)z'\psi\mathrm{d}x - \int_{A_+} (z')^{p-1}\,\frac{x-x_2}{|x-x_2|}\cdot\nabla\psi\mathrm{d}x\\
&= \omega\int_{A_+} b'(z)z'\psi\mathrm{d}x - \int_{\partial A_+}\left[(z')^{p-1}\,\frac{x-x_2}{|x-x_2|}\right]\cdot\nu(x-x_2)\psi\mathrm{d}\sigma(x) + \int_{A_+}\mathrm{div}\left[(z')^{p-1}\,\frac{x-x_2}{|x-x_2|}\right]\psi\mathrm{d}x\\
&= \omega\int_{A_+} b'(z)z'\psi\mathrm{d}x \qquad - \int_{|x-x_2|=R+\omega(t-t_0+\tau)}\left[(z')^{p-1}\,\frac{x-x_2}{|x-x_2|}\right]\cdot\frac{x-x_2}{|x-x_2|}\psi\mathrm{d}\sigma(x)\\
&\quad + \int_{|x-x_2|=R}\left[(z')^{p-1}\,\frac{x-x_2}{|x-x_2|}\right]\cdot\frac{x-x_2}{|x-x_2|}\psi\mathrm{d}\sigma(x)\\
&\quad + \int_{A_+}\left[-(p-1)(z')^{p-2}z''\frac{x-x_2}{|x-x_2|}\cdot\frac{x-x_2}{|x-x_2|} + (z')^{p-1}\mathrm{div}\frac{x-x_2}{|x-x_2|}\right]\psi\mathrm{d}x\\
&= \omega\int_{A_+} b'(z)z'\psi\mathrm{d}x - 0 + 0 + \int_{A_+}\left[-(p-1)(z')^{p-2}z'' + (z')^{p-1}\frac{N-1}{r}\right]\psi\mathrm{d}x,
\end{aligned}
$$

owing to $z'(R+\omega(t-t_0+\tau)-|x-x_2|) = z'(0) = 0$ if $|x-x_2| \geq R+\omega(t-t_0+\tau)$, and $\psi(x) = 0$ if $|x-x_2| = R$.

Furthermore, using the substitution $r = |x-x_2| \geq 0$ for $x \in \mathbb{R}^N$, we continue by calculating the following pointwise estimates for $R < r < R+\omega(t-t_0+\tau)$:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_A b(v)\psi\mathrm{d}x + \int_A |\nabla v|^{p-2}\,\nabla v\cdot\nabla\psi\mathrm{d}x\\
&= \omega\int_{A_+} b'(z)z'\psi\mathrm{d}x + \int_{A_+}\left[-(p-1)(z')^{p-2}z'' + (z')^{p-1}\frac{N-1}{r}\right]\psi\mathrm{d}x\\
&= \int_{A_+} z'\left\{\omega b'(z) - (p-1)(z')^{p-2}\left[f_\varepsilon'(z) - \frac{N-1}{(p-1)r}\right]\right\}\psi\mathrm{d}x,
\end{aligned}
\tag{2.7}
$$

where $z = z(R+\omega(t-t_0+\tau)-|x-x_2|) = z_\varepsilon(R+\omega(t-t_0+\tau)-r) > 0$ thanks to $R < r < R+\omega(t-t_0+\tau)$, and the expression in the curly bracket is nonpositive,

$$
\{\dots\} \stackrel{\mathrm{def}}{=} \omega b'(z) - (p-1)(z')^{p-2}\left[f_\varepsilon'(z) - \frac{N-1}{(p-1)r}\right] \leq 0,
\tag{2.8}
$$

by the following calculations: First, we choose $\eta_1 \in (0,\eta]$ such that

$$
\frac{s^{2-p}b'(s)}{|\log s|^{p-1}} \leq \frac{p-1}{2\omega} \quad \text{for all } s \in (0,\eta_1].
$$

Next, we choose $\eta_2 \in (0,\eta_1]$ such that $\eta_2 < 1$ and

$$
\frac{1}{4}\,|\log\eta_2| \geq \max\left\{1+\varepsilon_1, \frac{N-1}{(p-1)R}\right\} > 1.
\tag{2.9}
$$

This choice guarantees

$$
\begin{aligned}
f_\varepsilon'(s) &= (|\log s| - 1 - \varepsilon)\,|\log s|^\varepsilon \geq \left(|\log s| - \frac{1}{4}\,|\log\eta_2|\right)|\log s|^\varepsilon\\
&\geq \frac{3}{4}|\log s|^{1+\varepsilon} \quad \text{whenever } 0 < \varepsilon \leq \varepsilon_1 \text{ and } 0 < s \leq \eta_2.
\end{aligned}
$$

Taking also $R < r < R + \omega(t - t_0 + \tau)$, we arrive at

$$f'_\varepsilon(s) - \frac{N-1}{(p-1)r} \geq \frac{3}{4}\,|\log s|^{1+\varepsilon} - \frac{1}{4}\,|\log \eta_2| \geq \frac{1}{2}|\log s|^{1+\varepsilon}. \tag{2.10}$$

Subsequently, we choose $\varepsilon_2 \in (0, \varepsilon_1]$ such that

$$z_\varepsilon(R^*) \leq \eta_2 \ (\leq \eta_1 \leq \eta) \quad \text{whenever } 0 < \varepsilon \leq \varepsilon_2.$$

Substituting

$$s = z(R + \omega(t - t_0 + \tau) - |x - x_2|) = z_\varepsilon(R + \omega(t - t_0 + \tau) - r)$$

with $R < r < R + \omega(t - t_0 + \tau)$, we observe that $0 < s \leq z_\varepsilon(\omega\tau + R) = z_\varepsilon(R^*) \leq \eta_2$ whenever $0 < \varepsilon \leq \varepsilon_2$. Hence, we may combine in Eq. (2.10) with $z' = f_\varepsilon(z)$ from Eq. (2.3) and formula (2.4), to estimate

$$\begin{aligned}(p-1)(z')^{p-2}\left[f'_\varepsilon(z) - \frac{N-1}{(p-1)r}\right] &\geq \frac{p-1}{2}\,(f_\varepsilon(z))^{p-2}\,|\log z|^{1+\varepsilon} \\ &= \frac{p-1}{2} z^{p-2}|\log z|^{(p-1)(1+\varepsilon)} \geq \frac{p-1}{2} z^{p-2}|\log z|^{p-1},\end{aligned}$$

whence, by Eq. (2.8),

$$\{\dots\} \leq \frac{p-1}{2} z^{p-2}|\log z|^{p-1} - \frac{p-1}{2} z^{p-2} \cdot |\log z|^{p-1} = 0.$$

The last inequality shows that the right-hand side of Eq. (2.7) is nonpositive, as claimed, provided $0 < \varepsilon \leq \varepsilon_2$. Consequently, the function v defined in Eq. (2.2) satisfies the parabolic inequality (2.6) in $Q = A \times (t_0 - \tau, t_0]$.

In order to prove the last claim of the theorem, let $\xi \in \Omega$ be such that $u_0(\xi) > 0$. Thanks to the continuity of u, the number τ defined in Eq. (1.3) is positive. Applying the main claim of the theorem with an arbitrary fixed $t_0 \in (0, \tau)$, we conclude that $u(\xi, t_0) > 0$ implies $u(x, t_0) > 0$ for every $x \in \Omega$. Hence, we have proved that $u > 0$ holds throughout the entire cylinder $\Omega \times (0, \tau)$. □

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Appendix A4

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ORIGIN OF THE p-LAPLACIAN AND A. MISSBACH

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Abstract. We describe the historical process of derivation of the p-Laplace operator from a nonlinear Darcy law and the continuity equation. The story begins with nonlinear flows in channels and ditches. As the nonlinear Darcy law we use the power law discovered by Smreker and verified in experiments by Missbach for flows through porous media in one space dimension. These results were generalized by Christianovitch and Leibenson to porous media in higher space dimensions. We provide a brief description of Missbach's experiments.

1. Introduction

The authors of this article have often been confronted with the question on the origin of the p-Laplace operator. The main goal of the present work is to answer this question at satisfactory technical and historical levels. We do not attempt to provide or claim complete answers to many questions that arise in our investigation of the available resources. In particular, we leave the question of *competitiveness* of mathematical models with the p-Laplacian to alternative mathematical models still widely open in practical applications [51, 58].

2. The Filtration Problem and the Equation

An important task of hydrodynamics engineering throughout the 18^{th} century was to build reliable water supplies for fast growing urban centers. The need for water sparked a number new directions in theoretical research on hydrodynamics and hydrology. Numerous interesting mathematical problems in this area are derived and formulated in the monograph by Jacob Bear [3]. Among them we are interested in *filtration of fluids through porous media* and *unsaturated flow*; see [3, Sect. 5.2, 5.10, 5.11] and [3, Sect. 9.4], respectively. A mathematical model for such phenomena is presented in J. I. Díaz and F. de Thélin [14]. It is described by the following *nonlinear* initial-boundary value problem of parabolic type for the

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unknown function $u = u(x,t)$ of the space and time variables, x and t, respectively:

$$\begin{gathered} \frac{\partial}{\partial t}\, b(u) - \operatorname{div} \phi\big(\nabla u - K(b(u))\mathbf{e}\big) + g(x,u) = f(x,t) \quad \text{in } \Omega \times (0,\infty)\,, \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,\infty)\,, \\ b(u(x,0)) = b(u_0(x)) \quad \text{in } \Omega\,. \end{gathered} \tag{2.1}$$

Here, $\Omega \subset \mathbb{R}^N$ is a bounded open subset of the N-dimensional Euclidean space $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, $b : \mathbb{R} \to \mathbb{R}$, $K : \mathbb{R} \to \mathbb{R}$, and $g(x, \cdot\,) : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying some additional hypotheses ([14, Sect. 1]), such as b being monotone increasing with $b(0) = 0$, $\mathbf{e}$ denotes a given unit vector in $\mathbb{R}^N$, and for some $1 < p < \infty$,

$$\phi(\boldsymbol{\zeta}) = |\boldsymbol{\zeta}|^{p-2}\boldsymbol{\zeta} \quad \text{for every } \boldsymbol{\zeta} \in \mathbb{R}^N\,. \tag{2.2}$$

As usual, $t \in \mathbb{R}_+ := [0,\infty)$. Finally, $f : \Omega \times (0,\infty) \to \mathbb{R}$ is (typically) a Lebesgue-integrable function standing for sources (if $f(x,t) > 0$) and sinks (if $f(x,t) < 0$), whereas $u_0 : \Omega \to \mathbb{R}$ stands for the prescribed initial data, usually assumed to be Lebesgue-measurable and (essentially) bounded.

For filtration of fluids through porous media in laminar regime one begins with

$$\text{the continuity equation} \qquad \frac{\partial\theta}{\partial t} + \operatorname{div}\mathbf{v} = 0 \tag{2.3}$$

$$\text{and the Darcy law} \qquad \mathbf{v} = -K(\theta)\,\nabla\Phi(\theta)\,, \tag{2.4}$$

where $\theta = \theta(x,t)$ is the volumetric moisture content, $K = K(\theta)$ is the hydraulic conductivity, and the potential Φ is given by $\Phi(\theta) = \psi(\theta) + z$ with $\psi(\theta)$ being the hydrostatic potential and z the gravitational potential. For instance, if $N = 3$ then we fix the unit vector $\mathbf{e} = (0,0,-1) \in \mathbb{R}^N$ in the direction opposite (but parallel) to the gravitational force, perpendicular to the horizontal plane $(x_1,x_2,0)$, so that the gravitational potential $z = z(x) = gx_3 + \text{const}$ at the point $x = (x_1,x_2,x_3) \in \mathbb{R}^3$ yields the gravitational force

$$\mathbf{G} = \mathbf{G}(x) = -\nabla z = \big(0, 0, \ -\tfrac{\partial z}{\partial x_3}\big) = (0,0,-g) = -g\mathbf{e} \in \mathbb{R}^3\,.$$

To simplify our notation, we normalize the gravitational constant to one, $g = 1$; hence, $\mathbf{G} = -\nabla z = -\mathbf{e} \in \mathbb{R}^3$. Thus, we obtain

$$\nabla\Phi(\theta) = \psi'(\theta)\,\nabla\theta - \mathbf{e}$$

which, after being inserted into Darcy's law (2.4), yields

$$\mathbf{v} = -K(\theta)\,\psi'(\theta)\,\nabla\theta + K(\theta)\mathbf{e} = -\nabla\varphi(\theta) + K(\theta)\mathbf{e} \in \mathbb{R}^3 \tag{2.5}$$

where

$$\varphi(\theta) := \int_0^\theta K(\vartheta)\,\psi'(\vartheta)\,\mathrm{d}\vartheta \quad \text{for } \theta \in \mathbb{R}\,.$$

In general, the vector field $\mathbf{v}$ stands for the seepage flow which, in our applications, will be proportional to the fluid velocity, thus denoted by $\mathbf{v}$. It is reasonable to assume $K(\vartheta) > 0$ and $\psi'(\vartheta) > 0$ (see Bear [3]), so that also $\varphi'(\vartheta) > 0$ holds. As a consequence, $\varphi : \mathbb{R} \to \mathbb{R}$ is a strictly monotone increasing, continuously differentiable function.

Beginning in the 1870s, many engineers concerned with fluid dynamics (including the works in [20, 21, 22, 23, 24, 32, 36, 37, 38, 43, 44, 45, 46, 52, 54, 55, 56, 57, 62, 64, 66, 67]) have discovered that, if the fluid flow is in *turbulent regime*, the linear

Darcy law (2.4) does not provide the correct relationship between the *pressure slope* (force),

$$\mathbf{F} = -\nabla\varphi(\theta) + K(\theta)\mathbf{e}\,,$$

on the right-hand side and the *velocity*, $\mathbf{v}$, on the left-hand side of Darcy's law (2.4). Oscar Smreker [55, Eqs. (5)–(7), pp. 361–362] shows by rigorous calculations how linear Darcy's law leads to a contradiction in a practical problem (dug well, "Schachtbrunnen" in German). Among several "correction" alternatives to Darcy's law, O. Smreker [54, 55, 56] suggested the following *power law*:

$$\mathbf{F} = -K(\theta)\,\nabla\Phi(\theta) = -\nabla\varphi(\theta) + K(\theta)\mathbf{e} \tag{2.6}$$

is given by

$$\mathbf{F} = \text{const}\cdot|\mathbf{v}|^{p'-2}\mathbf{v} \quad \text{with some } p' > 2\,, \tag{2.7}$$

with the power $s = (p'-2)+1 = p'-1$, where the multiplicative constant is set to one, const $= 1$, for simplicity. Smreker's work [54] suggests $p'-1 = 3/2$, i.e., $p' = 2.5$, whereas Reynolds's measurements [52] show $p'-1 = 1.723$. A. M. White [62] proposed an analogous relation with $p'-1 = 1.8$. All such corrections to Darcy's law allow only the power range $1 \le p'-1 \le 2$. Denoting by $p = p'/(p'-1)$ the conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$, we thus have to deal with the range $3/2 \le p \le 2$ and the velocity

$$\mathbf{v} = |\mathbf{F}|^{p-2}\mathbf{F}\,. \tag{2.8}$$

Inserting (2.6) and (2.8) into the continuity equation (2.3) we finally arrive at problem (2.1), where $b = \varphi_{-1}$ denotes the inverse function to φ and $f \equiv g \equiv 0$.

We refer an interested reader to J. I. Díaz and F. de Thélin [14] for how to obtain problem (2.1) in a model dealing with *unsaturated flow* (gas flow, typically). There, $p = 3/2$.

It is now evident, that the *p-Laplace operator* Δ_p,

$$\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\,, \quad \text{for } 1 < p < \infty\,, \tag{2.9}$$

is created by the nonlinear power law (2.7) or, equivalently, by (2.8). The continuity equation (2.3) is standard for both, linear and nonlinear Darcy's laws. This means that the origin of the p-Laplacian Δ_p is closely tied to *who was the first* to plug the power law (2.8) into the continuity equation (2.3) or at least into its stationary special case $\operatorname{div}\mathbf{v} = 0$. There seems to be a wide-spread agreement in the literature that the power law (2.8) with $p = 5/3$ was suggested first by Oscar Smreker [54] [1] in 1878 in the equivalent form (2.7) with $p' = 2.5$. A number of "power laws" (with a more general exponent $s = p'-1$) by various authors followed afterwards. We will discuss the most important ones in the following two sections.

In this context ("*Who was the first?*"), we should mention the articles by Smreker [56] from 1881 (used also in his doctoral dissertation [57] in 1914) and by N. E. Zhukovskii [64] from 1889 (reprinted in his collected works [65] in 1937), in which they give the explicit formula for the radially symmetric solution, $u(x) \equiv u(|x|)$, of the so-called *p-harmonic equation*, $\Delta_p u = 0$, for any $1 < p < \infty$, $p \ne N$,

$$u(r) = C_0 + \text{const}\cdot r^{1-\mu} \quad \text{for every } r = |x| > 0\,, \tag{2.10}$$

$$\mu = \frac{N-1}{p-1} \ge 0, \quad \mu \ne 1,$$

[1] Ref. [6] by M. Brenčič provides a "Short description of life and work of Oskar Smreker".

see [57, Eq. (I), p. 36] and [65, Eq. (13), p. 19], respectively. Here, $C_0 \in \mathbb{R}$ is a constant; $C_0 = u(0)$ if $\mu < 1$ and $C_0 = u(+\infty) := \lim_{r\to+\infty} u(r)$ if $\mu > 1$. Although this formula is valid in any dimension $N \geq 1$, both engineers, in [56, 64], treat only the planar case ($N = 2$) given by the hydroengineering model. They had *never* written down the p-harmonic equation ($\Delta_p u = 0$) explicitly throughout their entire articles [56, 64]; rather, they preferred to refer to Smreker's work in [54] for the power law. In fact, since every radially symmetric solution $u(x) \equiv u(|x|)$ to the p-harmonic equation $\Delta_p u = 0$ in the plane ($N = 2$) satisfies the stationary case of the continuity equation (2.3), $\operatorname{div} \mathbf{v} = 0$, which is equivalent to

$$\Delta_p u(x) = r^{1-N} \cdot \frac{\mathrm{d}}{\mathrm{d}r} \left(r^{N-1} \, |u'(r)|^{p-2} u'(r) \right) = 0 \quad \text{for every } r = |x| > 0 \,,$$

both, Smreker [56] and Zhukovskii [64], may have very easily used an alternative way (e.g., the surface integral over a sphere) to obtain in the plane ($N = 2$),

$$r \, |u'(r)|^{p-2} u'(r) = \text{const} \qquad \text{for all } \ r = |x| > 0 \,, \ x \in \mathbb{R}^2 \,,$$

whence (2.10) follows with $N = 2$ and $\mu = 1/(p-1) > 0$ (recall that $p \neq N = 2$).

3. Flow in a Channel or Porous Media

The rapid development of hydrology in the late 18^{th} and early 19^{th} centuries required new theoretical background and related new measurement techniques. Much of this research, particularly by French engineers closely connected with the famous Parisian engineering school *École des ponts et chaussées*, was published in 1804 in the monograph by one of its former directors, baron Gaspard Riche de Prony [50]. This book is a very comprehensive description of French research on water flow through channels and large pipes. Some studies treat also smaller (thinner) pipes and hoses which, towards the end of the 19^{th} century, developed into research on filtration through soil, sand, and other similar porous materials. Mathematically, all models in this research are set in space dimension one. The spectrum of specialists involved in the 18^{th} century research begins with *civil engineers* (count Pierre Louis George du Buat [8] and Pierre-Simon Girard [26]), continues with *theoretical engineers* and *applied mathematicians* like de Prony himself and Antoine de Chézy [9], and ends up with *mathematicians* (marquess Pierre-Simon de Laplace [33]). The author, de Prony [50], describes and further develops the research findings of his former teacher, Antoine de Chézy [9], published in 1775 which contains also his famous mathematical formula on the average flow velocity. De Prony's book [50] was further influenced by the work of P. L. G. du Buat [8] and P.-S. Girard [26]. One of their most important discoveries was the formula for the resistance force due to adhesion of the fluid to the contact surface, cf. G. R. de Prony [50, pp. 44, 58]:

If u stands for the average flow velocity, then this resistance force, $\chi\delta s\, \phi(u)$, is proportional to a polynomial function $\phi = \phi(u)$ of degree one to three, where χ, δ, and s are some positive constants that describe the adhesion to the contact surface, and

$$\phi(u) = c + \alpha u + \beta u^2 + \gamma u^3 \tag{3.1}$$

with some nonnegative constants c, α, β, and γ. We refer to pages 44 and 58 of de Prony's book [50]. Calculation of these constants from available measurements was a subject of strong theoretical and practical interest to civil engineers working on the constructions of channels and water pipelines throughout entire France ([50, pp. 65–90]).

The transformation of the research interests in water flow through channels and large pipes into research on filtration through porous materials began in mid-19$^{\text{th}}$ century in the work by Henry Darcy [11] in 1856, a French hydroengineer working in Dijon, with his famous (linear) *Darcy law*, and by Jules Dupuit [18], another French engineer and economist, published in 1863, who, in contrast, works with de Prony's quadratic law (3.1) (where $\gamma = 0$) for the dependence of the resistance force or pressure loss (difference) on the average flow velocity, u. While *Darcy's law* became quickly a very popular, simple tool for calculating the dependence of force or pressure on the velocity u for small absolute values of u, de Prony's quadratic law has turned out to fit the filtration problems much more accurately also with higher velocities required to filter a sufficient amount of liquid (water) needed by a large urban community. Towards the end of the 19$^{\text{th}}$ century, several civil engineers throughout Western Europe have adopted de Prony's polynomial formula (3.1) (typically quadratic or cubic) in their investigation of fluid filtration phenomena; Oscar Smreker [54, 55][2] (an Austrian-born engineer based in the city of Mannheim, Germany, and active in several neighboring countries) seems to be the first of them in 1878–1879 (with another work [56] in 1881), followed by C. Kröber [32] in 1884 and Philipp Forchheimer [20] in 1886 and [21] in 1901 (another Austrian engineer active also in Germany). Especially Forchheimer's latter article, [21], became a landmark in nonlinear fluid dynamics. Owing to Forchheimer's tremendous theoretical and practical activity in filtration problems, which includes several lecture notes and comprehensive textbooks [22, 23, 24], de Prony's and Smreker's quadratic law (3.1), $\gamma = 0$, in filtration theory is called *Forchheimer's equation*. We will stick to this terminology in the rest of this article while keeping in mind earlier contributions by de Prony and Smreker. Smreker's main merit is an early application of Forchheimer's quadratic formula (3.1) in civil engineering, particularly in the construction of a water supply system to the Alsatian city of Strasbourg (France) ([54], see the sketches following p. 128). This engineering project plays the key role in Smreker's works [54, 55, 56] mentioned above (in 1878–1881). This work (and from his other articles to follow it) is collected in his doctoral dissertation [57] (Dr.–Ing.) from 1914 at the age of sixty. By then he had designed and/or built numerous water supply systems in various European cities: Belgrade, Ljubljana, Lvow (Lemberg), Mannheim, Prague, Trieste, Vilnius, etc. Greater details on his achievements can be found in M. Brenčič's survey [6].

Nevertheless, it was Oscar Smreker [54] again who has discovered that, at "low" velocity levels v, neither the linear *Darcy law* nor the quadratic (or cubic) *de Prony--Forchheimer* law (3.1) describes the relation between the pressure loss and the velocity v accurately. He suggested the following correction for the *(pressure) slope* h/ℓ,

$$\frac{h}{\ell} = \frac{v^2}{2g} \cdot \xi \quad \text{where} \quad \xi = f(v) \quad \text{for} \;\; v > 0\,, \tag{3.2}$$

[2] In fact, the latter article, [55], was intended to be an introduction to the former one, [54]. The temporal order of publication is publisher's mistake; see publisher's remark at the end of the latter [55].

with the gravitational constant (acceleration) g given by $g = 9.81$ (m/s^2) and the function $\xi = f(v)$ taking the "hyperbolic" form

$$f(v) = \alpha + \frac{\beta}{\sqrt{v}} \quad \text{for } v > 0 \tag{3.3}$$

with some positive constants α and β. The (positive) quantities h and ℓ, respectively, stand for the difference h of water levels before and after the (horizontal) filter of length ℓ; cf. Forchheimer [21, Fig. 1, p. 1736] and Smreker [55, pp. 358–360]. Formulas (3.2) and (3.3) yield a very special, but important case of the famous *power law*,

$$\frac{h}{\ell} = \frac{v^{3/2}}{2g} \cdot \big(\alpha\sqrt{v} + \beta\big) \approx \frac{\beta}{2g} \cdot v^{3/2} \quad \text{for } v > 0\,, \tag{3.4}$$

with the approximation by the power $(\beta/2g) \cdot v^{3/2}$ being valid for small velocities $v > 0$. In his work [54, p. 127], Smreker suggests also a much more general relation, namely,

$$\xi = f(v) = \alpha + \sum_{n=1}^{\infty} \beta_n\, v^{-1/n} \quad \text{for } v > 0 \tag{3.5}$$

with some nonnegative constants α and β_n. This is how the *power law*

$$\frac{h}{\ell} = \text{const} \cdot v^s \quad (1 < s < 2) \quad \text{for } v > 0 \tag{3.6}$$

was discovered for the (pressure) slope h/ℓ. Starting with the articles [21, 54], the precise value of the constant $s \in (1, 2)$ was the subject of numerous measurements and theoretical investigations; $s > 1$ shows the tendency to approach one ($s \searrow 1$). Of course, the case $s = 1$ renders (linear) *Darcy's law*. The power law (3.6) for soil permeability and *high* water velocity v was confirmed in the experiments performed by F. Zunker [66] in 1920 with $s = 3/2$; see also Zunker's survey article [67]. He claims that Darcy's law is applicable to *medium* water velocities v. In Great Britain, the two nonlinear Darcy laws, the quadratic law (3.1) (where $\gamma = 0$) and the power law (3.6) (where $s = 1.723$), appear for the first time in 1883 in the work by Osborne Reynolds [52, Sect. III, §37, pp. 973–976]. He considers very briefly also Smreker's general problem (3.2) (cf. [54, p. 119]). However, the relation of his research findings to those of O. Smreker [54] is unclear[3].

It was not until mid-1930s when Smreker's power law (3.6) was verified by Alois Anton Missbach [43] – [46] in many laboratory experiments with sugar juice and water penetrating a medium consisting of tiny glass balls of constant diameter. The final comparison of Smreker's power law with A. Missbach's laboratory experiments were published in the (now) famous article [46]. His experiments are so well-documented in the series of articles [43] – [46][4] that many researchers in nonlinear fluid dynamics, especially in the "West" (Americas, Australia, Europe, and New Zeeland), consider A. Missbach's article [46] as the *verification* of Smreker's power law (3.6). For this reason, this power law is often called *Missbach's equation* in Western literature (or the *Darcy-Missbach equation* in [51]). We will use this terminology in the rest of this article, although many authors from Russia, the mainland China, and Taiwan prefer to attribute the power law to, e.g., the prolific Russian engineer S. V. Izbash [29, 30]; see also S. V. Izbash and Kh. Yu. Khaldre

[3] Reynolds [52] seems to be unaware of Smreker's results in [54] published five years earlier.

[4] Part VI (Ref. [45]) of Missbach's work appeared before Part V (Ref. [44]).

[31] and H. Watanabe [61]. A. Missbach's work [46] summarizes the results of a large research program sponsored by several sugar refineries in Czechoslovakia in the early 1930s on efficient sugar juice filtration. It is the final part (Part VII) of the series of seven articles on *Filtration ability of separated and saturated juices* inspired by the scientific and industrial activities of Missbach's doctoral adviser, Jaroslav Dědek, who himself also contributed to this article series (Part III), cf. J. Dědek and D. Ivančenko [12]. The findings of the research reported in Missbach's article [46], albeit obtained with penetrating water rather than sugar juice, were immediately incorporated into industrial sugar production. This article is written in two parallel originals, Czech and German. Further details on his professional involvement with the Czechoslovakian sugar producing industry will be provided in Section 9. A very practical application of *Missbach's equation* to nonlinear Darcy flow (also called *non-Darcian flow*) is provided in P. M. Quinn, J. A. Cherry, and B. L. Parker [51]. This flow occurs in high-precision straddle packer tests conducted in boreholes in a fractured dolostone aquifer using constant rate injection step tests to identify the conditions of change from Darcian to non-Darcian flow. An interesting comparison of *Forchheimer's* and *Missbach's equations*, (3.1) and (3.6), respectively, is available in the survey article by K. P. Stark and R. E. Volker [58] who, unfortunately, seem to be unaware of O. Smreker's pioneering work [54, 55, 56, 57].

4. The Russian School

Significant contributions to the filtration problem in porous materials by Russian (or Soviet) engineers and scientists began in early 1920s by N. N. Pavlovskii [48] in a hand-written monograph of 753 pages. It provides a very well-written, up--to-date introduction to hydraulics from a (mostly) theoretical point of view, with plenty of valuable references to the literature. In Russia, this time is characterized by massive industrialization (1920s and 1930s). In the first chapter, Pavlovskii surveys constitutive laws (Darcy's law, Forchheimer's quadratic and cubic laws, and the power law). In the second chapter, he suggests a criterion based on the Reynolds number to establish the validity range of the linear Darcy law and the range where a nonlinear law must be used instead. According to V. I. Aravin and S. N. Numerov [2], p. 4 and also p. 33 with a detailed explanation, Pavlovskii's work [48] is the first one to use Reynolds number for this purpose. Despite of the fact that the monograph [48] thoroughly discusses various constitutive laws in its first two chapters, the partial differential equations used throughout the book to study the seepage are only linear.

Serious interests in nonlinear (and non-Newtonian) fluid dynamics in the former Soviet Union began in early 1930s with the works by S. V. Izbash [29, 30], who has published the power law (3.6) already in 1931 in a monograph available only in Russian. Decisive contributions to fluid dynamics were made by N. E. Zhukovskii (see his collected works [65] from 1937), the most relevant for us being [64] from 1889. As we have already mentioned in Section 2, he gives the explicit formula for the radially symmetric solution, $u(x) \equiv u(|x|)$, of the p-harmonic equation, $\Delta_p u = 0$, see [64, Eq. (13), p. 19]. In the same article, [64], Zhukovskii discusses applicability of various constitutive laws to filtration of water through sandy soil known to that date, i.e., Darcy's, Kröber's, and Smreker's power-type laws [11, 32, 54], and compares them to scores of available experimental results. For instance, he derives Laplace's equation by inserting the (linear) Darcy law into the differential equation of continuity. Using the Laplace equation he studies several configurations of water wells scattered in the field (standalone well, wells in a row, and wells on a circle). For the standalone case, he finds out that the discrepancy between theoretical predictions from the formula based on the solution of Laplace's equation and the reality (measured data) is too large. To fix this problem, he suggests to use the velocity v given by Kröber's and Smreker's power law [32, 54] $-\frac{\mathrm{d}u}{\mathrm{d}x} = i = \mathrm{const} \cdot v^{p'-1}$, $2 < p' < \infty$, cf. eq. (2.8), to be plugged into the stationary case of the continuity equation (2.3) as described above. In particular, eq. (2.7) plays the role of the constitutive law.

To the best of our knowledge, all work on nonlinear (and non-Newtonian) fluid dynamics until 1940, throughout the entire world, treated only spatially one-dimensional problems. (Smreker's and Zhukovskii's radially symmetric planar solution in [56, 64] mentioned above is essentially one-dimensional.) It was the Russian scientist S. A. Christianovitch [10] who employed nonlinear constitutive laws (Forchheimer's quadratic and cubic laws and Missbach's power law) to derive nonlinear partial differential equations for the seepage movement of underground water. He restricts himself to the spatially two-dimensional case. In the case of the power law, he obtains the following equation (re-written in contemporary notation):

$$\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0\,,$$

for the unknown function $u = u(x, y)$. Since he works in two space dimensions, he can use methods of complex analysis and suggest analytical techniques to obtain approximations of the solution to this equation with the so-called *p-Laplace operator* Δ_p, $1 < p < \infty$. The common (linear) Laplace operator Δ is obtained for the (linear) Darcy law ($p = 2$).

Another notable person in the Russian hydraulic engineering school was L. S. Leibenson who investigated seepage of oil and gas in the oil and gas fields near the city of Baku (now Azerbaijan, formerly Soviet Union). Much of his research from the 1920s and early 1930s was published not only in brief article form, but also as a survey monograph [35]. His most important findings concern turbulent filtration of gas in porous medium [36, 37] (see also [40]). It was his article [36] where the *doubly nonlinear parabolic equation*,

$$\frac{\partial u^m}{\partial t} = c\,\Delta_p u \quad \text{for } (x, y, z, t) \in \mathbb{R}^3 \times (0, T)\,, \tag{4.1}$$

with $m + 1 = p = 3/2$, appeared for the first time. Here, $u = u(x, y, z, t)$ is the unknown function of space and time, and $c > 0$ is some constant. Thanks to $m = p - 1$, eq. (4.1) is called *$(p-1)$-homogeneous*. He used the separation of space and time variables,

$$u(x, y, z, t) = v(t)\, w(x, y, z)\,,$$

in order to obtain the following equation with the so-called *1-Laplacian*,

$$\operatorname{div}\left(\frac{\nabla w}{|\nabla w|}\right) + A\sqrt{w} = 0\,, \tag{4.2}$$

where $w = w(x, y, z)$ is the unknown function of space and $A > 0$ is a constant. This article, [36], published in 1945 seems to be the first one to derive and consider a quasilinear *parabolic* (time-dependent) problem, eq. (4.1), with the *p-Laplace operator* Δ_p in space dimension three (defined in eq. (2.9)), albeit for $p = 3/2$

only. For the p-harmonic equation, $\Delta_p u = 0$ with $p = 3/2$, Leibenson [36] finds solutions in the spatially one-dimensional and radially symmetric cases. In contrast, S. A. Christianovitch [10] (in 1940) treated only a quasilinear *elliptic* (stationary) problem, $\Delta_p u = 0$, in two space dimensions, but for any $1 < p < \infty$.

In his next work [37], immediately following [36], L. S. Leibenson allows for a wider range of values of p, $3/2 \le p \le 2$. Also his doubly nonlinear parabolic equation (4.1) becomes more general,

$$\frac{\partial}{\partial t}\Big(u^{\frac{1}{m+1}}\Big) = c\,\Delta_p u \quad \text{for} \ \ (x,y,z,t) \in \mathbb{R}^3 \times (0,T)\,, \tag{4.3}$$

with $m > 0$, which is no longer $(p-1)$-homogeneous. This equation results from Leibenson's studies [37] of filtration of turbulent polytropic gas flow through porous medium; $m > 0$ is called the *polytropic index* of the gas. It is a direct generalization of an earlier work by L. S. Leibenson [34] which still uses the linear Darcy law, whereas [37] uses Smreker's power law[5] (3.6). Practically all Leibenson's results we have mentioned above are very carefully collected and explained in his monograph [39] published in 1947; his scientific articles [34, 36, 37, 38] are reprinted in [40].

An important member of the Russian school was also P. Ya. Polubarinova-Kochina. Her Russian monograph [49] from 1952 (translated into English in 1962) became quickly a widely used textbook by hydrogeologists all over the world.

5. From Darcy's law to Forchheimer's equation (from linear to nonlinear diffusion)

Although fluid flow through channels, large pipes, and hoses had occupied theoretical hydrologists since the 18^{th} century (see de Prony's equation (3.1)), fluid flow through porous media attracted major attention much later, in mid-19^{th} century. We recall from Section 3 the research on filtration through porous materials by Henry Darcy [11] in 1856 (the linear *Darcy law*) and by Jules Dupuit [18] in 1863 (working with de Prony's quadratic law). The idea of the quadratic law (3.1) was picked up by Ph. Forchheimer who, in his groundbreaking work [21], developed applications of de Prony's quadratic law to filtration through porous materials (soil, in particular),

$$i = av + bv^2\,. \tag{5.1}$$

Here, the quantity i is the (negative) total piezometric head gradient, $i = -\frac{\mathrm{d}u}{\mathrm{d}x}$, v stands for the average seepage velocity, and a and b are nonnegative constants determined by the properties of the fluid and medium; typically, $a > 0$ and $b > 0$. His article [21], published in 1901, meant also the introduction of nonlinear diffusion after several decades of intensive studies of linear diffusion prompted by Darcy's law. A number of workers have inferred that Forchheimer's equation has sound physical backing apart from its attraction as a relatively simple nonlinear expression. We refer the reader to J. Bear, D. Zaslavsky, and S. Irmay [4], for example, who have derived the Forchheimer relation by inferred arguments from the fundamental Navier-Stokes equations for the general case when inertia terms are considered; see also Irmay [28]. A few decades later, in 1930, Ph. Forchheimer [24] extended his nonlinear Darcy law to

$$i = av + bv^m\,, \tag{5.2}$$

[5] Leibenson [37] was apparently not aware of Missbach's work [46].

where m is a constant typically taking values in the interval $(1,2]$, i.e., $1 < m \le 2$.

Remark 5.1. From the point of view of Mathematical Physics, relation (**??**) means that if $a > 0$, then the head gradient i has nearly linear, nontrivial growth

$$i(v) - i(0) = i = av\Big(1 + \frac{b}{a}v\Big) \approx av \tag{5.3}$$

for low velocity v. On one hand, this phenomenon was confirmed for certain types of fluids and media from both theoretical and experimental viewpoints, e.g., in the work of V. I. Aravin and S. N. Numerov [2], E. Lindquist [41], and J. C. Ward [60]. On the other hand, the nontrivial growth (5.1) ($a > 0$), which yields

$$v = v(i) = -\frac{a}{2b} + \sqrt{\big(\frac{a}{2b}\big)^2 + \frac{i}{b}} = \frac{a}{2b}\Big(-1 + \sqrt{1 + \big(\frac{2b}{a}\big)^2 \frac{i}{b}}\Big) > 0 \quad \text{if also } b > 0\,,$$

whence $v \approx i/a$ for $i \ge 0$ small, does not occur for other types of fluids and media studied in M. Anandakrishnan and G. H. Varadarajulu [1], C. R. Dudgeon [16], C. R. Dudgeon and C. N. Yuen [17], L. Escande [19], A. Missbach [43, 44, 45, 46], A. K. Parkin [47], A. M. White [62], and J. K. Wilkins [63].

6. Missbach's power law (nonlinear, power-type diffusion)

In contrast with Forchheimer's approach to generalizing Darcy's law, Alois Missbach [46] based his approach to the porous medium problem on numerous *experimental results* that became available in the 1930s in various rapidly developing industries, such as sugar and petroleum (oil) production, where certain types of fluids are filtered through special porous media. Missbach's experiments were prompted by theoretical and experimental results obtained much earlier by C. Kröber [32], O. Reynolds [52], O. Smreker [54], and F. Zunker [66]. The experimental results obtained during the sugar beet campaign of 1935 in Czechoslovakia led A. Missbach [46] to verifying the *power law relation*

$$i = c\,v^m \tag{6.1}$$

between the head gradient and the velocity, i and v, respectively, published in 1937. The power m typically takes values in the interval $(1,2)$. A couple of years before Missbach's article appeared, in 1935, A. M. White [62] proposed an analogous relation with $m = 1.8$. As a porous medium, Missbach used gravels, sands, and packings of uniform spheres (e.g., tiny glass balls), while in his starting experiments [43] – [45] the fluid was represented by sugar juice of various sugar contents. However, in his most important work for us, [46], he used water as the penetrating fluid (Figure 1 below). He found out that the power m stays in $(1,2)$ and tends to 1 with the decreasing diameter of the spheres. C. R. Dudgeon [16] carried out tests on coarse materials serving as porous medium (gravels, sands, and packings of uniform spheres) and confirmed that while the results followed closely an expression of Missbach's form (6.1) the values of c and m were not constant for the particular material for all fluid flow conditions. These and other experimental results have confirmed Missbach's equation (6.1). A theoretical derivation of the special case of Missbach's equation (6.1) for $m = 3/2$ has been given in E. Skjetne and J.-L. Auriault [53]. The authors of the present article have not been able to find any reference concerned with a theoretical derivation of Missbach's equation (6.1) for an arbitrary power $m \in (1,2)$. The article by A. Brieghel-Müller [7] thoroughly surveys almost all results concerning constitutive laws for filtration known up to 1940 and discusses their applicability to filtration processes in sugar production.

Since experiments and measurements play a decisive role in A. Missbach's work [43] – [46], we provide a brief description of his apparatus. A. Missbach [46] calls his experimental laboratory equipment "Apparatus for testing the hydraulic conductivity (permeability, porosity) through a layer of glass balls".

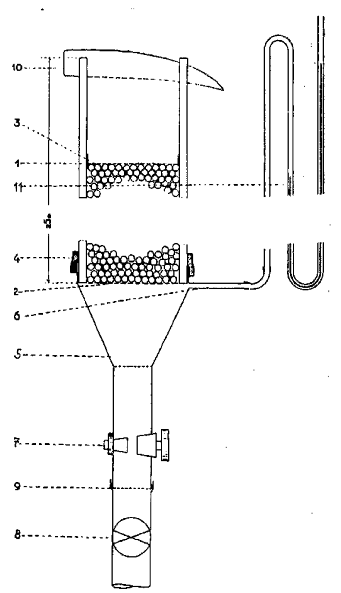

FIGURE 1. Apparatus for testing the hydraulic conductivity through a layer of glass balls.

Figure 1 is a scanned copy of the original figure from Missbach's work [46], p. 294, Obr. 1 (in the Czech edition) and p. 424, Abb. 1 (in the German edition). Missbach [46] credits the use of tiny glass balls to Zunker [66].

Figure 1 **description**:

(1) Glass tube with strong walls of internal diameter 45 mm, slightly longer than 200 mm.
(2) Lower sieve.
(3) Upper sieve with a steel spring.
(4) Connecting rubber hose with strong walls.
(5) Tin funnel with a sieve insole.
(6) Thin connection pipe for the differential water manometer.
(7) Faucet for flow regulation.
(8) Outlet for flow regulation.
(9) Screw thread with an inserted filter cloth.
(10) Trench for draining overflowing liquid.
(11) Manometer.

In contrast with earlier filtration experiments (e.g., F. Zunker [66, 67]) which used a system of parallel capillary tubes having undesirable side effects, A. Missbach [46] decided to construct an apparatus of a relatively large diameter (45 mm) whose walls do not influence (obstruct, slow down) the fluid flow through the layer of tiny glass balls. He used glass balls of four (4) different sizes (A, B, C, D; specified in [46, Table I]) and varied both, the thickness (height) of the layer of glass balls and the pressure of the fluid penetrating through the layer. The fluid used in this experiment was tap water, carefully filtered, with no air bubbles and other "pollutants". The filtered water was pumped through the outlet for flow regulation (8) from the bottom, under the atmospheric pressure of up to 0.5 atm, then led to penetrate through the layer of glass balls upwards. In order to guarantee a constant fluid flow velocity, v, throughout the horizontal cross section of the glass tube, a sieve insole (2) is inserted into the glass tube. The upper sieve with a steel spring (3) prevents the glass balls from being moved upwards by the penetrating fluid. Finally, the overflowing liquid is drained into the trench (10) and its volume is measured in a cylindrical vessel.

The thickness of the layer of glass balls, the size of the balls (A, B, C, D), the vertical pressure difference in the layer, the flow velocity, and many other important measurements are carefully recorded in [46, Tables II through V]. These experiments provide evidence for Missbach's power law relation (6.1).

7. Comparison of the Forchheimer and Missbach equations (two different types of nonlinear diffusion)

Both, Forchheimer's and Missbach's models have been very useful in a number of various situations. Which of the two nonlinear models is better (i.e., more accurate) depends strongly on the fluid properties and the velocity v. A brief comparison of the two models has been carried out e.g. in P. M. Quinn, J. A. Cherry, and B. L. Parker [51], K. P. Stark and R. E. Volker [58], and numerically in R. E. Volker [59]. The experimental conditions in [51] seem to be slightly more favorable for Missbach's model. We refer to Figure 5 in [51, Chapt. 9, pp. 9–12] for a detailed comparison of the two models. It is interesting to observe that the authors in [58, Chapt. 5, pp. 131–196] slightly favor Forchheimer's model for water penetrating a porous medium *between two horizontal plates* (see [58, pp. 144, 185–186, and 196]), whereas A. Missbach [46] obtains highly favorable results for filtration of water through a porous medium *in a vertical cylinder* described in the previous section (with applications to filtration of saturated sugar juice). Although the laminar flow regime often obeys the linear Darcy law, it is always nonlinear in character. Thus, Missbach's equation applies also to the laminar flow regime and in the transition to a turbulent regime.

8. Some basic analytic and numerical results for the p-Laplacian

A comprehensive survey on only basic analytic and numerical results for the p-Laplacian would have to contain literally hundreds of references. As this is not the purpose of our present article, we have decided to mention only a few ones. Perhaps the very basic monograph on modern (nonlinear) functional-analytic methods for the p-Laplacian and similar quasilinear partial differential operators is the classical book by J.-L. Lions [42]. Besides methods of Nonlinear Analysis it contains also many applications to various mathematical models. Among important topics are the *global climate modelling* treated in J.-I. Díaz, G. Hetzer, and L. Tello [13] and *nonlinear fluid dynamics* in J. I. Díaz and F. de Thélin [14].

The spectrum of the (positive) p-Laplace operator $-\Delta_p$ on the Sobolev space $W_0^{1,p}(\Omega)$ (that is, a monotone nonlinear operator with the zero Dirichlet boundary conditions) has been an interesting open problem for decades, with the exception of the first eigenvalue; see the monograph by S. Fučík, J. Nečas, J. Souček, and V. Souček [25]. The Fredholm alternative at the first eigenvalue is studied in P. Drábek, P. Girg, P. Takáč, and M. Ulm [15] in a bounded domain $\Omega \subset \mathbb{R}^N$ and in J. Benedikt, P. Girg and P. Takáč [5] in a bounded open interval $\Omega \subset \mathbb{R}^1$. Bifurcations at the first eigenvalue are treated in P. Girg and P. Takáč [27].

9. A short sketch of Missbach's biography

A. Missbach (full name Alois Anton Missbach) was born on the 11$^{\text{th}}$ of June, 1897 in Plenkovice near Znojmo, Moravia (present Czech Republic), and baptized on June 13$^{\text{th}}$, 1897. According to the population statistics office ("matrika") in the town of Libáň in Eastern Bohemia (Czech Republic), A. Missbach had moved to Libáň in 1923 and stayed there until July 26$^{\text{th}}$, 1945. He was employed as a technical engineer from 1923 through 1945 in the sugar refinery in Libáň where he performed his research reported in Refs. [43] – [46]. While working full time as an engineer (the second technical adjunct), he defended his doctoral thesis on June 26$^{\text{th}}$, 1936 at the Czech Technical University in Brno, Moravia. He received the degree of Doctor of Technical Sciences (Dr. techn.). His thesis advisor was the well-known expert in Chemistry and sugar production, prof. Ing. Dr. techn. et Dr. agr. h.c. Jaroslav Dědek.

A. Missbach got married in 1928 in the famous Old Town Hall in the historic center of Prague, then the capital of Czechoslovakia. According to the statistics office in Libáň, he moved out to Havraň near the town of Most in Northwestern Bohemia (Czech Republic). As far as we know from the municipal office of Havraň, several months later he moved to the nearby village of Lenešice, also near the town of Most. He was the director of the sugar refinery in Havraň at least during his stay there. His last residence known to us was the town of Most starting on August 12$^{\text{th}}$, 1953. Both sugar refineries, in Libáň and Havraň, have been closed down several decades ago.

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local museum library, particularly Refs. [12] and [43] – [46] in both Czech and German.

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Appendix A5

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DIFFERENTIABILITY PROPERTIES OF p-TRIGONOMETRIC FUNCTIONS

PETR GIRG, LUKÁŠ KOTRLA

Abstract. p-trigonometric functions are generalizations of the trigonometric functions. They appear in context of nonlinear differential equations and also in analytical geometry of the p-circle in the plain. The most important p-trigonometric function is $\sin_p(x)$. For $p > 1$, this function is defined as the unique solution of the initial-value problem

$$(|u'(x)|^{p-2}u'(x))' = (p-1)|u(x)|^{p-2}u(x), \quad u(0) = 0,\ u'(0) = 1\,,$$

for any $x \in \mathbb{R}$. We prove that the n-th derivative of $\sin_p(x)$ can be expressed in the form

$$\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cos_p^{1-q_{k,n}}(x)\,,$$

on $(0, \pi_p/2)$, where $\pi_p = \int_0^1 (1-s^p)^{-1/p} \mathrm{d}s$, and $\cos_p(x) = \sin_p'(x)$. Using this formula, we proved the order of differentiability of the function $\sin_p(x)$. The most surprising (least expected) result is that $\sin_p(x) \in C^\infty(-\pi_p/2, \pi_p/2)$ if p is an even integer. This result was essentially used in the proof of theorem, which says that the Maclaurin series of $\sin_p(x)$ converges on $(-\pi_p/2, \pi_p/2)$ if p is an even integer. This completes previous results that were known e.g. by Lindqvist and Peetre where this convergence was conjectured.

1. Introduction

In the previous two decades, p-trigonometric functions have attracted attention of many researchers; see, e.g., [1, 5, 6, 7, 10, 11, 12, 13, 15, 16, 25], and references therein. The p-trigonometric functions arise from the study of the eigenvalue problem for the one-dimensional p-Laplacian. We assume $p > 1$ and say, that $\lambda \in \mathbb{R}$ is an eigenvalue of

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda |u|^{p-2}u = 0 \quad \text{in } (0, \pi_p)\,, \\ u(0) = u(\pi_p) = 0\,, \end{gathered} \tag{1.1}$$

if there is a nonzero function $u \in W^{1,p}(0, \pi_p)$ that satisfy (1.1) in a weak sense. Here

$$\pi_p = 2\int_0^1 \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s = \frac{2\pi}{p\sin(\pi/p)}\,. \tag{1.2}$$

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Let us note, that the problem can be considered on any bounded open interval, but the choice $(0,\pi_p)$ significantly simplifies the calculations. The discreetness of the spectrum of this eigenvalue problem was established already by Nečas [21]. This eigenvalue problem was later studied by means of the initial-value problem

$$\begin{gathered} -(|u'|^{p-2}u')' - \lambda|u|^{p-2}u = 0 \quad \text{in } (0,\infty)\,, \\ u(0)=0, \quad u'(0)=1\,; \end{gathered} \tag{1.3}$$

see Elbert [11] for initial work in this direction. Later it was independently studied by del Pino-Elgueta-Manasevich [8], Ôtani [22] and Lindqvist [14].

Let $\sin_p(x)$ denote the solution of (1.3) with $\lambda = (p-1)$. It follows from [11] that $\sin_p(x)$ is positive on $(0,\pi_p)$ and satisfies an identity

$$|\sin_p(x)|^p + |\sin_p'(x)|^p = 1 \quad \forall x \in \mathbb{R}\,, \tag{1.4}$$

which for $p=2$ becomes the familiar identity for sine and cosine. This suggest the definition $\cos_p(x) := \sin_p'(x)$ and justifies the notation $\sin_p(x)$ and $\cos_p(x)$. The identity (1.4) is called p-trigonometric identity. It also follows from [11] that the eigenvalues of (1.3) form a sequence $\lambda_k = k^p(p-1), k \in \mathbb{N}$ and corresponding eigenfunctions are functions $\sin_p(kx), k \in \mathbb{N}$. Thus all the eigenfunctions are determined by the function $\sin_p(x)$. It comes as no surprise that the properties of the function $\sin_p(x)$ were studied extensively in the previous 30 years. It was shown in [11] that $\sin_p(x)$ can be expressed on $[0,\pi_p/2]$ (the p-trigonometric identity (1.4) can be thought of as the first integral of (1.3)) as the inverse of

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s\,, \quad x \in [0,1]\,, \tag{1.5}$$

which is extended to $[0,\pi_p]$ by reflection $\sin_p(x) = \sin_p(\pi_p - x)$ and to $[-\pi_p,\pi_p]$ as the odd function. Finally, it is extended to $\mathbb{R}$ as the $2\pi_p$-periodic function. The function $\operatorname{arcsin}_p(x)$ from (1.5) is extended to $[-1,1]$ as an odd function. Then

$$\sin_p(\operatorname{arcsin}_p(x)) = x \quad \forall x \in [-1,1]\,. \tag{1.6}$$

Note that for $p=2$, we obtain classical arcsine and sine from this definition. The (now familiar) notation $\sin_p$ appears in [8] for the first time, where the authors studied homotopic deformation along p to calculate the degree of trivial solutions of (1.1) in order to establish existence results for the nonlinear problem $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0)=u(T)=0$, $p>1$, $T>0$. The homotopy result from [8] initiated development of bifurcation theory for quasilinear bifurcations.

As a historical remark, let us mention that generalizations of arcsine similar to (1.5) were studied in a very different context by Lundberg [17] in 1879. It is interesting to mention that the p-trigonometric functions satisfy certain relations to geometrical objects such as arclength and area of a circle in a noneuclidean metric; see Elbert [11], and Lindqvist [15]. The p-trigonometric functions also possesses some approximation properties in certain function spaces; see, e.g., Binding-Boulton-Čepička-Drábek-Girg [1], Lang-Edmunds [13] for theoretical research, and Boulton-Lord [6] for a very interesting computational application in evolutionary PDEs. In Wood [27], the particular case $p=4$ was studied and "p-polar" coordinates in the xy-plane were proposed.

In this article we focus on the differentiability and analyticity properties of p-trigonometric functions. One can immediately see from (1.2), (1.5), and (1.6) that $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$ for all $p>1$. From (1.4) and the definition of

$\cos_p(x)$, we obtain $\cos_p(0) = 1$ and $\cos_p(\pi_p/2) = 0$. It follows from the results in [11, 15, 22] that the possible differentiability issues are located at $x = 0$ and $x = \pi_p/2$. There are several results concerning differentiability and asymptotic behaviour of $\sin_p(x)$ at $x = 0$ and $x = \pi_p/2$ in Manásevich-Takáč [19] and Benedikt-Girg-Takáč [2]. In Peetre [25], generalized formal Maclaurin series for $\sin_p(x)$ were studied and their convergence was conjectured on $(-\pi_p/2, \pi_p/2)$. The local convergence of the generalized Taylor series (and/or the generalized Maclaurin series) for $\sin_p(x)$ follows from Paredes-Uchiyama [24]. Taking into account that the point $x = 0$ is often considered as the center for the Taylor (i.e. the Maclaurin) series or the generalized Taylor (i.e. the generalized Maclaurin) series for $\sin_p(x)$, we decided to provide detailed study of the convergence of these series towards $\sin_p(x)$ on $(-\pi_p/2, \pi_p/2)$. We were also motivated by work of Ôtani [23], where he studies properties of the solutions of

$$\begin{gathered} (|u'|^{p-2}u')' + |u|^{q-2}u = 0 \quad \text{in } (a,b)\,, \\ u(a) = u(b) = 0\,, \end{gathered} \tag{1.7}$$

for general exponents $p, q \in (1, +\infty)$ with $p \neq q$. Among other properties he proved that for $p = \frac{2m+2}{2m+1}, m \in \{0\} \cup \mathbb{N}$ and for q even, any solution of (1.7) belongs to $C^\infty(a, b)$. In our case, $p = q$ we find that $\sin_p(x)$ belongs to $C^\infty(-\pi_p/2, \pi_p/2)$ if and only if p is even. Let us also remark that local analytic solutions of the radial variant of (1.7) were studied in Bognár [4].

Though we are aware that our methods are elementary mathematics, we are sure that our results will help to better understand the behavior of $\sin_p(x)$ and its derivatives in the vicinity of 0. This behavior is crucial in establishing asymptotic estimates such as those in the proof of the Fredholm alternative for the p-Laplacian in the degenerate case Benedikt-Girg-Takáč [2, 3]. Moreover, knowledge of the convergence/nonconvergence of the Taylor and/or the Maclaurin series is very important in the development of numerical methods for calculating approximations of function values of p-trigonometric functions. Recently, Marichev [20] from the Wolfram Research, Inc., pointed out to the first author of this paper in a personal communication that Mathematica from version 8.0 has a capability to effectively compute coefficients for $\sin_p(x)$ for formal generalized Maclaurin power series by means of the Bell Polynomials. With few lines of Mathematica code one can obtain partial sums of generalized Maclaurin series for $\sin_p(x)$ of large order in a couple of minutes. Thus the question of the convergence of the partial sums of the Maclaurin series is becoming quite urgent. This was our main motivation to address this topic.

Our main result provides convergence of these partial sums. We treat two cases separately, $p > 2$ is an even integer and $p > 2$ is an odd integer. Namely, for the particular case $\sin_{2(m+1)}(x)$, $m \in \mathbb{N}$, $x \in (-\pi_p/2, \pi_p/2)$, we show that the Maclaurin series converges towards the values $\sin_{2(m+1)}(x)$ on the interval $(-\pi_p/2, \pi_p/2)$. On the other hand, we show that the Maclaurin series converge towards $\sin_{2m+1}(x)$, $m \in \mathbb{N}$, for $x \in (0, \pi_p/2)$ and does not for $x \in (-\pi_p/2, 0)$. More precisely, the Maclaurin series converges on $x \in (-\pi_p/2, \pi_p/2)$, but not towards values of $\sin_{2m+1}(x)$, $m \in \mathbb{N}$ for $x \in (-\pi_p/2, 0)$.

The article is organized as follows. In Section 2, we give a definition of the function $\sin_p(x)$ by means of a differential equation and also introduce other useful notation. In Section 3, we state and discuss our main results concerning differentiability and/or non-differentiability of $\sin_p(x)$ and convergence of Maclaurin series of

$\sin_p(x)$. In Section 4, we express higher derivatives of $\sin_p(x)$ by means of powers of $\sin_p(x)$ and $\cos_p(x)$. Finally, in Section 5, we prove our main results using formulas for higher derivatives of $\sin_p(x)$ from Section 4. In Section 6, we conclude with remarks and open problems.

2. Definitions of p-trigonometric functions

Proposition 2.1. *The initial-value problem*

$$\begin{gathered} -(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0 \\ u(0) = 0, \quad u'(0) = 1\,, \end{gathered} \tag{2.1}$$

has the unique local solution and moreover any local solution to (2.1) *can be continued to* $(-\infty, +\infty)$.

For uniqueness of the solution see [8, Sect. 3], and for the existence of global solutions see [9, Lemma A.1].

Definition 2.2. The function $\sin_p(x)$ is defined as the unique solution of the initial-value problem (2.1) on $\mathbb{R}$.

For any $q > 1$ and $z \in \mathbb{R}$ we define

$$\varphi_q(z) = \begin{cases} |z|^{q-2}z & \text{for} z \neq 0\,, \\ 0 & \text{for } z = 0\,. \end{cases} \tag{2.2}$$

Note that $\varphi_{p'}(\varphi_p(z)) = \varphi_p(\varphi_{p'}(z)) = z$ provided $p > 1$ and $1/p + 1/p' = 1$. With this notation, we can rewrite the initial-value problem (2.1) as an equivalent first-order system

$$\begin{gathered} u'(x) = \varphi_{p'}(v(x))\,, \\ v'(x) = -(p-1)\varphi_p(u(x))\,, \\ u(0) = 0, \quad v(0) = 1\,. \end{gathered} \tag{2.3}$$

Clearly, from the definition of Carathéodory solution, it follows that $u(x) = \sin_p(x)$ and $v(x) = \varphi_p(\sin_p'(x))$ must be absolutely continuous on any compact interval $[-K, K]$, $K > 0$. Thus $\sin_p'(x) = \varphi_{p'}(v(x))$ is continuous on any $[-K, K]$, $K > 0$, which entails that $\sin_p'(x) = \varphi_{p'}(v(x))$ is continuous on $(-\infty, +\infty)$. Thus the following definition makes sense.

Definition 2.3. For $x \in \mathbb{R}$, we define $\cos_p(x) = \sin_p'(x)$.

Since $\cos_p(0) = \sin_p'(0) = 1$ and $\cos_p(x)$ is continuous, there exists an interval $(-c, c)$ such that $\cos_p(x) > 0$ on $(-c, c)$, $c > 0$. Moreover, since $\sin_p'(0) = 1$ and $\sin_p \in C^1(\mathbb{R})$, there exists an interval $[0, s)$, $s > 0$, such that $\sin_p(x) \geq 0$ on $[0, s)$.

Definition 2.4. For $p > 1$, let π_p denote

$$2\sup\{s > 0 : \forall x \in (0, s) \text{ holds } \sin_p(x) > 0 \wedge \cos_p(x) > 0\}\,.$$

It was shown in [11], that

$$\pi_p = 2\int_0^1 \frac{1}{(1 - x^p)^{1/p}}\,\mathrm{d}x = \frac{2\pi}{p \cdot \sin(\pi/p)}\,,$$

for $p > 1$. It was also shown in [11], that $\sin_p(x)$ can be expressed on $[0, \pi_p/2]$ as the inverse of

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s \quad x \in [0,1]\,, \tag{2.4}$$

and, moreover, it extends to $[0, \pi_p]$ by reflection $\sin_p(x) = \sin_p(\pi_p - x)$ and to $[-\pi_p, \pi_p]$ as the odd function. Finally, it extends to $\mathbb{R}$ as the $2\pi_p$-periodic function.

Remark 2.5. In the following text, formulas containing higher order derivatives and powers of $\sin_p(x)$ and $\cos_p(x)$ appear. We try to keep our notation as close as possible to the usual notation for classical trigonometric functions. Thus the derivatives are denoted by, e.g., $\sin_p'(x), \dots, \sin_p'''(x), \sin_p^{(\mathrm{iv})}(x)$ (primes and roman numerals) and/or, e.g., $\sin_p^{(n)}(x)$, $\sin_p^{(2n-1)}$ and $\sin_p^{(2n)}$ for $n \in \mathbb{N}$. On the other hand, the powers are denoted by $\sin_p^2(x), \sin_p^3(x), \sin_p^q(x)$, $q \in \mathbb{R}$. Where a confusion may happen, we denote the powers by, e.g., $(\sin_p(x))^m$, $m \in \mathbb{N}$, to distinguish them clearly from derivatives. For the convenience of the reader, we write the values of p as explicit as possible, with a few exceptions such as in the proofs of Theorems 3.3 and 3.4, where this approach would produce very lengthy formulas.

3. Main results

In the sequel, we study derivatives of $\sin_p(x)$ for $p \in \mathbb{N}$, $p > 2$ on the interval $x \in (-\pi_p/2, \pi_p/2)$. We distinguish two cases p is even, i.e., $p = 2(m+1)$ and $m \in \mathbb{N}$, and p is odd; i.e., $p = 2m+1$ and $m \in \mathbb{N}$. In the first case $p = 2(m+1)$, the p-trigonometric identity (1.4) takes form

$$(\sin_{2(m+1)}(x))^{2(m+1)} + (\cos_{2(m+1)(x)})^{2(m+1)} = 1\,, \tag{3.1}$$

which is valid for any $x \in \mathbb{R}$ and hence on $(-\pi_p/2, \pi_p/2)$. Note that there is no absolute value, since there are even powers.

In the second case $p = 2k+1$, we have to distinguish two subcases. For $0 < x < \frac{\pi_p}{2}$, the p-trigonometric identity takes form

$$(\sin_{2m+1}(x))^{2m+1} + (\cos_{2m+1}(x))^{2m+1} = 1\,. \tag{3.2}$$

On the other hand, for $-\pi_p/2 < x < 0$, the p-trigonometric identity takes form

$$-(\sin_{2m+1}(x))^{2m+1} + (\cos_{2m+1}(x))^{2m+1} = 1\,. \tag{3.3}$$

Since there is only one identity (3.1) for $p = 2(m+1)$, this case has nice smoothness properties on $(-\pi_p/2, \pi_p/2)$ and we obtain a rather surprising result concerning smoothness of function $\sin_p(x)$ for even p.

Theorem 3.1. *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then*

$$\sin_{2(m+1)}(x) \in C^\infty\big(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2}\big)\,.$$

On the other hand, for $p = 2m+1$, we have to distinguish two subcases (3.2) and (3.3), which has damaging effect on the differentiability of $\sin_p(x)$. Thus the smoothness is lost when p is odd. The smoothness is also lost if p is not an integer.

Theorem 3.2. *Let $p \in \mathbb{R} \setminus \{2m\}$, $m \in \mathbb{N}$, $p > 1$. Then*

$$\sin_p(x) \in C^{\lceil p \rceil}(-\pi_p/2, \pi_p/2)\,,$$

but

$$\sin_p(x) \notin C^{\lceil p \rceil + 1}(-\pi_p/2, \pi_p/2)\,.$$

Here $\lceil p \rceil := \min\{k \in \mathbb{N} : k \geq p\}$.

Our last result gives an explicit radius of convergence of the Maclaurin series for even $p > 2$. To the best of our knowledge, all previous results concerning convergence of series for $\sin_p(x)$ were only local; see, e.g., [24].

Theorem 3.3. *Let $p = 2(m+1)$ for $m \in \mathbb{N}$. Then the Maclaurin series of $\sin_{2(m+1)}(x)$ converges on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$.*

Theorem 3.4. *Let $p = 2m+1$, $m \in \mathbb{N}$. Then the formal Maclaurin series of $\sin_{2m+1}(x)$ converges on $(-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$. Moreover, the formal Maclaurin series of $\sin_p(x)$ converges towards $\sin_{2m+1}(x)$ on $[0, \frac{\pi_{2m+1}}{2})$, but does not converge towards $\sin_{2m+1}(x)$ on $(-\frac{\pi_{2m+1}}{2}, 0)$.*

The proofs of Theorems 3.1–3.4 are postponed to Section 5.

4. Derivatives of $\sin_p(x)$

The following lemma summarizes basic properties of $\sin_p(x)$ and $\cos_p(x)$.

Lemma 4.1. *Let $p \in \mathbb{R}, p > 1$. Functions $\sin_p(x)$ and $\cos_p(x)$ have the following basic properties.*

1. *$\sin_p(x) > 0$ on $(0, \pi_p)$, $\sin_p(0) = 0$, $\sin_p(x) = \sin_p(\pi_p - x)$ for $x \in (\frac{\pi_p}{2}, \pi_p)$, and $\sin_p(x) = -\sin_p(-x)$ on $(-\pi_p, 0)$. The function $\sin_p(x)$ extends to $\mathbb{R}$ as $2\pi_p$-periodic function.*
2. *$\sin_p(x)$ is strictly increasing on $(-\pi_p/2, \pi_p/2)$.*
3. *$\cos_p(x) > 0$ on $(-\pi_p/2, \pi_p/2)$, $\cos_p(-\frac{\pi_p}{2}) = \cos_p(\frac{\pi_p}{2}) = 0$ and $\cos_p(x) < 0$ on $\left[-\pi_p, -\frac{\pi_p}{2}\right) \cup \left(\frac{\pi_p}{2}, \pi_p\right]$.*
4. *For all $n \in \mathbb{N}$, if $\sin_p^{(2n-1)}(x)$ exists on $(-\pi_p/2, \pi_p/2)$, then it is even function on $(-\pi_p/2, \pi_p/2)$.*
5. *For all $n \in \mathbb{N}$, if $\sin_p^{(2n)}(x)$ exists on $(-\pi_p/2, \pi_p/2)$, then it is odd function on $(-\pi_p/2, \pi_p/2)$.*

Statements 1–3 follows from [11]. Statements 4, and 5 are trivial consequence of statement 1.

Lemma 4.2. *For all $p \in \mathbb{R}, p > 1$*

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \quad \text{for } x \in (0, \pi_p/2)\,, \tag{4.1}$$

$$\sin_p''(x) = \sin_p^{p-1}(-x) \cdot \cos_p^{2-p}(x) \quad \text{for } x \in (-\pi_p/2, 0)\,. \tag{4.2}$$

Proof. The identity (4.1) is obtained by a straightforward calculation; see, e.g., [13]. For $x \in (-\pi_p/2, 0)$, we obtain from Lemma 4.1 statement 1 and 3 and the identity (1.4)

$$\sin_p^p(-x) + \cos_p^p(x) = |-\sin_p(-x)|^p + |\cos_p(x)|^p = |\sin_p(x)|^p + |\cos_p(x)|^p = 1. \tag{4.3}$$

Taking

$$\sin_p^p(-x) + \cos_p^p(x) = 1 \tag{4.4}$$

into derivative we obtain

$$-p \cdot \sin_p^{p-1}(-x) \cdot \cos_p(-x) + p \cdot \cos_p^{p-1}(x) \cdot \sin_p''(x) = 0\,. \tag{4.5}$$

From Lemma 4.1, statements 3 and 4, we obtain

$$\sin_p^{p-1}(-x) \cdot \cos_p(x) = \cos_p^{p-1}(x) \cdot \sin_p''(x)$$

which yields

$$\sin_p''(x) = \sin_p^{p-1}(-x) \cdot \cos_p^{2-p}(x)\,.$$

□

Lemma 4.3. *Let $p \in \mathbb{R} \setminus \{2\}$ such that $p > 1$.*

1. *If $p > 2$, then the function $\sin_p(x) \in C^1(\mathbb{R})$ and $\sin_p(x) \notin C^2(\mathbb{R})$.*
2. *If $p \in (1,2)$, then the function $\sin_p(x) \in C^2(\mathbb{R})$ and $\sin_p(x) \notin C^3(\mathbb{R})$.*

Proof. By the definition of $\cos_p(x)$, $\sin_p'(x) = \cos_p(x)$. The function $\cos_p(x) \in C(\mathbb{R})$, for all $p > 1$. Thus $\sin_p(x) \in C^1(\mathbb{R})$. By Lemma 4.2,

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \quad \text{for } x \in (0, \pi_p/2)\,.$$

Taking into account that

$$\lim_{x \to \frac{\pi_p}{2}-} \sin_p^{p-1}(x) = 1 \quad \text{and} \quad \lim_{x \to \frac{\pi_p}{2}-} \cos_p^{2-p}(x) = +\infty \quad \text{for } p > 2\,,$$

we find that

$$\lim_{x \to \frac{\pi_p}{2}-} \sin_p''(x) = -\infty\,.$$

Thus the continuity of $\sin_p''(x)$ fails at $x = \pi_p/2$ for $p > 2$ and the statement 1 of Lemma 4.3 follows.

From (2.3), we find that the function $v'(x) = -(p-1)\varphi_p(\sin_p(x))$ is continuous on $\mathbb{R}$ as $\sin_p(x)$ is continuous on $\mathbb{R}$. We also find that $\cos_p(x) = \varphi_{p'}(v(x))$ from (2.3). Taking into account that $\varphi_{p'} \in C^1(\mathbb{R})$ for $p \in (1,2)$ (observe that $p' = \frac{p}{p-1} > 2$ in this case), we infer that $\cos_p'(x) = \varphi_{p'}'(v(x)) \cdot v'(x)$ is continuous on $\mathbb{R}$. Thus $\sin_p(x)$ is two times continuously differentiable on $\mathbb{R}$ for $p \in (1,2)$. On the other hand, taking

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \quad \text{on } (0, \frac{\pi_p}{2})$$

into derivative, we obtain

$$\sin_p'''(x) = -(p-1)\sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) - (2-p) \cdot \sin_p^{p-1}(x) \cdot \cos_p^{1-p}(x) \cdot \sin_p''(x)\,.$$

Substituting for $\sin_p''(x)$ from the later equation into the former, we have

$$\sin_p'''(x) = -(p-1)\sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) + (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x)\,.$$

Since $\lim_{x\to 0+} \sin_p(x) = 0$ and $\lim_{x \to 0+} \cos_p(x) = 1$, we obtain

$$\lim_{x \to 0+} \sin_p'''(x) = -\infty$$

for $p \in (1,2)$. This concludes the proof of statement 2 of Lemma 4.3. □

Let us define the following 'symbolic' operators (rewriting rules) defined on expressions of the form

$$a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) \quad \text{ with } a, q \in \mathbb{R} \tag{4.6}$$

as follows

$$\mathrm{D}_s\, a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) := \begin{cases} a \cdot q \cdot \sin_p^{q-1}(x) \cdot \cos_p^{1-(q-1)}(x) & q \neq 0\,, \\ 0 & q = 0\,. \end{cases} \tag{4.7}$$

$$\mathrm{D}_c\, a \cdot \sin_p^q(x) \cdot \cos_p^{1-q}(x) := \begin{cases} -a \cdot (1-q) \cdot \sin_p^{q+p-1}(x) \cdot \cos_p^{1-(q+p-1)}(x) & q \neq 1\,, \\ 0 & q = 1\,. \end{cases} \tag{4.8}$$

Let us observe that the results of application D_s and D_c have the form (4.6). Hence they are also in the domain of definition of D_s and D_c. Thus we can consider compositions of D_c and D_s of arbitrary length. We will show that the first derivative of $\sin_p^q(x)\cdot\cos_p^{1-q}(x)$ (here $a=1$) can be written using these symbolic operators as follows

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\sin_p^q(x)\cdot\cos_p^{1-q}(x)\\&=\mathrm{D}_s\sin_p^q(x)\cdot\cos_p^{1-q}(x)+\mathrm{D}_c\sin_p^q(x)\cdot\cos_p^{1-q}(x)\,.\end{aligned}$$

To show this, we have to distinguish three cases $q\in\mathbb{R}\setminus\{0,1\}$, $q=1$, and $q=0$.
Case $q\in\mathbb{R}\setminus\{0,1\}$. Here

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\sin_p^q(x)\cdot\cos_p^{1-q}(x)\\&=q\sin_p^{q-1}(x)\cdot\cos_p^{1-(q-1)}(x)-(1-q)\sin_p^{q+p-1}(x)\cdot\cos_p^{1-(q+p-1)}(x)\\&=\mathrm{D}_s\sin_p^q(x)\cdot\cos_p^{1-q}(x)+\mathrm{D}_c\sin_p^q(x)\cdot\cos_p^{1-q}(x)\,.\end{aligned}$$

Note that the distance between the exponents of $\sin_p(x)$ in the resulting terms, i.e., $\sin_p^{q_0-1}(x)\cdot\cos_p^{2-q_0}(x)$ and $\sin_p^{q_0+p-1}\cdot\cos_p^{2-p-q_0}(x)$, is exactly p. This is crucial in the sequel of the paper, because in a sum of the type

$$c_0\sin_p^{q_0}(x)\cdot\cos_p^{1-q_0}(x)+c_1\sin_p^{q_0+p}(x)\cdot\cos_p^{1-(q_0+p)}(x)$$

the terms combine together as in the diagram depicted on Fig. 1
Case $q=1$. In this case the term $\sin_p^q(x)\cdot\cos_p^{1-q}(x)=\sin_p(x)$. Thus the derivative of this term is the *single* term $\cos_p(x)$. By the definitions of $\mathrm{D}_s,\mathrm{D}_c$, we find that $\mathrm{D}_s\sin_p(x)=\cos_p(x)$ and $\mathrm{D}_c\sin_p(x)=0$. Thus $\frac{\mathrm{d}}{\mathrm{d}x}\sin_p(x)=\mathrm{D}_s\sin_p(x)+\mathrm{D}_c\sin_p(x)$. The fact $\mathrm{D}_c\sin_p(x)=0$ will be reflected in our diagrams by omitting 'right-down' edge departing from this node, see Figure 2.
Case $q=0$. This case corresponds to $\sin_p^q(x)\cdot\cos_p^{1-q}(x)=\cos_p(x)$. Thus the derivative of this term is the *single* term $-\sin_p^{p-1}(x)\cos_p^{1-(p-1)}(x)$. By the definitions of $\mathrm{D}_s,\mathrm{D}_c$, we find that $\mathrm{D}_s\cos_p(x)=0$ and

$$\mathrm{D}_c\cos_p(x)=-\sin_p^{p-1}(x)\cos_p^{1-(p-1)}(x)\,.$$

Thus $\frac{\mathrm{d}}{\mathrm{d}x}\cos_p(x)=\mathrm{D}_s\cos_p(x)+\mathrm{D}_c\cos_p(x)$. The fact $\mathrm{D}_s\cos_p(x)=0$ will be reflected in our diagrams by omitting 'left-down' edge departing from this node, see Figure 3. Note that since in our diagrams we write powers only, the node corresponding to $-\sin_p^{p-1}(x)\cos_p^{1-(p-1)}(x)$ is labeled by $s_p^{p-1}\,c_p^{1-(p-1)}$.

In the same way, we can express higher order derivatives, thus, e.g., the second derivative of $\sin_p^q(x)\cdot\cos_p^{1-q}(x)$ (here $a=1$) can be written as

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin_p^q(x)\cdot\cos_p^{1-q}(x)\\&=(\mathrm{D}_s\circ\mathrm{D}_s)\sin_p^q(x)\cdot\cos_p^{1-q}(x)+(\mathrm{D}_c\circ\mathrm{D}_s)\sin_p^q(x)\cdot\cos_p^{1-q}(x)\\&\quad+(\mathrm{D}_s\circ\mathrm{D}_c)\sin_p^q(x)\cdot\cos_p^{1-q}(x)+(\mathrm{D}_c\circ\mathrm{D}_c)\sin_p^q(x)\cdot\cos_p^{1-q}(x).\end{aligned}$$

To better understand our methods of proof, it is good to have in mind the diagrams Figures 1–3.

The way how the term in the n-th derivative on the k-th position was derived from $\sin_p''(x)$ can be recovered from n and k as follows. First let us recall some notation from formal languages.

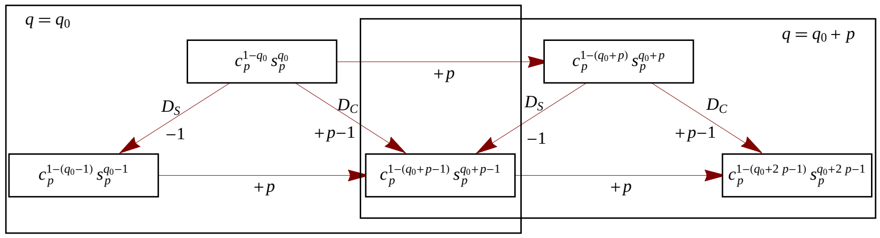

FIGURE 1. Rewriting diagram of the first derivative of $c_0 \sin_p^{q_0}(x) \cdot \cos_p^{1-q_0}(x) + c_1 \sin_p^{q_0+p}(x) \cdot \cos_p^{1-(q_0+p)}(x)$. For the lack of space, we *do not* write the coefficients standing in front of these terms and *use short-cuts*, i.e., we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$

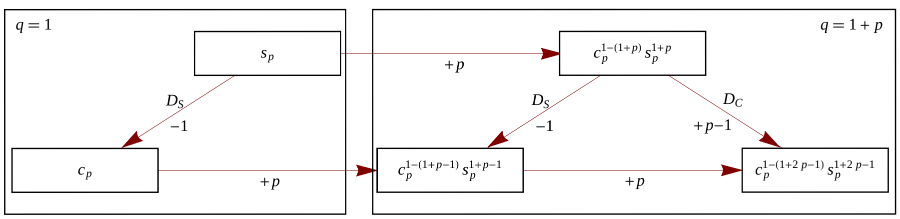

FIGURE 2. Rewriting diagram of the case $q = 1$. Recall that we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$ and do not write the coefficients

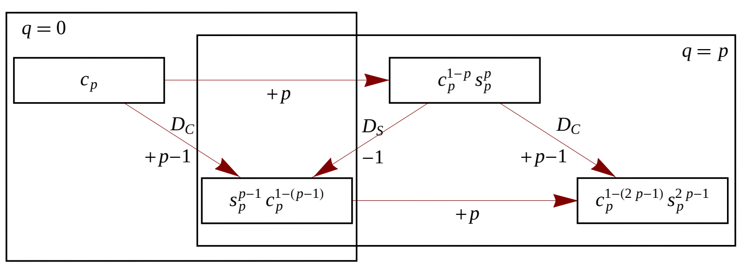

FIGURE 3. Rewriting diagram of the case $q = 0$. Recall that we write s_p^q instead of $\sin_p^q(x)$ and c_p^{1-q} instead of $\cos_p^{1-q}(x)$ and do not write the coefficients

Definition 4.4. (Salomaa-Soittola [26, I.2, p. 4,], and/or Manna [18, p. 2–3, p. 47, p. 78]) An *alphabet* (denoted by V) is a finite nonempty set of letters. A *word* (denoted by w) over an alphabet V is a finite string of zero or more letters from the alphabet V. The word consisting of zero letters is called the *empty word*. The set of all words over an alphabet V is denoted by V^* and the set of all nonempty words over an alphabet V is denoted by V^+. For strings w_1 and w_2 over V, their juxtaposition w_1w_2 is called *catenation* of w_1 and w_2, in operator notation cat : $V^* \times V^* \to V^*$ and $\mathrm{cat}(w_1, w_2) = w_1w_2$. We also define the length of the word w, in operator notation len : $V^* \to \{0\} \cup \mathbb{N}$, which for a given word w yields the number of letters in w when each letter is counted as many times as it occurs in w. We also use *reverse function* rev : $V^* \to V^*$ which reverses the order of the letters in any word w (see [18, p. 47, p. 78]).

For our purposes here, we consider the alphabet $V = \{0, 1\}$ and the set of all nonempty words V^+. Thus words in V^+ are, e.g.,

$$\text{“}\mathtt{0}\text{”}, \text{“}\mathtt{1}\text{”}, \text{“}\mathtt{01}\text{”}, \text{“}\mathtt{10}\text{”}, \text{“}\mathtt{11}\text{”} \dots .$$

For instance, cat(“1110”, “011”) = “1110011”, and

$$\mathrm{rev}(\text{“}\mathtt{010011000}\text{”}) = \text{“}\mathtt{000110010}\text{”}\,,$$
$$\mathrm{len}(\text{“}\mathtt{010011000}\text{”}) = 9\,.$$

Let $n \in \mathbb{N}$, $k \in \{0\} \cup \mathbb{N}, 0 \le k \le 2^{n-2} - 1$ and $(k)_{2,n-2}$ be the string of bits of the length $n - 2$ which represents binary expansion of k (it means, e.g., for $k = 3$ and $n = 5$, $(3)_{2,5-2} =$ “011”). Now we are ready to define $D_{k,n}$ in two steps as follows.

Step 1 We create an ordered $n-2$-tuple $d_{k,n-2} \in \{\mathrm{D}_s, \mathrm{D}_c\}^{n-2}$ (cartesian product of sets $\{\mathrm{D}_s, \mathrm{D}_c\}$ of length $n-2$) from $\mathrm{rev}((k)_{2,n-2})$ such that for $1 \le i \le n-2$, $d_{k,n-2}$ contains D_s on the i-th position if $\mathrm{rev}((k)_{2,n-2})$ contains “0” on the i-th position, and $d_{k,n}$ contains D_c on the i-th position if $\mathrm{rev}((k)_{2,n-2})$ contains “1” on the i-th position (it means, e.g., for $k = 3$, and $n = 5$, we obtain $d_{3,5-2} = (\mathrm{D}_c, \mathrm{D}_c, \mathrm{D}_s)$).

Step 2 We define $D_{k,n}$ as the composition of operators $\mathrm{D}_s, \mathrm{D}_c$ in the order they appear in the ordered n-tuple $d_{k,n-2}$ (it means, e.g., for $k = 3$, and $n = 5$, we obtain $D_{3,5} = (\mathrm{D}_c \circ \mathrm{D}_c \circ \mathrm{D}_s)$).

The following Lemma implies that

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} D_{k,n} \sin_p''(x) \tag{4.9}$$

for all $x \in (0, \pi_p/2)$.

Lemma 4.5. *Let $p \in \mathbb{R}$, $p > 1$, $n \in \mathbb{N}$. Then $\sin_p^{(n)}(x)$ exists on $(0, \pi_p/2)$ and it is continuous. Moreover,*

$$\textit{for } n = 1: \quad \sin_p'(x) = \cos_p(x)\,, \tag{4.10}$$

$$\textit{for } n = 2: \quad \sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)\,, \tag{4.11}$$

and for $n = 3, 4, 5, \dots$, $k = 0, 1, 2, 3, \dots, 2^{n-2} - 1$ there exists $a_{k,n} \in \mathbb{R}$, $l_{k,n}, m_{k,n} \in \mathbb{Z}$ such that

$$\mathrm{D}_{k,n} \sin_p''(x) = a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1-p \cdot l_{k,n} - m_{k,n}}(x)\,, \tag{4.12}$$

and

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1-p \cdot l_{k,n} - m_{k,n}}(x)\,. \tag{4.13}$$

Moreover, let $j(k) \in \{0\} \cup \mathbb{N}$ *be the digit sum of the binary expansion of* $k = 0, 1, 2, \dots, 2^{n-2}-1$ *(thus* $j(k)$ *is the number of occurrences of* D_c *in* $\mathrm{D}_{k,n}$*) and let* $\mathrm{D}_{k,n} \sin_p''(x) \not\equiv 0$. *Then, for* $k = 0, 1, 2, \dots, 2^{n-2}-1$*, the exponents*

$$q_{k,n} := p \cdot l_{k,n} + m_{k,n} \tag{4.14}$$

satisfy

$$q_{k,n} = j(k)(p-1) + (n-2-j(k))(-1) + p - 1\,. \tag{4.15}$$

Proof. The cases $n = 1$ and $n = 2$ follows immediately from the definition of $\cos_p(x)$ and from Lemma 4.2.

We proceed by induction to prove the validity of the statement for $n = 3, 4, 5, \dots$.

Step 1. Taking (4.11) into derivative, we obtain

$$\sin_p'''(x) = -(p-1) \cdot \sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) + (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x)\,.$$

For $k = 0, 1$ we obtain $a_{0,3} = -(p-1)$, $a_{1,3} = (2-p)$, $l_{0,3} = 1$, $l_{1,3} = 2$ $m_{0,3} = -2$, and $m_{1,3} = -2$. Hence

$$\sin_p'''(x) = \sum_{k=0}^{1} a_{k,3} \cdot \sin_p^{p \cdot l_{k,3} + m_{k,3}}(x) \cdot \cos_p^{1-p \cdot l_{k,3} - m_{k,3}}(x)\,.$$

Since we assume $p > 1$ we obtain $p - 1 \neq 0$ and thus by the definition of D_s and $\mathrm{D}_{k,n}$

$$\begin{aligned}
\mathrm{D}_{0,3} \sin_p''(x) &= \mathrm{D}_s(-\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)) \\
&= -(p-1) \cdot \sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) \\
&= a_{0,3} \cdot \sin_p^{p \cdot l_{0,3} + m_{0,3}}(x) \cdot \cos_p^{1-p \cdot l_{0,3} - m_{0,3}}(x).
\end{aligned}$$

Analogously, by the definition of D_c and $\mathrm{D}_{k,n}$ for $p \neq 2$, we find

$$\begin{aligned}
\mathrm{D}_{1,3} \sin_p''(x) &= \mathrm{D}_c(-\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)) \\
&= -(-1) \cdot (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x) \\
&= a_{1,3} \cdot \sin_p^{p \cdot l_{1,3} + m_{1,3}}(x) \cdot \cos_p^{1-p \cdot l_{1,3} - m_{1,3}}(x)\,,
\end{aligned}$$

and for $p = 2$, we obtain

$$\mathrm{D}_{1,3} \sin_p''(x) = \mathrm{D}_c(-\sin_2(x) \cdot \cos_2^0(x)) = 0\,.$$

Hence,

$$\begin{aligned}
\sin_p'''(x) &= \mathrm{D}_s \sin_p''(x) + \mathrm{D}_c \sin_p''(x) \\
&= \mathrm{D}_{0,3} \sin_p''(x) + \mathrm{D}_{1,3} \sin_p''(x) \\
&= \sum_{k=0}^{1} \mathrm{D}_{k,3} \sin_p''(x)\,.
\end{aligned}$$

Step 2. Let us assume that $\sin_p^{(n)}(x)$ exists, it is continuous on $(0, \pi_p/2)$, and for all $k = 0, 1, 2, \dots, 2^{n-2}-1$ there exist $a_{k,n} \in \mathbb{R}$, $l_{k,n}, m_{k,n} \in \mathbb{Z}$ such that

$$\mathrm{D}_{k,n} \sin_p^{(n)}(x) = a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1-p \cdot l_{k,n} - m_{k,n}}(x)\,, \tag{4.16}$$

and

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{p \cdot l_{k,n} + m_{k,n}}(x) \cdot \cos_p^{1-p \cdot l_{k,n} - m_{k,n}}(x)\,. \tag{4.17}$$

By the additivity rule of the derivative, we find that

$$\begin{aligned}\sin_p^{(n+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{k=0}^{2^{n-2}-1} a_{k,n}\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}}(x)\cdot\cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x)\\ &= \sum_{k=0}^{2^{n-2}-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(a_{k,n}\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}}(x)\cdot\cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x)\right).\end{aligned} \tag{4.18}$$

For all $k=0,1,2,\dots,2^{n-2}-1$, we find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}(a_{k,n}\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}}(x)\cdot\cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x))\\ &= a_{k,n}\cdot(p\cdot l_{k,n}+m_{k,n})\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}-1}(x)\cdot\cos_p^{1-(p\cdot l_{k,n}+m_{k,n}-1)}(x)\\ &\quad + a_{k,n}(1-p\cdot l_{k,n}-m_{k,n})\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}}(x)\cdot\cos_p^{1-p\cdot l_{k,n}-m_{k,n}-1}(x)\sin_p''(x)\\ &= a_{k,n}\cdot(p\cdot l_{k,n}+m_{k,n})\cdot\sin_p^{p\cdot l_{k,n}+m_{k,n}-1}(x)\cdot\cos_p^{1-(p\cdot l_{k,n}+m_{k,n}-1)}(x)\\ &\quad - a_{k,n}(1-p\cdot l_{k,n}-m_{k,n})\cdot\sin_p^{p\cdot(l_{k,n}+1)+m_{k,n}-1}(x)\cdot\cos_p^{1-(p\cdot(l_{k,n}+1)+m_{k,n}-1)}(x).\end{aligned} \tag{4.19}$$

For $k=0,1,2,\dots,2^{n-2}-1$, we denote

$$a_{2k,n+1} := a_{k,n}\cdot(p\cdot l_{k,n}+m_{k,n}), \tag{4.20}$$

$$a_{2k+1,n+1} := -a_{k,n}\cdot(1-p\cdot l_{k,n}-m_{k,n}), \tag{4.21}$$

$$l_{2k,n+1} := l_{k,n}, \tag{4.22}$$

$$m_{2k,n+1} := m_{k,n}-1, \tag{4.23}$$

$$l_{2k+1,n+1} := l_{k,n}+1, \tag{4.24}$$

$$m_{2k+1,n+1} := m_{k,n}-1. \tag{4.25}$$

Hence from (4.18), (4.19), and (4.20)–(4.25) we obtain

$$\sin_p^{(n+1)}(x) = \sum_{k'=0}^{2^{n-1}-1} a_{k',n+1}\cdot\sin_p^{p\cdot l_{k',n+1}+m_{k',n+1}}(x)\cdot\cos_p^{1-p\cdot l_{k',n+1}-m_{k',n+1}}(x). \tag{4.26}$$

Note that $\sin_p(x)>0$ and $\cos_p(x)>0$ for $x\in(0,\pi_p/2)$ by Lemma 4.1, statements 1 and 3, and continuous by Lemma 4.3. Moreover, the function $z\mapsto z^q$, defined for $z>0$ and $q\in\mathbb{R}$ belongs to $C^\infty(0,+\infty)$. Thus the function on the right-hand side of (4.26) is continuous for $x\in(0,\pi_p/2)$ which implies the continuity of $\sin_p^{(n+1)}(x)$ for $x\in(0,\pi_p/2)$.

Now, we show that for all $k'=0,1,2,\dots,2^{n-2}-1$: $a_{k',n+1}\in\mathbb{R}$, $l_{k',n+1},m_{k',n+1}\in\mathbb{Z}$ and, moreover,

$$\mathrm{D}_{k',n+1}\sin_p^{(n)}(x) = a_{k',n+1}\cdot\sin_p^{p\cdot l_{k',n+1}+m_{k',n+1}}(x)\cdot\cos_p^{1-p\cdot l_{k',n+1}-m_{k',n+1}}(x). \tag{4.27}$$

Let us set

$$\mathrm{D}_{2k,n+1} := \mathrm{D}_s\circ\mathrm{D}_{k,n}, \tag{4.28}$$

$$\mathrm{D}_{2k+1,n+1} := \mathrm{D}_c\circ\mathrm{D}_{k,n}. \tag{4.29}$$

Then it follows easily from corresponding binary expansion of k and $2k$ that

$$(2k)_{2,n-1} = \mathrm{cat}((k)_{2,n-2}, \text{“}\mathtt{0}\text{”}),$$

$$(2k+1)_{2,n-1} = \mathrm{cat}((k)_{2,n-2}, \text{“1”})$$

and also that (4.28), (4.29) cover all 2^{n-1} of $k' = 0, 1, \dots 2^{n-1} - 1$. Thus our definitions (4.28) and (4.29) conform the relation between binary expansion of $k' = 2k$ and/or $k' = 2k + 1$ and order of compositions of $\mathrm{D}_s, \mathrm{D}_c$ in $\mathrm{D}_{k',n+1}$.

For $k' = 0, 2, 4, \dots, 2^{n-1} - 2$ even,

$$\mathrm{D}_{k',n+1} \sin_p''(x) = \mathrm{D}_{2k,n+1} \sin_p''(x) = \mathrm{D}_s \circ \mathrm{D}_{k,n} \sin_p''(x)\,. \tag{4.30}$$

From the induction assumption (4.16), the definition of D_s (4.7) and (4.20), (4.22), (4.23), we find

$$\begin{aligned}
&\mathrm{D}_s(\mathrm{D}_{k,n} \sin_p''(x)) \\
&= \mathrm{D}_s(a_{k,n} \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}}(x) \cdot \cos_p^{1-p\cdot l_{k,n}-m_{k,n}}(x)) \\
&= a_{k,n} \cdot (p \cdot l_{k,n} + m_{k,n}) \cdot \sin_p^{p\cdot l_{k,n}+m_{k,n}-1}(x) \cdot \cos_p^{1-(p\cdot l_{k,n}+m_{k,n}-1)}(x) \\
&= a_{2k,n+1} \cdot \sin_p^{p\cdot l_{2k,n+1}+m_{2k,n+1}}(x) \cdot \cos_p^{1-p\cdot l_{2k,n+1}-m_{2k,n+1}}(x).
\end{aligned}$$

We can treat $k' = 1, 3, 5, \dots, 2^{n-1} - 1$ in the same way using D_c instead of D_s and (4.8) and (4.21), (4.24), (4.25). This concludes the proof by induction.

It remains to show (4.15). In fact, from the definition (4.8) of D_c, each occurrence of the symbolic operator D_c in $\mathrm{D}_{k,n}$ increases the exponent q of $\sin_p^q(x)$ by $p - 1$. Analogously, from the definition of (4.7) of D_s, each occurrence of the symbolic operator D_s in $\mathrm{D}_{k,n}$ decreases the exponent q of $\sin_p^q(x)$ by 1. Taking into account these facts and also that $q_{1,2} = p - 1$, the formula (4.15) follows. This concludes the proof of Lemma 4.5. □

Lemma 4.6. *Let $p \in \mathbb{N}$, $p > 1$, and for all $n \in \mathbb{N}$, $n \geq 2$*

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x)\,. \tag{4.31}$$

Then for all $n \in \mathbb{N}$, $n \geq 2$, and all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$

$$q_{k,n} \in \{0\} \cup \mathbb{N}\,. \tag{4.32}$$

Proof. From the definitions (4.7) and (4.8),

$$\begin{aligned}
q_{2k,n+1} &= q_{k,n} - 1 \quad \text{(we applied } \mathrm{D}_S \text{ on the expression)} \\
q_{2k+1,n+1} &= q_{k,n} + p - 1 \quad \text{(we applied } \mathrm{D}_\mathrm{C} \text{ on the expression)}
\end{aligned} \tag{4.33}$$

The proof proceeds by induction in n.

Step 1. From Lemma 4.2, for $\sin_p''(x)$ on $(0, \pi_p/2)$ we obtain the formula

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)\,.$$

Thus $q_{1,2} = p - 1$. By assumption $p \in \mathbb{N}$, $p > 1$ we find $q_{1,2} \in \mathbb{N}$.

Step 2. We distinguish two cases, $q_{k,n} \in \mathbb{N}$ and $q_{k,n} = 0$. Let $q_{k,n} \in \mathbb{N}$. Then from (4.33), $p \in \mathbb{N}$, $p > 1$, we obtain

$$\begin{aligned}
q_{2k,n+1} &= q_{k,n} - 1 \in \{0\} \cup \mathbb{N}, \\
q_{2k+1,n+1} &= q_{k,n} + p - 1 \in \mathbb{N},
\end{aligned}$$

which satisfies (4.32). Let $q_{k,n} = 0$. Then the corresponding term in (4.31) has form

$$a_{k,n} \cdot \cos_p(x)\,, \tag{4.34}$$

since $\sin_p^0(x) \equiv 1$ for $x \in (0, \pi_p/2)$. Taking (4.34) into derivative, we find

$$a_{k,n} \cdot \cos_p'(x) = -a_{k,n} \cdot \sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x)$$

and $q_{2k+1,n+1} = p - 1 \in \mathbb{N}$, because $p \in \mathbb{N}$, $p > 1$. This concludes the proof by induction. □

Lemma 4.7. *Let $p \in \mathbb{N}$, $p \geq 3$. Then for all $n \in \mathbb{N}$, $n \geq 2$*

$$\sin_p^{(n)}(x) \leq 0 \quad \text{on} \quad (0, \frac{\pi_p}{2})\,.$$

Proof. By Lemma 4.5 and substitution (4.14), we have

$$\sin_p^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x)\,. \tag{4.35}$$

Let Q_n denote the set of all values of $q_{k,n}$ attained in the previous expression (this is to handle possible multiplicities), i.e.,

$$Q_n = \{q_{k,n} : k = 0, \dots, 2^{n-2} - 1\}\,. \tag{4.36}$$

By Lemma 4.6 for all $n \geq 2$ and for all $k \leq 2^{n-2} - 1$, we have $q_{k,n} \in \{0\} \cup \mathbb{N}$. Clearly, $Q_n \subset \{0\} \cup \mathbb{N}$ has at most 2^{n-2} elements and thus there exists $i_0 \in \mathbb{N} : 0 < i_0 \leq 2^{n-2} - 1$ and bijective mapping

$$\overline{q}_n : \{0, 1, 2, \dots i_0\} = Q_n \tag{4.37}$$

satisfying the order condition

$$\forall i, j = 0, 1, \dots, i_0 : i < j \Rightarrow \overline{q}_i < \overline{q}_j\,. \tag{4.38}$$

In the sequel, $\overline{q}_{i,n}$ stands for $\overline{q}_n(i)$. With this at hand, we add together the coefficients in (4.35) corresponding to the same value of powers $q_{k,n}$ and for any $i = 0, 1, \dots, i_0$ define

$$c_{i,n} := \sum_{\substack{k=0,1,\dots 2^{n-2}-1\\ q_{k,n}=\overline{q}_{i,n}}} a_{k,n}\,. \tag{4.39}$$

Now, we rewrite (4.35) using coefficients $c_{i,n}$:

$$\sin_p^{(n)}(x) = \sum_{i=0}^{i_0} c_{i,n} \cdot \sin_p^{\overline{q}_{i,n}}(x) \cdot \cos_p^{1-\overline{q}_{i,n}}(x)\,. \tag{4.40}$$

Later, we will prove by induction that

$$\forall i = 0, 1, \dots, i_0 : c_{i,n} \leq 0\,. \tag{4.41}$$

By Lemma 4.1 statements 1 and 3, $\sin_p(x) > 0$ and $\cos_p(x) > 0$ on $(0, \frac{\pi_p}{2})$, which implies that for all $q, r \in \{0\} \cup \mathbb{N}$ and $x \in (0, \pi_p/2)$

$$\sin_p^q(x) \cdot \cos_p^r(x) > 0\,. \tag{4.42}$$

Thus from (4.40)–(4.42) the statement of Lemma 4.7 follows.

Now it remains to prove by induction in n that (4.41) holds.

Step 1. By Lemma 4.2 we find that

$$\sin_p''(x) = -\sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \tag{4.43}$$

for all $x \in (0, \pi_p/2)$ and so $c_{0,2} = -1 < 0$.

Taking the derivative of (4.43) (and after some straightforward rearrangements),

$$\sin_p'''(x) = -(p-1) \cdot \sin_p^{p-2}(x) \cdot \cos_p^{3-p}(x) + (2-p) \cdot \sin_p^{2p-2}(x) \cdot \cos_p^{3-2p}(x) \tag{4.44}$$

for $x \in (0, \pi_p/2)$. Since $p \geq 3$, we have $c_{0,3} = -(p-1) \leq -2 \leq 0$ and $c_{1,3} = (2-p) \leq -1 \leq 0$ as desired. Taking the derivative (4.44),

$$\begin{aligned}
\sin_p^{(\mathrm{iv})} &= -(p-1)\cdot(p-2)\cdot\sin_p^{p-3}(x)\cdot\cos_p^{4-p}(x)+ \\
&\quad + (p-1)\cdot(3-p)\cdot\sin_p^{2p-3}(x)\cdot\cos_p^{4-2p}(x) \\
&\quad + (2-p)\cdot(2p-2)\cdot\sin_p^{2p-3}(x)\cdot\cos_p^{4-2p}(x) \\
&\quad - (2-p)\cdot(3-2p)\cdot\sin_p^{3p-3}(x)\cdot\cos_p^{4-3p}(x) \\
&= -(p-1)\cdot(p-2)\cdot\sin_p^{p-3}(x)\cdot\cos_p^{4-p}(x) \\
&\quad + ((p-1)\cdot(3-p)+(2-p)\cdot(2p-2))\cdot\sin_p^{2p-3}(x)\cdot\cos_p^{4-2p}(x) \\
&\quad - (2-p)\cdot(3-2p)\cdot\sin_p^{3p-3}(x)\cdot\cos_p^{4-3p}(x)
\end{aligned} \tag{4.45}$$

for all $x \in (0, \pi_p/2)$. Since $p \geq 3$ we have $c_{0,4} = -(p-1)\cdot(p-2) \leq -2 \leq 0$, $c_{1,4} = (p-1)\cdot(3-p)+(2-p)\cdot(2p-2) \leq -4 \leq 0$, and $c_{2,4} = -(2-p)\cdot(3-2p) \leq -3 \leq 0$

Step 2. Let us assume that $\sin_p^{(n)}(x)$ for $n \geq 4$ can be written in the form (4.40) and

$$\forall i \leq i_0 \, : c_{i,n} \leq 0\,. \tag{4.46}$$

The proof falls naturally into two parts.

Case 1. If

$$\overline{q}_{i,n} \geq 1\,, \tag{4.47}$$

then taking the i-th term of (4.40), which is

$$c_{i,n}\cdot\sin_p^{\overline{q}_{i,n}}(x)\cdot\cos_p^{1-\overline{q}_{i,n}}(x)\,, \tag{4.48}$$

into derivative we obtain

$$\begin{aligned}
&c_{i,n}\cdot\overline{q}_{i,n}\cdot\sin_p^{\overline{q}_{i,n}-1}(x)\cdot\cos_p^{1-\overline{q}_{i,n}+1}(x) \\
&+ c_{i,n}\cdot(1-\overline{q}_{i,n})\cdot\sin_p^{\overline{q}_{i,n}}(x)\cdot\cos_p^{1-\overline{q}_{i,n}-1}(x)\cdot\sin_p''(x)\,.
\end{aligned}$$

Substituting (4.43) for $\sin_p''(x)$ into the previous expression, we obtain

$$\begin{aligned}
&c_{i,n}\cdot\overline{q}_{i,n}\cdot\sin_p^{\overline{q}_{i,n}-1}(x)\cdot\cos_p^{2-\overline{q}_{i,n}}(x) \\
&- c_{i,n}\cdot(1-\overline{q}_{i,n})\cdot\sin_p^{\overline{q}_{i,n}+p-1}(x)\cdot\cos_p^{-\overline{q}_{i,n}-p+2}(x)\,.
\end{aligned}$$

Let us denote

$$\begin{aligned}
a'_{2i-1,n+1} &:= c_{i,n}\cdot\overline{q}_{i,n}\,, \\
a'_{2i,n+1} &:= c_{i,n}\cdot(\overline{q}_{i,n}-1)\,.
\end{aligned}$$

By the induction assumption (4.46) and assumption of Case 1 (4.47), we have $a'_{2i-1,n+1}, a'_{2i,n+1} \leq 0$.

Case 2. If $\overline{q}_{i,n} = 0$, then $i = 0$ (by the ordering) and the corresponding term of (4.40) is

$$c_{0,n}\cdot\sin_p^0(x)\cdot\cos_p^1(x). \tag{4.49}$$

Taking derivatives in (4.49) we find

$$-\,c_{0,n}\cdot\sin_p^{p-1}(x)\cdot\cos_p^{2-p}(x)\,. \tag{4.50}$$

Denote $a'_{1,n+1} := -c_{0,n}$ which is clearly nonnegative by the induction assumption (4.46). We consider the second term of (4.40) ($i = 1$) and take the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}x}c_{1,n}\cdot\sin_p^{\overline{q}_{1,n}}(x)\cdot\cos_p^{1-\overline{q}_{1,n}}(x)$$

$$= \mathrm{D}_s\, c_{1,n} \cdot \sin_p^{\overline{q}_{1,n}}(x) \cdot \cos_p^{1-\overline{q}_{1,n}}(x) + \mathrm{D}_c\, c_{1,n} \cdot \sin_p^{\overline{q}_{1,n}}(x) \cdot \cos_p^{1-\overline{q}_{1,n}}(x) .$$

Since $\overline{q}_{0,n} = 0$, $\overline{q}_{1,n} = p$ (see Figure 4). Note that the right-hand side of

$$\mathrm{D}_s\, c_{1,n} \sin_p^p(x) \cdot \cos_p^{1-p}(x) = p \cdot c_{1,n} \cdot \sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) \tag{4.51}$$

has the same exponent $q = p-1$ as (4.50) has. It remains to prove that $p \cdot c_{1,n} - c_{0,n} \le 0$. Using $(n-2)$-th derivative of $\sin_p(x)$ we obtain (4.50),

$$\begin{aligned}(\mathrm{D}_c \circ \mathrm{D}_s \circ \mathrm{D}_s) c_{0,n-2} \cdot \sin_p^2(x) \cos_p^{-1}(x) &= (\mathrm{D}_c \circ \mathrm{D}_s) 2 \cdot c_{0,n-2} \cdot \sin_p(x) \\ &= \mathrm{D}_c\, 2 \cdot c_{0,n-2} \cdot \cos_p(x) \\ &= -2 \cdot c_{0,n-2} \cdot \sin_p^{p-1}(x) \cos_p^{2-p}(x)\end{aligned} \tag{4.52}$$

and (4.51),

$$\begin{aligned}&(\mathrm{D}_s \circ \mathrm{D}_s \circ \mathrm{D}_c) c_{0,n-2} \cdot \sin_p^2(x) \cos_p^{-1}(x) \\ &= (\mathrm{D}_s \circ \mathrm{D}_s) c_{0,n-2} \cdot \sin_p^{1+p}(x) \cos_p^{-p}(x) \\ &= \mathrm{D}_s (1+p) \cdot c_{0,n-2} \cdot \sin_p^p(x) \cdot \cos_p^{1-p}(x) \\ &= p \cdot (1+p) \cdot c_{0,n-2} \cdot \sin_p^{p-1}(x) \cdot \cos_p^{2-p}(x) .\end{aligned} \tag{4.53}$$

Comparing (4.52) with (4.50), we find that

$$-c_{0,n} = -2 \cdot c_{0,n-2} .$$

In addition, comparing (4.51) and (4.53), we find

$$p \cdot c_{1,n} = p \cdot (p+1) \cdot c_{0,n-2} .$$

From the induction assumption, $c_{0,n-2} \le 0$ and for $p \ge 3$, we easily find

$$p \cdot c_{1,n} - c_{0,n} = (p \cdot (p+1) - 2) c_{0,n-2} \le 0$$

by adding the previous two identities.

In the definition (4.39) of $c_{i,n}$, we are adding coefficients

$$a'_{k,n} , \quad k = 0, 1, \dots, 2(i_0 + 1)$$

corresponding to the same value of exponent $\overline{q}$. From the both cases, we obtain $c_{i,n+1} \le 0$ for all $i \in \mathbb{N}$, $i \le i_1$, $0 < i_1 \le 2^{n-1} - 1$. This concludes the proof by induction. $\square$

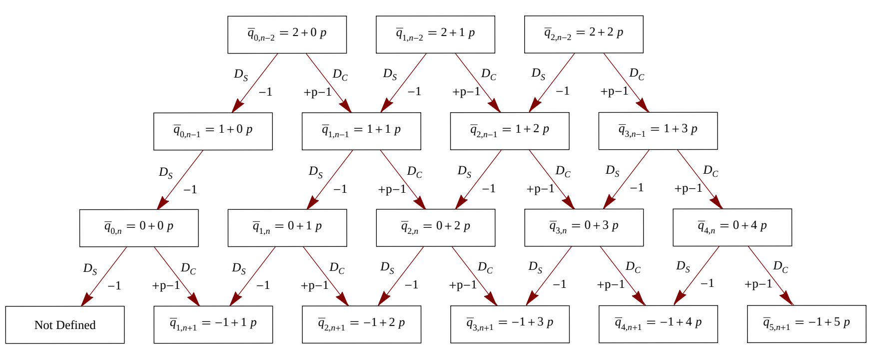

FIGURE 4. Rewriting diagram - starting with $\overline{q}_{0,n-2}, \overline{q}_{1,n-2}, \overline{q}_{2,n-2}$

5. Proofs of main results

Proof of Theorem 3.1. By Lemma 4.5 and substitution (4.14), we can write

$$\sin_{2(m+1)}^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_{2(m+1)}^{q_{k,n}}(x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(x)\,,$$

where

$$q_{k,n} = (2(m+1)-1)\cdot j(k) + (n - j(k) - 2) + 2(m+1) - 1\,,$$

and $j(k)$ has the same meaning as in Lemma 4.5. Thus $a_{k,n} \in \mathbb{Z}$.

From Lemma 4.1, statement 4 and 5, we also know that $\sin_{2(m+1)}^{(n)}(x)$ is even function for n odd and $\sin_{2(m+1)}^{(n)}(x)$ is odd function for n even. It follows that for $x \in (-\frac{\pi_{2(m+1)}}{2}, 0)$

$$\sin_{2(m+1)}^{(n)}(x) = \begin{cases} -\sin_{2(m+1)}^{(n)}(-x) & \text{for } n \text{ even}\,, \\ \sin_{2(m+1)}^{(n)}(-x) & \text{for } n \text{ odd}\,. \end{cases} \tag{5.1}$$

Now we assume $p = 2(m+1)$, $m \in \mathbb{N}$, and

$$\begin{aligned} q_{k,n} &= (2(m+1)-1)j(k) + (n - j(k) - 2) + 2(m+1) - 1 \\ &= (2(m+1)-1)(j(k)+1) + j(k) + 2 - n \\ &= 2(m+1)(j(k)+1) - n + 1 \end{aligned}$$

which implies $q_{k,n}$ is *odd* for n even. Thus we obtain

$$\begin{aligned} -\sin_{2(m+1)}^{(n)}(-x) &= -\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(-x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(-x) \\ &= \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(x)\,. \end{aligned} \tag{5.2}$$

Analogously, $q_{k,n}$ is *even* for n odd and

$$\begin{aligned} \sin_{2(m+1)}^{(n)}(-x) &= \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(-x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(-x) \\ &= \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(x)\,. \end{aligned} \tag{5.3}$$

Hence from (5.2), (5.3), we obtain

$$\sin_{2(m+1)}^{(n)}(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_{2(m+1)}^{q_{k,n}}(x) \cdot \cos_{2(m+1)}^{1-q_{k,n}}(x) \tag{5.4}$$

for all $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2}) \setminus \{0\}$.

Now, we prove the continuity of $\sin_{2(m+1)}^{(n)}(x)$ for all $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$ by induction in n.

Step 1. For $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$ the function

$$v(x) = \varphi_{2(m+1)}(\cos_{2(m+1)}(x)) > 0$$

and so we can take the first equation in (2.3) into its derivative and obtain

$$u''(x) = \varphi'_{p'}(v(x))v'(x)\,, \text{ where } p' = \frac{2m+1}{2m}\,.$$

Since v' is continuous and $\varphi_{p'} \in C^1(0,+\infty)$ ($\varphi_{p'}(z) = z^{p-1}$ for $z > 0$), we obtain continuity of $\sin^{(n)}_{2(m+1)}(x)$ for $n = 2$.

Step 2. Let us assume that $\sin^{(n)}_{2(m+1)}(x)$ is continuous on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$. From Lemma 4.5 we know that $\sin^{(n+1)}_{2(m+1)}(x)$ is continuous on $(0, \frac{\pi_{2(m+1)}}{2})$. Now we distinguish two cases: $n+1$ is odd then $\sin^{(n+1)}_{2(m+1)}(x)$ is even by Lemma 4.1, statement 4, and $n+1$ is even then $\sin^{(n+1)}_{2(m+1)}(x)$ is odd by Lemma 4.1, statement 5. In both cases, $\sin^{(n+1)}_{2(m+1)}(x) \in C(0, \frac{\pi_{2(m+1)}}{2})$ implies $\sin^{(n+1)}_{2(m+1)}(x) \in C(-\frac{\pi_{2(m+1)}}{2}, 0)$. It remains to prove the continuity at $x = 0$. From (5.4) we know that

$$\lim_{x\to 0-} \sin^{(n+1)}_{2(m+1)}(x) = \lim_{x\to 0+} \sin^{(n+1)}_{2(m+1)}(x)\,. \tag{5.5}$$

At the end we compute the derivative of $\sin^{(n)}_{2(m+1)}(0)$ from its definition:

$$\sin^{(n+1)}_{2(m+1)}(0) = \lim_{h\to 0} \frac{\sin^{(n)}_{2(m+1)}(h) - \sin^{(n)}_{2(m+1)}(0)}{h}\,.$$

It is a limit of the type "0/0". Since the limit $\lim_{h\to 0} \sin^{(n+1)}_{2(m+1)}(h)$ exists, we obtain $\sin^{(n+1)}_{2(m+1)}(0) = \lim_{h\to 0} \sin^{(n+1)}_{2(m+1)}(h)$ by L'Hôspital's rule. Note that by Lemma 4.6, $q_{k,n} \geq 0$ for all $n \in \mathbb{N}$, $n \geq 2$, and all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$, these limits are finite and we obtain continuity. This proves the continuity of $\sin^{(n+1)}_{2(m+1)}(x)$ for all $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$. $\square$

Proof of Theorem 3.2. By Lemma 4.5 and substitution (4.14), we have

$$\sin^{(n)}_p(x) = \sum_{k=0}^{2^{n-2}-1} a_{k,n} \cdot \sin_p^{q_{k,n}}(x) \cdot \cos_p^{1-q_{k,n}}(x) \quad \text{on } (0, \frac{\pi_p}{2})\,.$$

Moreover, by Lemma 4.1, statement 4 and 5, we obtain

$$\sin^{(n)}_p(x) = \begin{cases} -\sin^{(n)}_p(-x) & \text{for } n \text{ even}\,, \\ \sin^{(n)}_p(-x) & \text{for } n \text{ odd}\,, \end{cases} \tag{5.6}$$

for $x \in (-\pi_p/2, 0)$. Since $\sin^{(n)}_p(x)$ is continuous for $x \in (0, \pi_p/2)$, it is also continuous on $x \in (-\pi_p/2, 0)$ by (5.6). Thanks to (5.6) it is enough to study the behavior of $\sin_p(x)$ in the right neighborhood of 0. From Lemma 4.5, we have that

$$q_{k,n} = j(k)\cdot(p-1) + (-1)\cdot(n-2-j) + p - 1 = p\cdot(j(k)+1) + 1 - n\,. \tag{5.7}$$

for all $n \in \mathbb{N}$, $n \geq 2$ and all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$. Since $j(k) \in \{0\} \cup \mathbb{N}$ we find that

$$q_{k,n} \geq p + 1 - n\,.$$

Then, for $n < p + 1$, we have $q_{k,n} > 0$ for all $k \in \{0\} \cup \mathbb{N}$, $k \leq 2^{n-2} - 1$. And so using the theorem of the algebra of the limits from any classical analysis textbook, we find that

$$\lim_{x\to 0+} \sin^{(n)}_p(x) = 0\,.$$

From (5.6),

$$\lim_{x\to 0-}\sin_p^{(n)}(x)=\begin{cases}-\lim_{x\to 0+}\sin_p^{(n)}(x)=0 & \text{for } n \text{ even}\,,\\ \lim_{x\to 0+}\sin_p^{(n)}(x)=0 & \text{for } n \text{ odd}\,.\end{cases}$$

The continuity at $x=0$ follows from L'Hôspital's rule used recurrently from $n=2$ to $n=\lceil p\rceil$.

By Lemma 4.5, $\sin_{2(m+1)}^{(2m+2)}(x)$ satisfies

$$\sin_p^{(\lceil p\rceil+1)}(x)=\sum_{k=0}^{2^{\lceil p\rceil-1}-1} D_{k,\lceil p\rceil+1}\sin_p''(x)\quad \text{on } (0,\frac{\pi_p}{2})\,.$$

Since $q_{k,n}>0$ for all $n<\lceil p\rceil$ and all $k\in\{0\}\cup\mathbb{N}$, $k<2^{\lceil p\rceil}-1$, the function $\mathrm{D}_S\, a_{k,n}\cdot\sin_p^{q_{k,n}}(x)\cdot\cos_p^{1-q_{k,n}}(x)$ does not vanish identically. Thus $a_{0,\lceil p\rceil+1}\neq 0$. Since $a_{0,\lceil p\rceil+1}\neq 0$, we can apply (5.7) for $j(0)=0$ which gives

$$q_{0,\lceil p\rceil+1}=p-\lceil p\rceil\le 0\,.$$

From the fact that $j(k)>j(0)$ for all $k\in\{0\}\cup\mathbb{N}$, $k\le 2^{\lceil p\rceil-1}-1$ and from (5.7) we know that

$$q_{k,\lceil p\rceil+1}>q_{0,\lceil p\rceil+1}\,.$$

Moreover from (5.7),

$$q_{k,\lceil p\rceil+1}=(j(k)+1)\cdot p+1-(\lceil p\rceil+1)=(j(k)+1)\cdot p-\lceil p\rceil>0$$

for $j(k)\ge 1$ and $p>1$. Since, for all $q_{k,n}>0$,

$$\lim_{x\to 0} a_{k,n}\cdot\sin_p^{q_{k,n}}(x)\cdot\cos_p^{1-q_{k,n}}(x)=0\,,$$

we obtain

$$\begin{aligned}\lim_{x\to 0+}\sin_p^{(\lceil p\rceil+1)}(x)&=\lim_{x\to 0+} a_{0,\lceil p\rceil+1}\cdot\sin_p^{p-\lceil p\rceil}(x)\cdot\cos_p^{1-p+\lceil p\rceil}(x)\\ &\quad+\sum_{k=1}^{2^{\lceil p\rceil-1}-1} a_{k,\lceil p\rceil+1}\cdot\sin_p^{q_{k,\lceil p\rceil+1}}(x)\cdot\cos_p^{1-q_{k,\lceil p\rceil+1}}(x)\\ &=\lim_{x\to 0+} a_{0,\lceil p\rceil+1}\cdot\sin_p^{p-\lceil p\rceil}(x)\cdot\cos_p^{1-p+\lceil p\rceil}(x)\end{aligned}\tag{5.8}$$

by the theorem of the algebra of the limits.

Now the proof falls into two cases, $p=2m+1$ and $p\in\mathbb{R}\setminus\mathbb{N}$, $p>1$.

Case 1. For $p=2m+1$, we have by (5.8)

$$\lim_{x\to 0+}\sin_{2m+1}^{(2m+2)}(x)=\lim_{x\to 0+} a_{0,2m+2}\cdot\cos_p(x)=a_{0,2m+2}\neq 0\,.$$

Since $2m+2$ is even, $\sin_{2m+1}^{(2m+2)}(x)$ is odd function by Lemma 4.1, statement 5. Thus

$$\lim_{x\to 0-}\sin_{2m+1}^{(2m+2)}(x)=-a_{0,2m+2}\,.$$

Hence $\sin_{2m+1}^{(2m+2)}(x)$ is not continuous at $x=0$.

Case 2. Since for $p\in\mathbb{R}\setminus\mathbb{N}$, $p>1$, we have

$$\lim_{x\to 0+}\sin_p^{(\lceil p\rceil+1)}(x)=\lim_{x\to 0+} a_{0,\lceil p\rceil+1}\cdot\sin_p^{p-\lceil p\rceil}(x)\cdot\cos_p^{1-p+\lceil p\rceil}(x)=+\infty$$

from (5.8). Hence $\sin_p^{\lceil p\rceil+1}(x)$ is discontinuous at $x=0$. This concludes the proof. $\square$

Proof of Theorem 3.3. It follows from [24, Thm. 1.1, consider $p = q$ and $\sigma = 0$] that there exists a unique analytic function $F(z)$ near origin such that the unique solution $u(x) = \sin_p(x)$ of the initial value problem (2.1); i.e.,

$$-(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0$$
$$u(0) = 0, \quad u'(0) = 1\,,$$

takes the form $\sin_p(x) = u(x) = x \cdot F(|x|^p)$. Note that for $p = 2(m+1)$ and $m \in \mathbb{N}$,

$$\sin_p(x) = x \cdot F(|x|^p) = x \cdot F(x^p) = \sum_{l=0}^{+\infty} \alpha_l \cdot x^{l\cdot p+1}, \quad \text{where } F(z) = \sum_{l=0}^{+\infty} \alpha_l z^l\,,$$

which is also an analytic function in a neighborhood of $x = 0$. In the sequel of this proof $p = 2(m+1)$, $m \in \mathbb{N}$. By the uniqueness of the Maclaurin series of analytic function, we see that

$$\sum_{l=0}^{+\infty} \alpha_l \cdot x^{l\cdot p+1} = \sum_{l=0}^{+\infty} \frac{\sin_p^{(l\cdot p+1)}(0)}{(l\cdot p+1)!} \cdot x^{l\cdot p+1}\,,$$

where the right-hand side also converges to $\sin_p(x)$ on some neigbourhood of $x = 0$. Note that $\sin_p^{(k)}(0) = 0$ for any $k \in \mathbb{N}$ such that

$$\forall l \in \{0\} \cup \mathbb{N} : k \neq l \cdot p + 1$$

as it follows from Lemma 4.5 and Lemma 4.6.

Since the restriction of $\sin_p(x)$ to $[-\frac{\pi_p}{2}, \frac{\pi_p}{2}]$ is the inverse function of $\operatorname{arcsin}_p(x)$, by the identity (1.6); i.e.,

$$\forall x \in [-1, 1] : \sin_p(\operatorname{arcsin}_p(x)) = x\,.$$

It is well known see, e.g., [13] that

$$\begin{aligned}\operatorname{arcsin}_p(x) &= \int_0^x (1 - s^p)^{-\frac{1}{p}} \,\mathrm{d}s \\ &= \frac{s \ \cdot \ {}_2F_1(1, \frac{1}{p}; 1 + \frac{1}{p}; s^p)}{p} \\ &= \sum_{n=0}^{+\infty} \frac{\Gamma(n + \frac{1}{p})}{\Gamma(\frac{1}{p})(n \cdot p + 1)} \cdot \frac{1}{n!} \cdot x^{n\cdot p+1}\end{aligned} \tag{5.9}$$

for $x \in (0, 1)$. Observe that for our special case $p = 2(m + 1)$ with $m \in \mathbb{N}$, this formula is valid on $[-1, 1]$. Note also that in our special case, (5.9) is in fact the Maclaurin series for $\operatorname{arcsin}_p(x)$ and, moreover, all coefficients are nonnegative (the explicitly written coefficients are positive, the other ones are zero).

To apply the formula for composite formal power series, we need to consider series for $\sin_p(x)$ and $\operatorname{arcsin}_p(x)$ including the zero terms. For this reason, we define for all $j \in \mathbb{N}$

$$\alpha'_j := \sin_p^{(j)}(0)/j! = \begin{cases} \alpha_i & \text{if } j = ip + 1 \text{ for some } i \in \{0\} \cup \mathbb{N}\,, \\ 0 & \text{otherwise} \end{cases} \tag{5.10}$$

and

$$\beta'_j := \begin{cases} \frac{\Gamma(n+\frac{1}{p})}{\Gamma(\frac{1}{p})(n\cdot p+1)} \cdot \frac{1}{n!} & \text{if } j = ip + 1 \text{ for some } i \in \{0\} \cup \mathbb{N}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{5.11}$$

Thus by well-known composite formal power series formula

$$\sin_p(\arcsin_p(x)) = \sum_{n=1}^{+\infty} c_n x^n \,, \tag{5.12}$$

where

$$c_n = \sum_{\substack{k \in \mathbb{N},\ j_1, j_2, \dots, j_k \in \mathbb{N} \\ j_1 + j_2 + \dots + j_k = n}} \alpha'_k \cdot \beta'_{j_1} \cdot \beta'_{j_2} \cdot \dots \cdot \beta'_{j_k} \,. \tag{5.13}$$

Since both functions $\sin_p(x)$ and $\arcsin_p(x)$ are analytic in some neighborhood of $x = 0$, the series from (5.12) with coefficients given by (5.13) is convergent towards the identity $x \mapsto x$ on some neighborhood of $x = 0$. From this fact, we infer that $c_1 = 1$ and $c_n = 0$ for all $n \in \mathbb{N}, n \geq 2$. Thus for any $x \in \mathbb{R}$

$$x = \sum_{n=1}^{+\infty} x^n \sum_{\substack{k \in \mathbb{N},\ j_1, j_2, \dots, j_k \in \mathbb{N} \\ j_1 + j_2 + \dots + j_k = n}} \alpha'_k \cdot \beta'_{j_1} \cdot \beta'_{j_2} \cdot \dots \cdot \beta'_{j_k} \tag{5.14}$$

and in particular

$$1 = \sum_{n=1}^{+\infty} \sum_{\substack{k \in \mathbb{N},\ j_1, j_2, \dots, j_k \in \mathbb{N} \\ j_1 + j_2 + \dots + j_k = n}} \alpha'_k \cdot \beta'_{j_1} \cdot \beta'_{j_2} \cdot \dots \cdot \beta'_{j_k} \,. \tag{5.15}$$

Now we show that also

$$\sum_{n=1}^{+\infty} \sum_{\substack{k \in \mathbb{N},\ j_1, j_2, \dots, j_k \in \mathbb{N} \\ j_1 + j_2 + \dots + j_k = n}} |\alpha'_k \cdot \beta'_{j_1} \cdot \beta'_{j_2} \cdot \dots \cdot \beta'_{j_k}| \tag{5.16}$$

is convergent. By Lemma 4.7 and (5.10) we see that $\alpha'_j \leq 0$ for all $j \in \mathbb{N}, j \geq 2$ and $\alpha'_1 = \cos_p(0) = 1$. Moreover, from (5.11) it follows that $\beta'_j \geq 0$ for all $j \in \mathbb{N}$. Thus the product $\alpha'_k \cdot \beta'_{j_1} \cdot \beta'_{j_2} \cdot \dots \cdot \beta'_{j_k}$ is positive if and only if $k = 1$. All positive terms can be written as $\alpha'_1 \cdot \beta'_n = \beta'_n$ for $n \in \mathbb{N}$ (if $k = 1$ then $j_1 = n$ is the only decomposition of n). Since the sum of all positive terms in (5.15) is $\sum_{n=1}^{+\infty} \beta'_n = \arcsin_p(1) = \frac{\pi_p}{2} < +\infty$, the sum of all negative terms must be finite too and equals $1 - \frac{\pi_p}{2}$. Thus (5.16) converges. This means that the series (5.15) converges absolutely to 1 and any rearrangement of this series must converge. Also any subseries of any rearrangement of this series must converge absolutely. Let $s_M = \sum_{m=1}^{M} \beta'_m$. Then the series $\sum_{k=1}^{+\infty} \alpha'_k \cdot (s_M)^k$ is a subseries of one of the rearrangements of (5.15) and it is convergent. Observe that s_M is nondecreasing and converging to $\sum_{m=1}^{+\infty} \beta'_m = \pi_p/2$ as $M \to +\infty$. Thus the Maclaurin series for $\sin_p(x) = \sum_{k=1}^{+\infty} \alpha'_k \cdot x^k$ is convergent for any $x \in (-\pi_p/2, \pi_p/2)$ to some analytic function.

Now it remains to show that it converges towards $\sin_p(x)$ on $(-\pi_p/2, \pi_p/2)$. This last step follows from the formal identity (5.14), which on the established range of convergence holds also analytically and the fact that the function $\sin_p(x)$ is the only function that satisfies the identity (1.6). $\square$

Proof of Theorem 3.4. From [24, Thm. 1.1, consider $p = q$ and $\sigma = 0$] it follows that, for any $p > 1$, there exists a unique analytic function $F(z)$ near origin such that $\sin_p(x) = x \cdot F(|x|^p)$; thus we have

$$\sin_p(x) = x \cdot F(|x|^p) = \sum_{l=0}^{+\infty} \alpha_l \cdot x \cdot |x|^{l \cdot p} \quad , \text{where } F(z) = \sum_{l=0}^{+\infty} \alpha_l \cdot z^l \,.$$

Note that for $p = 2m + 1$, $m \in \mathbb{N}$, the series

$$\sum_{l=0}^{+\infty} \alpha_l \cdot x^{l \cdot p+1} \tag{5.17}$$

defines an analytic function $G(x)$ in a neighborhood of $x = 0$ and also that

$$\sin_p(x) = \sum_{l=0}^{+\infty} \alpha_l \cdot x^{l \cdot p+1} = G(x) \quad \text{for } x > 0 \tag{5.18}$$

on a neighborhood of 0. Our aim is to show that the radius of convergence of (5.17) is $\pi_p/2$ for $p = 2m + 1$, $m \in \mathbb{N}$. By (5.18), the following derivatives are equal

$$\sin_p^{(n)}(x) = G^{(n)}(x) = \sum_{l=\lceil \frac{n-1}{p} \rceil}^{+\infty} \alpha_l \cdot \frac{(l \cdot p + 1)!}{(l \cdot p + 1 - n)!} x^{l \cdot p - n + 1}$$

for $x > 0$ on the neighborhood of 0 where the series converges. Now take a one-sided limit from the right in the previous equation

$$\lim_{x \to 0+} \sin_p^{(n)}(x) = \lim_{x \to 0+} \sum_{l=\lceil \frac{n-1}{p} \rceil}^{+\infty} \alpha_l \cdot \frac{(l \cdot p + 1)!}{(l \cdot p + 1 - n)!} x^{l \cdot p - n + 1}.$$

For $j := \frac{n-1}{p} \in \{0\} \cup \mathbb{N}$, we obtain

$$\lim_{x \to 0+} \sum_{l=j}^{+\infty} \alpha_l \cdot \frac{(l \cdot p + 1)!}{(l \cdot p + 1 - n)!} x^{l \cdot p - n + 1} = \alpha_j \cdot \frac{(j \cdot p + 1)!}{(j \cdot p + 1 - n)!} \,.$$

Thus

$$\lim_{x \to 0+} \sin_p^{(n)}(x) = \alpha_j \cdot \frac{(j \cdot p + 1)!}{(j \cdot p + 1 - n)!}$$

for $j \in \{0\} \cup \mathbb{N}$. By Lemma 4.7, $\lim_{x \to 0+} \sin_p^{(n)}(x) \le 0$ for $n \ge 2$, $p \in \mathbb{N}$ and $p \ge 3$. Thus $\alpha_j \le 0$ for $j \in \mathbb{N}, j > 1$.

The rest of the proof of the theorem is identical to the proof of Theorem 3.3 and we find that the convergence radius of the series (5.17) is $\frac{\pi_p}{2}$ for $p = 2m + 1$, $m \in \mathbb{N}$. The only difference against the proof of Theorem 3.3 is that the series (5.17) converges towards $\sin_p(x)$ only on $(0, \pi_p/2)$ for $p = 2m + 1$, $m \in \mathbb{N}$. Note that the series is still convergent on $(-\pi_p/2, 0)$ towards $G(x) \neq \sin_p(x)$ for $x < 0$. The changes in the proof are obvious and are left to the reader. $\square$

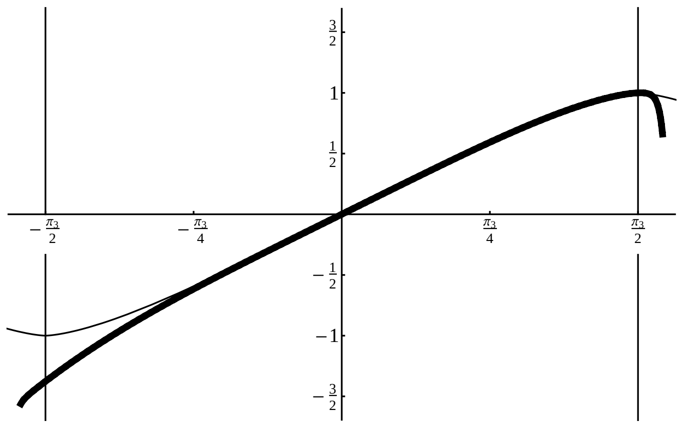

FIGURE 5. Graph of $\sin_3(x)$ obtained by high-precision numerical integration of (1.3) (thin line) versus graph of partial sum of the Maclaurin series for $\sin_3(x)$ up to the power x^{100} (thick line). Notice that the Maclaurin series does not converge to $\sin_3(x)$ for $x < 0$ and $x > \frac{\pi_3}{2}$

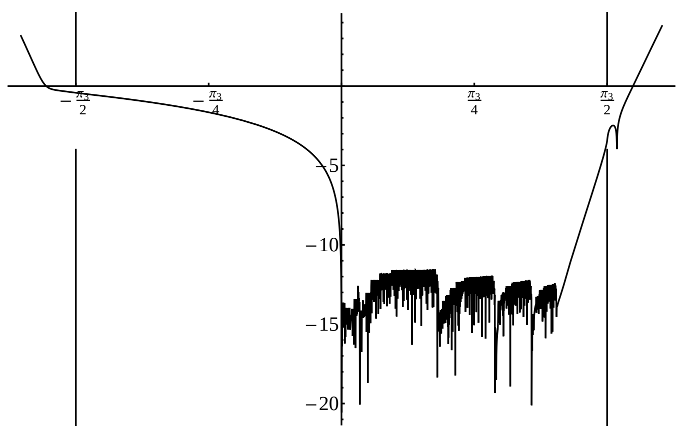

FIGURE 6. Graph of the function $\log_{10} |\sin_3(x) - \sum_{n=1}^{100} \alpha'_n x^n|$ where $\sum_{n=1}^{100} \alpha'_n x^n$ is the partial sum of the Maclaurin series of $\sin_3(x)$. The values of $\sin_3(x)$ were obtained by high-precision numerical integration of (1.3) using Mathematica command `NDSolve` with option `WorkingPrecision->50` which sets internal computations to be done up to 50-digit decadic precision. Notice that the Maclaurin series does not converge to $\sin_3(x)$ for $x < 0$ and $x > \pi_3/2$

6. Concluding remarks and open problems

As it was mentioned in the proofs of Theorems 3.3 and 3.4, it follows from [24, Thm. 1.1, consider $p = q$ and $\sigma = 0$] that, for any $p > 1$, there exists a unique

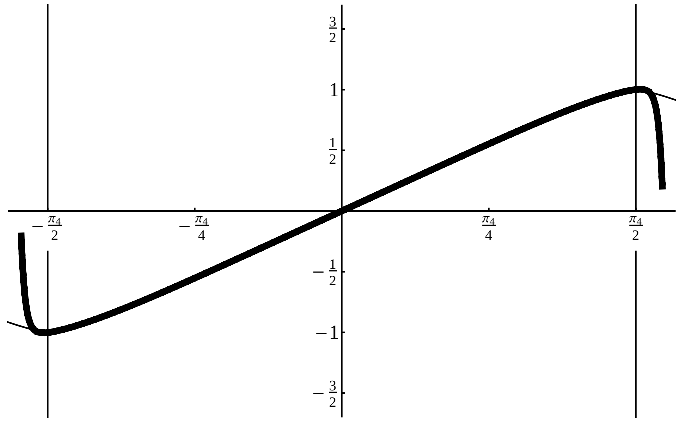

FIGURE 7. Graph of $\sin_4(x)$ obtained by high-precision numerical integration of (1.3) (thin line) versus graph of partial sum of the Maclaurin series for $\sin_4(x)$ up to the power x^{100} (thick line)

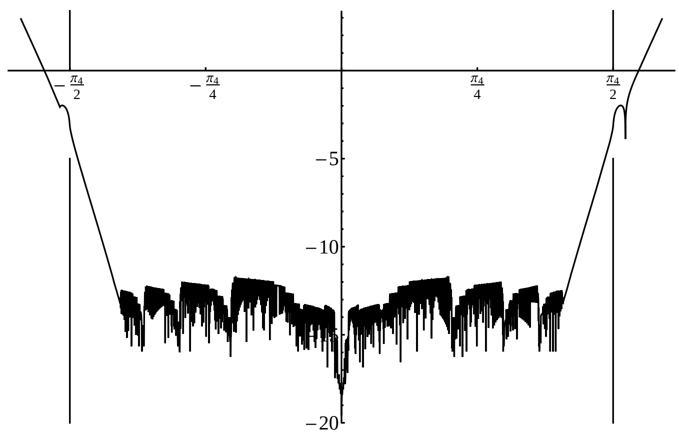

FIGURE 8. Graph of the function $\log_{10}|\sin_4(x) - \sum_{n=1}^{100} \alpha'_n x^n|$ where $\sum_{n=1}^{100} \alpha'_n x^n$ is the partial sum of the Maclaurin series of $\sin_4(x)$. The values of $\sin_4(x)$ were obtained by high-precision numerical integration of (1.3) using Mathematica command `NDSolve` with option `WorkingPrecision->50` which sets internal computations to be done up to 50-digit decadic precision. Notice that the Maclaurin series does not converge to $\sin_4(x)$ for $|x| > \pi_4/2$

analytic function $F(z)$ near origin such that

$$\sin_p(x) = x \cdot F(|x|^p)\,.$$

Thus the function $\sin_p(x)$ can be expanded into generalized Maclaurin series near the origin:

$$\sin_p(x) = x \cdot F(|x|^p) = \sum_{l=0}^{+\infty} \alpha_l \cdot x \cdot |x|^{l\cdot p}, \quad \text{where } F(z) = \sum_{l=0}^{+\infty} \alpha_l \cdot z^l \,.$$

Remark 6.1. (Convergence of generalized Maclaurin series) Let $p = 2m+1$ for $m \in \mathbb{N}$. It follows from the symmetry of the function $\sin_{2m+1}(x)$ with respect to the origin and from the proof of Theorem 3.4 that the generalized Maclaurin series $\sum_{l=0}^{+\infty} \alpha_l \cdot x \cdot |x|^{l\cdot(2m+1)}$ converges towards the values of $\sin_{2m+1}(x)$ on $(-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$.

Remark 6.2 (Complex argument for p even)**.** Let $p = 2(m+1)$ for $m \in \mathbb{N}$. It follows from the proof of Theorem 3.3 that the Maclaurin series $\sum_{l=0}^{+\infty} \alpha_l \cdot x^{l\cdot 2(m+1)+1}$ converges towards the values of $\sin_{2(m+1)}(x)$ on $(-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$ absolutely. This enables us to extend the range of definition of the function $\sin_{2(m+1)}(x)$ to the complex open disc

$$B_m = \{z \in \mathbb{C} : |z| < \frac{\pi_{2(m+1)}}{2}\}$$

by setting $\sin_{2(m+1)}(z) := \sum_{l=0}^{+\infty} \alpha_l \cdot z^{l\cdot 2(m+1)+1}$. Since all the powers of z are of positive-integer order $l \cdot 2(m+1)+1$, the function $\sin_{2(m+1)}(z)$ is an analytic complex function on B_m and thus is single-valued. Unfortunately, this easy approach works only for $p = 2(m+1)$ with $m \in \mathbb{N}$; cf [15].

Our methods for proving convergence of the Maclaurin or generalized Maclaurin series are based on the fact that p is an integer. Thus a natural question appears.

Open Problem 6.3 (Convergence for $p > 1$ not integer)**.** Consider $p > 1$, $p \notin \mathbb{N}$. Prove (or find a counterexample) that the generalized Maclaurin series corresponding to $\sin_p(x)$ 'suggests the convergence' on $(-\pi_p/2, \pi_p/2)$ towards the values of $\sin_p(x)$.

For the sake of completeness, we remark that [15] claims the convergence of the generalized Maclaurin series on $(-\pi_p/2, \pi_p/2)$ for any $p > 1$, but there is no proof nor any indication for the proof of this claim.

Moreover, we are not able to decide about the convergence at the endpoints. This is another open question.

Open Problem 6.4 (Endpoints of the interval)**.** Consider $p > 1$. Prove (or find a counterexample) that the generalized Maclaurin series of $\sin_p(x)$ converge at $-\frac{\pi_p}{2}$ and/or $\frac{\pi_p}{2}$.

Remark 6.5 (Function $\cos_p$ for p even)**.** Let $p = 2(m+1)$ for $m \in \mathbb{N}$. Since $\cos_p(x) = \sin_p'(x)$ by definition, the Maclaurin series for $\cos_{2(m+1)}(x)$ can be obtained by taking into derivative the Maclaurin series for $\sin_{2(m+1)}(x)$ term by term. The Maclaurin series for $\cos_{2(m+1)}(x)$ then converges towards the value $\cos_{2(m+1)}(x)$ for any $x \in (-\frac{\pi_{2(m+1)}}{2}, \frac{\pi_{2(m+1)}}{2})$.

Remark 6.6 (Function $\cos_p$ for p odd)**.** Let $p = 2m+1$ for $m \in \mathbb{N}$. In this case the Maclaurin series for $\cos_{2m+1}(x)$ can also be obtained by taking into derivative the Maclaurin series for $\sin_{2m+1}(x)$ term by term. This Maclaurin series then converges for $x \in (-\frac{\pi_{2m+1}}{2}, \frac{\pi_{2m+1}}{2})$. However, the Maclaurin series for $\cos_{2m+1}(x)$ converges towards the value $\cos_{2m+1}(x)$ for $x \in [0, \frac{\pi_{2m+1}}{2})$, but it does not converge towards the value $\cos_{2m+1}(x)$ for any $x \in (-\frac{\pi_{2m+1}}{2}, 0)$.

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Appendix A6

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GENERALIZED TRIGONOMETRIC FUNCTIONS IN COMPLEX DOMAIN

Petr Girg, Lukáš Kotrla, Plzeň

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Cordially dedicated to Professor Pavel Drábek on the occasion of his sixtieth birthday

Abstract. We study extension of p-trigonometric functions $\sin_p$ and $\cos_p$ to complex domain. For $p = 4, 6, 8, \ldots$, the function $\sin_p$ satisfies the initial value problem which is equivalent to

$$(*) \qquad -(u')^{p-2}u'' - u^{p-1} = 0, \quad u(0) = 0, \quad u'(0) = 1$$

in $\mathbb{R}$. In our recent paper, Girg, Kotrla (2014), we showed that $\sin_p(x)$ is a real analytic function for $p = 4, 6, 8, \ldots$ on $(-\pi_p/2, \pi_p/2)$, where $\pi_p/2 = \int_0^1 (1 - s^p)^{-1/p}$. This allows us to extend $\sin_p$ to complex domain by its Maclaurin series convergent on the disc $\{z \in \mathbb{C}\colon |z| < \pi_p/2\}$. The question is whether this extensions $\sin_p(z)$ satisfies $(*)$ in the sense of differential equations in complex domain. This interesting question was posed by Došlý and we show that the answer is affirmative. We also discuss the difficulties concerning the extension of $\sin_p$ to complex domain for $p = 3, 5, 7, \ldots$ Moreover, we show that the structure of the complex valued initial value problem $(*)$ does not allow entire solutions for any $p \in \mathbb{N}$, $p > 2$. Finally, we provide some graphs of real and imaginary parts of $\sin_p(z)$ and suggest some new conjectures.

Keywords: p-Laplacian; differential equations in complex domain; extension of $\sin_p$

MSC 2010: 33E30, 34B15, 34M05, 34M99

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1. Introduction

The initial value problem

$$-(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0, \quad u(0) = 0, \quad u'(0) = 1 \tag{1.1}$$

arises in connection with nonlinear boundary value problems for $p > 1$ (see e.g. [2], [3], [7], [9]). The solution of (1.1) is known as $\sin_p$, see e.g. [2], and $\cos_p \overset{\text{def}}{=} \sin_p'$. Since the functions $\sin_p$ and $\cos_p$ satisfy the well-known p-trigonometric identity, see e.g. [3],

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1, \tag{1.2}$$

they are also known as the p-trigonometric and/or generalized trigonometric functions. Note that (1.2) is in fact the so-called first integral of (1.1) (see e.g. [3]). It follows from this identity (see e.g. [3]) that

$$\int_0^{\sin_p(x)} (1 - s^p)^{-1/p}\,\mathrm{d}s = x$$

for $0 \leqslant x \leqslant \pi_p/2$, where $\sin_p(x) \geqslant 0$ and $\cos_p(x) \geqslant 0$ and where

$$\pi_p \overset{\text{def}}{=} 2\int_0^1 (1 - s^p)^{-1/p}\,\mathrm{d}s.$$

Thus it is natural to define

$$\operatorname{arcsin}_p(x) \overset{\text{def}}{=} \int_0^x (1 - s^p)^{-1/p}\,\mathrm{d}s \quad \text{for } 0 \leqslant x \leqslant 1, \tag{1.3}$$

and extend it to $[-1, 1]$ as an odd function. The function $\sin_p$ is the inverse function to $\operatorname{arcsin}_p(x)$ on $[-\pi_p/2, \pi_p/2]$. Moreover, $\sin_p(x) = \sin_p(\pi_p - x)$ for $x \in (\pi_p/2, \pi_p]$ and $\sin_p(x) = -\sin_p(-x)$ for $x \in [-\pi_p, 0]$. Finally, $\sin_p(x) = \sin_p(x + 2\pi_p)$ for all $x \in \mathbb{R}$ (see [3] for details).

Smoothness of $\sin_p$ on $(-\pi_p/2, \pi_p/2)$ for $p > 1$ was studied in [4]. The most surprising result of [4] is that $\sin_p$ is a real analytic function on $(-\pi_p/2, \pi_p/2)$ for $p = 4, 6, 8, \ldots$, i.e., $\sin_p(x)$ equals its Maclaurin on $(-\pi_p/2, \pi_p/2)$ for $p = 4, 6, 8, \ldots$ This approach naturally allows to extend $\sin_p$ for $p = 4, 6, 8, \ldots$ to an open disk

$$\{z \in \mathbb{C} \colon |z| < \pi_p/2\}$$

in the complex domain using power series (cf. [7], where the convergence of the series is conjectured without proof). When our recent result was presented at the conference "Nonlinear Analysis Plzeň 2013", O. Došlý posed an interesting question whether this extension satisfies (1.1) in the sense of differential equations in complex domain. This paper addresses his question. For $p = 4, 6, 8, \ldots$, the initial value problem (1.1) in $\mathbb{R}$ is equivalent to

$$-(u')^{p-2}u'' - u^{p-1} = 0, \quad u(0) = 0, \quad u'(0) = 1. \tag{1.4}$$

Note that for $p > 1$ real not being an even positive integer, we cannot get rid of the absolute values in (1.1). Thus the equation (1.1) does not make sense for general $p > 1$ in the complex domain. In this paper we consider the problem (1.4) in complex domain for integer $p > 2$. The complex valued ordinary differential equations are studied by means of power series (mostly Maclaurin series). Note that, by [4], Theorem 3.2 on page 5, $\sin_p^{(n)}(0)$ exists for $1 < n \leqslant p$, but $\sin_p^{(n)}(0)$ *does not exist* when $p \geqslant 3$ is an odd integer and $n > p$. Thus, by the formal Maclaurin series of $\sin_p(x)$, we mean a series calculated from the limits of the derivatives $\lim\limits_{x\to 0+} \sin_p^{(n)}(x)$, which were shown to exist in [4] for any $n \in \mathbb{N}$ and $p \geqslant 3$ an odd integer.

In Section 2, we prove that, for $p = 4, 6, 8, \ldots$, the function $\sin_p$ can be extended by its Maclaurin series to the disc $\{z \in \mathbb{C}\colon |z| < \pi_p/2\}$ and that this series solves the ordinary differential equation (1.4) in the sense of differential equations in the complex domain. On the other hand, in Section 3 we show that the complex valued formal Maclaurin series $M_{\sin_p}(z)$ of the real function $\sin_p(x)$ does not satisfy (1.4) in the sense of differential equations in the complex domain for odd powers $p = 3, 5, 7, \ldots$ In Section 4 we explain the relations between the real and imaginary components of the complex valued function $\sin_p(z)$ for $p = 2, 6, 10, \ldots$ and $p = 4, 8, 12, \ldots$, and also the complex valued formal Maclaurin series $M_{\sin_p}(z)$ of the real function $\sin_p(x)$ for $p = 3, 5, 7, \ldots$ In Section 5 we show that the fact that the function $\sin_p(z)$ cannot be extended as an entire function follows from an interesting connection between the p-trigonometric identity and some classical results from complex analysis. Finally, in Section 6 we visualize some of our result.

In the whole paper, the independent variable z stands for a complex number and the independent variable x stands for a real number. In the same spirit, $\sin_p(z)$ stands for a complex valued function and $\sin_p(x)$ stands for a function of one real variable.

2. Extension of $\sin_p$ for $p = 4, 6, 8, \ldots$ to complex domain

We assume that $p = 4, 6, 8, \ldots$ throughout this section unless specified differently. In [4], Theorem 3.3, we proved the following result.

Proposition 2.1 ([4], Theorem 3.3, page 6). *Let $p = 4, 6, 8, \ldots$ Then the Maclaurin series of $\sin_p(x)$ converges on $(-\pi_p/2, \pi_p/2)$.*

Let $M_{\sin_p}(x)$ denote the formal Maclaurin series of $\sin_p(x)$, $p = 3, 4, 5, 6, \ldots$ (any integer greater than 2). We also proved in the paper [4] that this Maclaurin series has the particular structure

$$M_{\sin_p}(x) = \sum_{k=0}^{\infty} \alpha_k x^{kp+1}, \tag{2.1}$$

where $\alpha_0 > 0$ and $\alpha_k \leqslant 0$.

The following result answers the question by O. Došlý in the affirmative way.

Theorem 2.1. *Let $p = 4, 6, 8, \ldots$, then the unique solution of the initial value problem* (1.4) *on $|z| < \pi_p/2$ is the Maclaurin series* (2.1).

In order to prove this result, we need to state several auxilliary results. First of all, let us note that the equation (1.4) is of second order. In order to apply the known theory, we rewrite (1.4) as an equivalent system. Using the substitution $u' = v$, we get the first order system

$$u' = v, \quad v' = -u^{p-1}/v^{p-2}, \quad u(0) = 0, \quad v(0) = 1. \tag{2.2}$$

To study systems of equations like (2.2) in complex domain, we need to use complex functions of several variables. We will often make use of the following result.

Proposition 2.2 ([6], Theorem 16, page 33). *Let f and g be holomorphic functions in an open set $M \subset \mathbb{C}^r$, $r \in \mathbb{N}$. Then the functions $f + g$, $f - g$ and fg are holomorphic in M. Moreover, if $g(z) \neq 0$ for all $z \in M$, then f/g is holomorphic on M.*

Let us consider the first order ODE system

$$\mathbf{y}' = \mathbf{f}(z, \mathbf{y}), \quad \mathbf{y}(z_0) = \mathbf{y}_0, \tag{2.3}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in \mathbb{C}^n$ and $\mathbf{f} = (f_1(z, \mathbf{y}), f_2(z, \mathbf{y}), \ldots, f_n(z, \mathbf{y}))^{\mathrm{T}} \in \mathbb{C}^n$ and the function $\mathbf{f}\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ is an analytic complex function of $n + 1$ complex variables. The folowing result concerning existence and uniqueness of the initial values problem in the complex domain is crucial in our proofs.

Proposition 2.3 ([5], Theorem 9.1, page 76). *Let a function* $\mathbf{f}\colon \mathbb{C}^{n+1} \to \mathbb{C}^n$ *be analytic and bounded in the region*

$$R\colon\ |z - z_0| < \alpha, \quad \|\mathbf{w} - \mathbf{w_0}\| < \beta,$$

where $\alpha > 0$, $\beta > 0$, *and let*

$$\mu \stackrel{\text{def}}{=} \sup_{(z,\mathbf{w})\in R} \|\mathbf{f}(z,\mathbf{w})\|, \quad \gamma \stackrel{\text{def}}{=} \min\Bigl(\alpha, \frac{\beta}{\mu}\Bigr).$$

Then there exists in the disk D_0, $|z - z_0| < \gamma$ *a unique analytic function* $\mathbf{w}\colon \mathbb{C} \to \mathbb{C}^n$ *which is the solution of* (2.3).

Lemma 2.1. *There is* $\delta > 0$ *such that in* $U_0 \stackrel{\text{def}}{=} \{z \in \mathbb{C}\colon\ |z| < \delta\}$ *the initial value problem* (1.4) *has the unique solution* $u(z)$ *which is an analytic function in* U_0.

P r o o f. Consider (2.2) in complex domain. Let us denote

$$f_1(z,\xi,\eta) \stackrel{\text{def}}{=} \eta$$

and (recall $p = 4, 6, 8, \ldots$ by assumption of this section)

$$f_2(z,\xi,\eta) \stackrel{\text{def}}{=} -\frac{\xi^{2m+1}}{\eta^{2m}}, \quad \text{where } z,\xi,\eta \in \mathbb{C} \text{ and } m \in \mathbb{N}.$$

Naturally, the functions $f = \xi$ and $g = \eta$ are holomorphic in the entire complex plane. Thus by Proposition 2.2, functions $f_1(z,\xi,\eta)$ and $f_2(z,\xi,\eta)$ are holomorphic on some neighborhood of $[0,0,1]$. Let R denote the maximal closed subset of this neigborhood. Then the functions f_1 and f_2 are holomorphic on the closed domain R and so they are continuous on R. Hence they are bounded on R (see [6], page 37). Therefore, the system (2.2) has a unique solution by Proposition 2.3. $\square$

The previous lemma yields a local solution $u(z)$ of (1.4) in a small neighborhood U_0 of 0 in $\mathbb{C}$. Since $u(z)$ is analytic in U_0, it can be written as a power series $u(z) = \sum\limits_{k=0}^{\infty} a_k z^k$, where this power series converges towards $u(z)$ for all $z \in U_0$. Our next aim is to show that the series corresponding to $u(z)$ has the same coefficients as the series corresponding to $\sin_p(x)$, which is the unique solution to the real-valued initial value problem (1.1). For this purpose, we will use the following result concerning the sums of two powers series.

Proposition 2.4 ([8], Theorem 16.6, page 352). *If the sums of two power series in the variable $z - z_0$ coincide on a set of points E for which z_0 is a limit point and $z_0 \notin E$, then identical powers of $z - z_0$ have identical coefficients, i.e., there is a unique power series in the variable $z - z_0$ with the given sum on the set E.*

Now we are ready to prove the main result of this section.

Proof of Theorem 2.1. By Lemma 2.1, $u(z) = \sum_{k=0}^{\infty} a_k z^k$ is the unique solution of (1.4) at any point $z \in U_0$. Observe that the solution $u(z) = \sum_{k=0}^{\infty} a_k z^k$ solves also the real-valued Cauchy problem (1.4) in the sense of real analysis. On the other hand, $\sin_p$ is the unique solution of the real-valued Cauchy problem (1.4). Since the Maclaurin series (2.1) of $\sin_p$ converges towards $\sin_p$ in $(-\pi_p/2, \pi_p/2)$ under the assumption of this section, we find that (2.1) satisfies (1.4) in $(-\pi_p/2, \pi_p/2)$. Moreover, convergence of (2.1) on $(-\pi_p/2, \pi_p/2)$ implies convergence of $\sum_{k=0}^{\infty} \alpha_k z^{kp+1}$ for all $z \in \mathbb{C}$, $|z| < \pi_p/2$. Therefore,

$$\sum_{j=0}^{\infty} a_j z^j = \sum_{k=0}^{\infty} \alpha_k z^{kp+1} \quad \text{for all } z \in U_0 \cap (-\pi_p/2, \pi_p/2).$$

Now we consider the set of points $z_n = \delta/(n+1)$, $n \in \mathbb{N}$. From the previous equation, we have

$$\sum_{j=0}^{\infty} a_j z_n^j - \sum_{k=0}^{\infty} \alpha_k z_n^{kp+1} = 0 = \sum_{j=0}^{\infty} 0 \cdot z_n^j.$$

By Proposition 2.4, we find that these two series must coincide on U_0. Hence the Maclaurin series (2.1) satisfies (1.4) on U_0. Let u be given by the series (2.1). Then u'', $(u')^{p-2}$, u^{p-1} have the radius of convergence $\pi_p/2$ for $p > 2$, $p \in \mathbb{N}$. Since any power series converges absolutely within the radius of its convergence, we see from (1.4) that

$$-\left[\left(\sum_{k=0}^{\infty} \alpha_k z_n^{kp+1}\right)'\right]^{p-2} \left(\sum_{k=0}^{\infty} \alpha_k z_n^{kp+1}\right)'' - \left(\sum_{k=0}^{\infty} \alpha_k z_n^{kp+1}\right)^{p-1} = 0 = \sum_{j=0}^{\infty} 0 \cdot z_n^j$$

for all $z_n = \delta/(n+1)$, $n \in \mathbb{N}$. Thus, by Proposition 2.4, u given by the series (2.1) is the solution of (1.4) on the disc $D = \{z \in \mathbb{C} \colon |z| < \pi_p/2\}$. $\square$

3. Obstacles for extension of $\sin_p$ for $p = 3, 5, 7, \dots$ to complex domain

Lindqvist [7] proposed an alternative definition of $\sin_p$ as the solution of

$$\frac{\mathrm{d}}{\mathrm{d}z}(w')^{p-1} + w^{p-1} = 0, \quad w(0) = 0, \quad w'(0) = 1 \tag{3.1}$$

in complex domain for $p > 1$ (considered only formally). In [7], Section 7, he conjectures the possibility that solutions to (3.1) and the real Cauchy problem

$$(|u'|^{p-2}u')' + |u|^{p-2}u, \quad u(0) = 0, \quad u'(0) = 1 \tag{3.2}$$

could produce different solutions on $\mathbb{R}$. We address this question in this section. However, we have definitions of π_p and $\sin_p$ in this paper different from those in [7]. Turning to our definitions of π_p and $\sin_p$, we get an equation corresponding to (3.1):

$$\frac{\mathrm{d}}{\mathrm{d}z}(w')^{p-1} + (p-1)w^{p-1} = 0, \quad w(0) = 0, \quad w'(0) = 1 \tag{3.3}$$

which is equivalent to (1.4), which is equivalent to (2.2). Since the $(p-1)$-st power is a multivalued complex function, we will limit ourselves to $p \in \mathbb{N}$, $p > 1$, in order to be able to perform rigorous analysis. The question is whether (3.3) produces a solution which is different from the solution (1.1) on $\mathbb{R}$. In the previous section we proved that for $p = 4, 6, 8, \dots$ (and of course for $p = 2$) the solutions of (3.3) and (1.1) are identical. Now we show that for $p = 3, 5, 7, \dots$ the solutions are different for negative arguments.

This proposition is crucial for the proof of the main result of this section.

Proposition 3.1 ([4], Theorem 3.4, page 6). *Let $p = 3, 5, 7, \dots$ Then the formal Maclaurin series of $\sin_p(x)$—the solution of the Cauchy problem (1.1)—converges on $(-\pi_p/2, \pi_p/2)$. Moreover, the formal Maclaurin series of $\sin_p(x)$ converges towards $\sin_p(x)$ on $[0, \pi_p/2)$, but does not converge towards $\sin_p(x)$ on $(-\pi_p/2, 0)$.*

Now we are ready to formulate the main result of this section.

Theorem 3.1. *Let $p = 3, 5, 7, \dots$ Then the unique solution $u(z)$ of the complex initial value problem (1.4) differs from the solution $\sin_p(x)$ of the Cauchy problem (1.1) for $z = x \in (-\pi_p/2, 0)$.*

P r o o f. Let us recall that (3.3) is equivalent to (2.2). There exists a unique solution of (2.2) on some nonempty open disc in $\mathbb{C}$ containing 0 by Proposition 2.3.

In the same way as in the proof of Theorem 2.1 (with obvious modifications), it follows that $M_{\sin_p}(z)$ solves (3.3) on the open disc $|z| < \pi_p/2$ and it is the unique solution on this disc. Since $\sin_p(x)$ is the unique solution of (1.1), $\sin_p(x) \neq M_{\sin_p}(x)$ for $x \in (-\pi_p/2, 0)$ by Proposition 3.1, we see that (1.1) and (3.3) produce different solutions on $\mathbb{R}$. □

4. Relations between real and imaginary parts

Let us mention an interesting relationship between real and imaginary parts of $\sin_p(z)$ for $p = 4, 8, 12, \ldots$ One can see in Figure 1 that the graph of the imaginary part of $\sin_4(z)$ is the graph of the real part, rotated by $-\pi/2$.

Theorem 4.1. *Let $p = 4, 8, 12, \ldots$ Then*

$$\Re[\sin_p(z)] = \Im[\sin_p(\mathrm{i}z)]$$

for all $z \in \mathbb{C}$, $|z| < \pi_p/2$.

Proof. Note that by (2.1)

$$\sin_p(z) = \sum_{k=0}^{\infty} \alpha_k z^{kp+1} = z \sum_{k=0}^{\infty} \alpha_k z^{kp}$$

for $z \in \mathbb{C}$, $|z| < \pi_p/2$. We assume $p = 4l$ where $l = 1, 2, 3, \ldots$ and thus

$$\sin_p(z) = z \sum_{k=0}^{\infty} \alpha_k z^{4kl}.$$

Substituting $\mathrm{i}z$ into this formula we find

$$\sin_p(\mathrm{i}z) = \mathrm{i}z \sum_{k=0}^{\infty} \alpha_k (\mathrm{i}z)^{4kl} = \mathrm{i} \sum_{k=0}^{\infty} \alpha_k z^{4kl+1} = \mathrm{i} \sin_p(z).$$

Now the result easily follows from comparison of the real and imaginary parts of $\sin_p(z)$ and $\mathrm{i}\sin_p(\dot{z})$. This completes the proof. □

Theorem 4.2. *Let $p = 2, 6, 10, 14, \ldots$ Then for all $\varphi \in [0, 2\pi)$ there exists $z \in \mathbb{C}$, $|z| < \pi_p/2$ such that*

$$\Re[\sin_p(z)] \neq \Im[\sin_p(\mathrm{e}^{\mathrm{i}\varphi} z)].$$

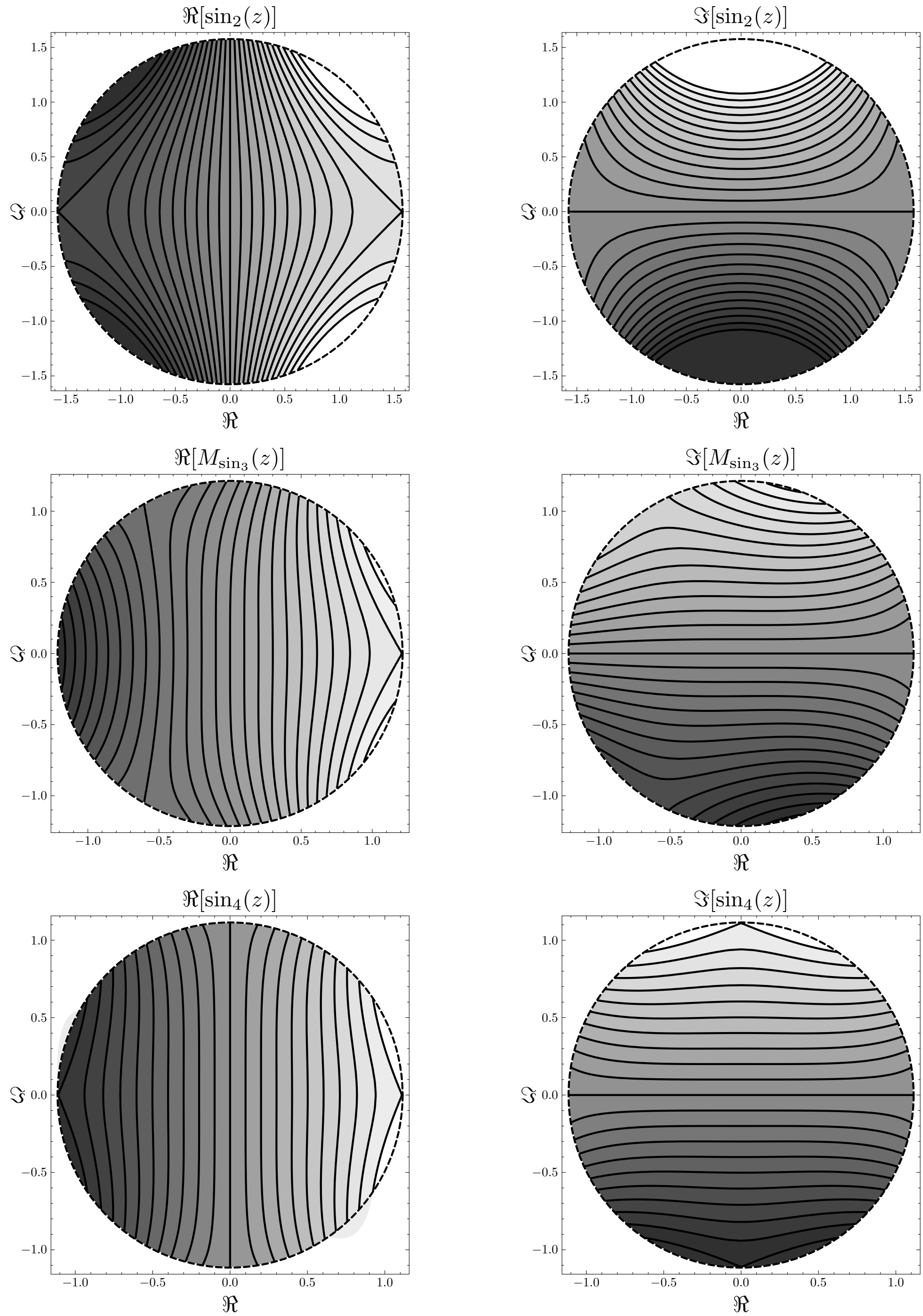

Figure 1. Continued on the next page.

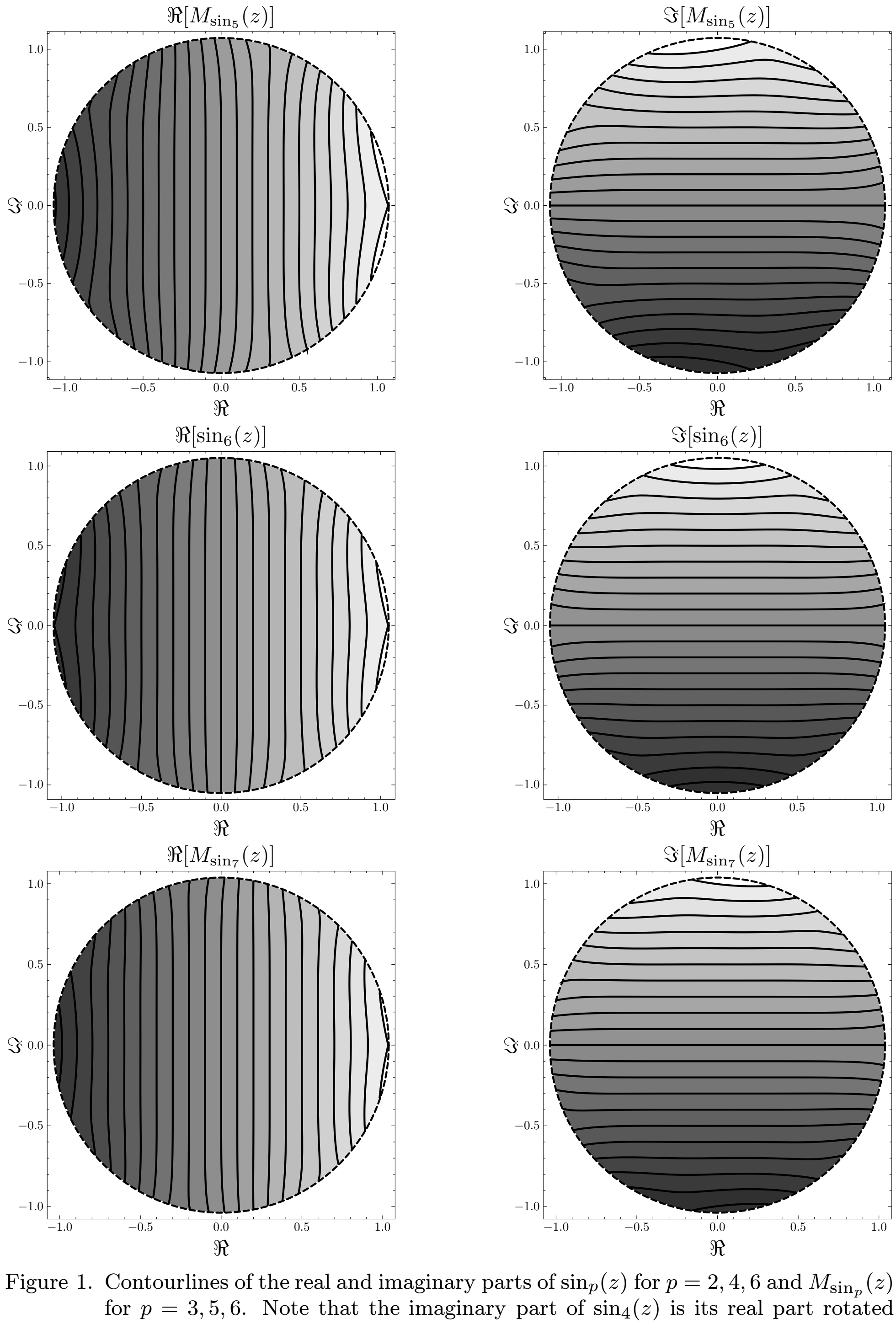

Figure 1. Contourlines of the real and imaginary parts of $\sin_p(z)$ for $p = 2, 4, 6$ and $M_{\sin_p}(z)$ for $p = 3, 5, 6$. Note that the imaginary part of $\sin_4(z)$ is its real part rotated by $-\pi/2$.

P r o o f. It is known from [4] that the series $M_{\sin_p}(z)$ has the form

$$M_{\sin_p}(z) = \sum_{k=0}^{\infty} \alpha_k z^{kp+1}.$$

First we show that $\alpha_0 = 1$ and $\alpha_1 = -1/(p(p+1)) < 0$ (cf e.g. [7]). In fact, evaluating the integral in (1.3) we see that

$$\operatorname{arcsin}_p(x) = \int_0^x (1-s^p)^{-1/p}\,\mathrm{d}s = {}_2\mathrm{F}_1\Big(\frac{1}{p}, \frac{1}{p}, 1+\frac{1}{p}, x^p\Big)x \quad \text{for } 0 \leqslant x \leqslant 1,$$

where ${}_2\mathrm{F}_1$ is the Gauss hypergeometric function. Using the known series

$${}_2\mathrm{F}_1(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!} \quad \text{for } |z| < 1,$$

where $(a)_k = \prod\limits_{j=0}^{k} (a+k-1)$ for any $a \in \mathbb{R}$ stands for the rising factorial, we find

$$\operatorname{arcsin}_p(w) = w \sum_{k=0}^{\infty} \frac{(1/p)_k^2 w^{kp}}{(1+1/p)_k k!} \quad \text{for } 0 < w < 1.$$

Hence

$$\operatorname{arcsin}_p(w) = w + \frac{1}{p(p+1)} w^{p+1} + O(w^{2p+1}) \quad \text{for } 0 < w < 1.$$

Denoting $w = \sin_p(x)$, we find

$$x = w + \frac{1}{p(p+1)} w^{p+1} + O(w^{2p+1}),$$

which yields

$$w = x - \frac{1}{p(p+1)} w^{p+1} + O(w^{2p+1}). \tag{4.1}$$

Substituting (4.1) into itself we obtain

$$w = x - \frac{1}{p(p+1)}\Big(x - \frac{1}{p(p+1)} + O(w^{2p+1})\Big)^{p+1} + O(w^{2p+1}).$$

Hence

$$\sin_p(x) = x - \frac{1}{p(p+1)} x^{p+1} + O(w^{2p+1}), \tag{4.2}$$

which gives the desired formulas for $\alpha_1 = 1$ and $\alpha_2 = -1/p(p+1)$. With this at hand, we can write

$$\text{(4.3)} \qquad M_{\sin_p}(z) = z - \frac{1}{p(p+1)} z^{p+1} + \sum_{m=2}^{\infty} \alpha_m z^{mp+1} = z - \frac{z^{p+1}}{p(p+1)} - z^{2p+1} \sum_{m=0}^{\infty} \alpha_{m+2} z^{mp}.$$

Let $z = a$, $a \in \mathbb{R}$, $0 < a < \pi_p/2$ for simplicity. Then $\varphi_0 = \pi/2$ is the unique angle in $[0, 2\pi)$ such that $\Re[z] = \Im[e^{i\varphi_0} z]$. The assumption on p of this theorem is that there exists $l \in \mathbb{N} \cup \{0\}$ such that $p = 4l + 2$. Thus $\Re[z^{p+1}] = \Re[z^{4l+3}] = \Re[a^{4l+3}]$. On the other hand, $\Im[(e^{i\varphi_0} z)^{p+1}] = \Im[(ia)^{4l+3}] = -a^{4l+3}$ for $\varphi_0 = \pi/2$. Inserting $z = a$ and $z = ia$ into (4.3), taking the real and imaginary part, respectively, and subtracting, we get

$$\text{(4.4)} \qquad \Re[M_{\sin_p}(a)] - \Im[M_{\sin_p}(ia)] = = -\frac{2a^{p+1}}{p(p+1)} + a^{2p+1} \left(\Re\left[\sum_{m=0}^{\infty} \alpha_{m+2} a^{mp} \right] - \Im\left[i^{2p+1} \sum_{m=0}^{\infty} \alpha_{m+2} (ia)^{mp} \right] \right).$$

Since the series on the right-hand side are convergent on the disc $\{z \in \mathbb{C} \colon |z| < \pi_p/2\}$,

$$A \overset{\text{def}}{=} \max_{\{z \in \mathbb{C} \colon |z| \leqslant \pi_p/4\}} \left| \left(\Re\left[\sum_{m=0}^{\infty} \alpha_{m+2} z^{mp} \right] - \Im\left[i^{2p+1} \sum_{m=0}^{\infty} \alpha_{m+2} (iz)^{mp} \right] \right) \right| < \infty$$

exists and from (4.4) we find

$$\left| \frac{\Re[M_{\sin_p}(a)] - \Im[M_{\sin_p}(ia)]}{a^{p+1}} - \frac{2}{p(p+1)} \right| \leqslant A a^p.$$

Taking $0 < a < \min\{\pi_p/4, (Ap(p+1))^{-1/p}\}$, we see that $\Re[M_{\sin_p}(a)] - \Im[M_{\sin_p}(ia)] \neq 0$. This concludes the proof. $\square$

5. Consequence of complex p-trigonometric identity

As was mentioned earlier, the maximal possible radius of convergence for the (formal) Maclaurin series for functions $\sin_p$ and $\cos_p$ is $\pi_p/2$. This fact was anticipated in [7] and studied in detail in [4]. In this section we explain that there was no hope for these series to have their radius of convergence infinite for $p = 3, 4, 5, 6, \dots$ Contrary to what one would think, we will show that it is not the absolute value in (1.1) that produces the main difficulty. It is a complex analogy of the p-trigonometric identity that produces the impossibility of $\sin_p$ to be an entire complex functions for $p = 3, 4, 5, 6, \dots$

Let us reconsider (1.4), i.e.,

$$-(u')^{p-2}u'' - u^{p-1} = 0, \quad u(0) = 0, \quad u'(0) = 1,$$

now for any $p = 3, 4, 5, 6, \dots$ in the complex domain. Let us assume that u is a solution which is a holomorphic function on some neighborhood U_0 of 0. Multiplying the equation of (1.4) by u' and integrating from 0 to $z \in U_0$, we obtain

$$(u'(z))^p - (u'(0))^p + (u(z))^p - (u(0))^p = 0.$$

Now using the initial conditions of (1.4) we get

$$(u'(z))^p + (u(z))^p = 1, \tag{5.1}$$

which is the first integral of (1.4), and we can think of it as a complex p-trigonometric identity for holomorphic solutions of (1.4) for $p = 3, 4, 5, 6, \dots$

Now we state the very classical result from complex analysis.

Proposition 5.1 ([1], Theorem 12.20, page 433). *Let f and g be entire functions satisfying for some positive integer n the identity*

$$f^n + g^n = 1.$$

(i) *If $n = 2$, then there is an entire function h such that $f = \cos \circ h$, $g = \sin \circ h$.*
(ii) *If $n > 2$, then both f and g are constants.*

It follows from this result that a holomorphic solution u of (1.4) cannot be an entire function for any $p = 3, 4, 5, 6, \dots$, since the derivative of an entire function is an entire function as well and u and u' must satisfy (5.1). Thus by Proposition 5.1 u and u' are constant, which contradicts $u'(0) = 1$.

In particular, for $p = 4, 6, 8, \ldots$, with $u(z) = \sin_p(z)$ and $u'(z) = \cos_p(z)$ this becomes

$$\cos_p^p(z) + \sin_p^p(z) = 1$$

and we see that $\sin_p$ and $\cos_p$ cannot be entire functions.

Note that there was an interesting internet discussion [10] that called our attention towards this connection between complex analysis (including the classical reference [1], Theorem 12.20) and p-trigonometric functions. It seems to us that this connection has been overlooked by the "p-trigonometric community".

6. Visualization of $\sin_p(z)$ and their Maclaurin series

In this section we visualize graphs of the extensions of $\sin_p(z)$ by its Maclaurin series for $p = 4, 6$ and the formal Maclaurin series for $p = 3, 5, 7$ and compare them with the classical result $\sin_p(z) = \sin_2(z)$, see Figure 2. To the best of our knowledge, these figures in complex domain are new and we believe that they will help to stimulate discussion on this topic. We also formulate some conjectures in the caption of Figure 3.

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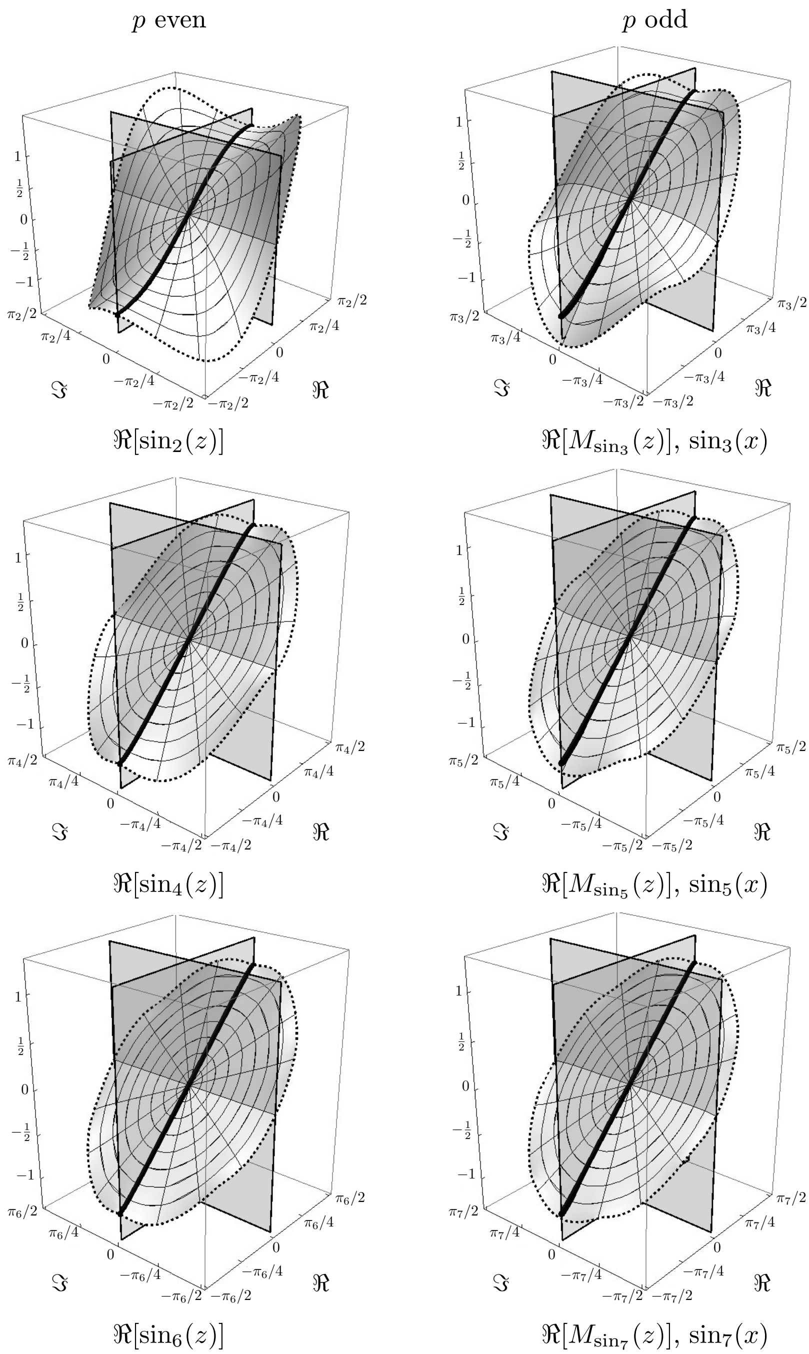

Figure 2. Comparison of real parts of $\sin_p(z)$ for p even (extended by the Maclaurin series) and the real parts of the formal Maclaurin series $M_{\sin_p}(z)$ and the real function $\sin_p(x)$ for p odd.

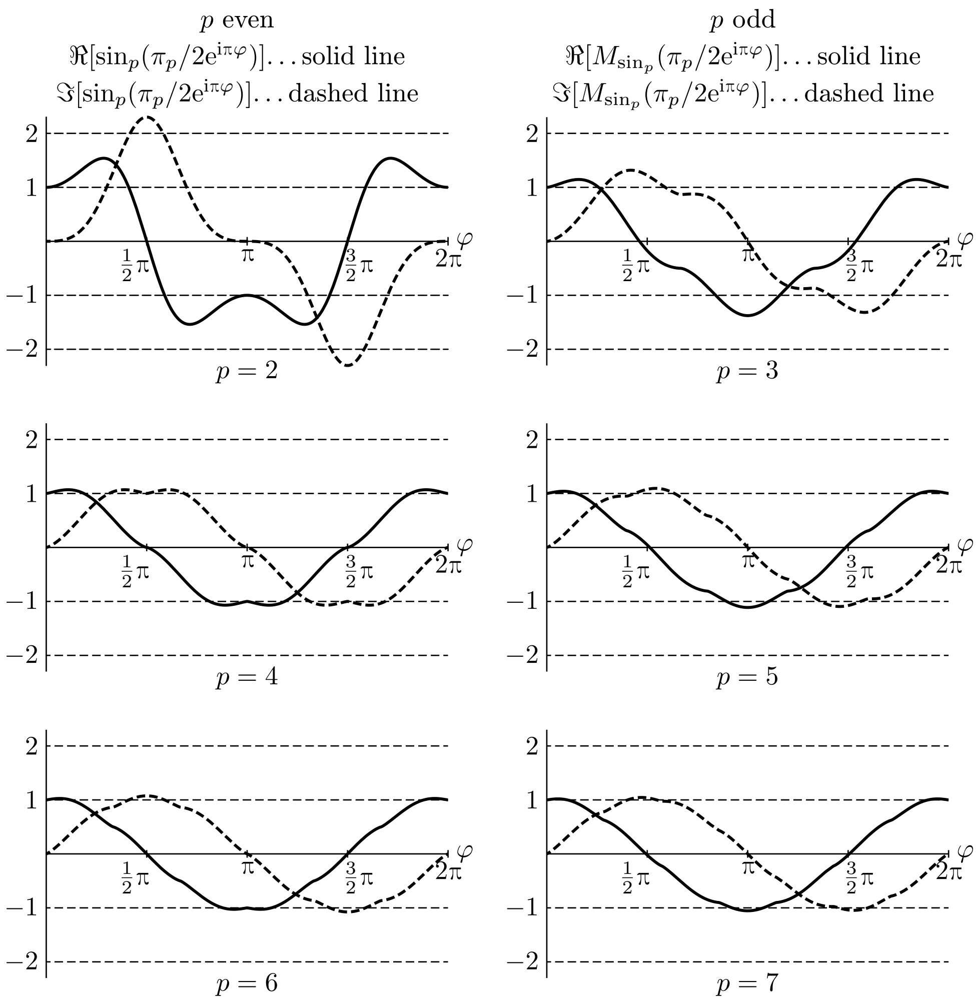

Figure 3. Numerical comparison of the real and imaginary parts of $\sin_p(\pi_p/2\mathrm{e}^{\mathrm{i}\pi\varphi})$ for $p = 2, 4, 6$ (extended by Maclaurin series) and the real and imaginary parts of $M_{\sin_p}(\pi_p/2\mathrm{e}^{\mathrm{i}\pi\varphi})$ for $p = 3, 5, 7$. Note that these graphs are only an illustration, because we know nothing about the convergence of the series for $z \in \mathbb{C}$, $|z| = \pi_p/2$. From these pictures we conjecture this convergence. It is interesting to note in these pictures that for larger p, the graph of the real part is a small perturbation of $\pi_p/2\cos\varphi$ and the graph of the imaginary part is a small perturbation of $\pi_p/2\sin\varphi$. We conjecture that this phenomena occur due to the fact that the Maclaurin series is $M_{\sin_p}(z) = z - z^{p+1}/(p(p+1)) + O(z^{2p+1})$ and for large p the higher order terms are negligible. Moreover, $\lim_{p\to\infty}\pi_p/2 = 1$. Thus we conjecture that these graphs tend to graphs of $\sin\varphi$ and $\cos\varphi$ for $p \to \infty$, respectively.

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Appendix A7

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Research Article

p-Trigonometric and p-Hyperbolic Functions in Complex Domain

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We study extension of p-trigonometric functions $\sin_p$ and $\cos_p$ and of p-hyperbolic functions $\sinh_p$ and $\cosh_p$ to complex domain. Our aim is to answer the question under what conditions on p these functions satisfy well-known relations for usual trigonometric and hyperbolic functions, such as, for example, $\sin(z) = -i\cdot\sinh(i\cdot z)$. In particular, we prove in the paper that for $p = 6, 10, 14, \ldots$ the p-trigonometric and p-hyperbolic functions satisfy very analogous relations as their classical counterparts. Our methods are based on the theory of differential equations in the complex domain using the Maclaurin series for p-trigonometric and p-hyperbolic functions.

1. Introduction

The p-trigonometric functions are generalizations of regular trigonometric functions sine and cosine and arise from the study of the eigenvalue problem for the one-dimensional p-Laplacian.

In recent years, the p-trigonometric functions were intensively studied from various points of views by many researchers; see, for example, monograph [1] for systematic survey and further references. The purpose of this paper is twofold. We begin with a short survey of results from [2, 3]. Then, we extend the ideas from [3] to define corresponding generalization of hyperbolic functions and study relations of p-trigonometric and p-hyperbolic functions on a disc in the complex domain.

More precisely, our goal is to generalize the hyperbolic functions such that the relations

$$\begin{aligned} \sin z &= -i \cdot \sinh(i \cdot z), \\ \cos z &= \cosh(i \cdot z), \end{aligned} \tag{1}$$

$$\begin{aligned} \cos z &= \sin' z, \\ \cosh z &= \sinh' z, \end{aligned} \tag{2}$$

$$\begin{aligned} \cos^2 z + \sin^2 z &= 1, \\ \cosh^2 z - \sinh^2 z &= 1, \end{aligned} \tag{3}$$

where $z \in \mathbb{C}$, have their counterparts for generalized p-trigonometric and p-hyperbolic functions. It turns out that this goal can be achieved only for even integer $p > 2$.

The p-trigonometric functions in the real domain $\mathbb{R}$ originate naturally from the study of the nonlinear eigenvalue problem

$$\begin{aligned} -\left(\left|u'\right|^{p-2} u'\right)' - \lambda |u|^{p-2} u = 0 \quad \text{in } \left(0, \pi_p\right), \\ u(0) = u\left(\pi_p\right) = 0, \end{aligned} \tag{4}$$

where $p > 1$, $\lambda \in \mathbb{R}$ is a parameter, and

$$\pi_p = 2 \int_0^1 \frac{1}{(1 - s^p)^{1/p}} \mathrm{d}s = \frac{2\pi}{p \sin(\pi/p)}. \tag{5}$$

It was shown in Elbert [4] that all eigenfunctions of (4) can be expressed in terms of solutions of the initial-value problem

$$\begin{aligned} -\left(\left|u'\right|^{p-2} u'\right)' - (p-1) |u|^{p-2} u = 0, \quad x \in \mathbb{R}, \\ u(0) = 0, \\ u'(0) = 1. \end{aligned} \tag{6}$$

Indeed, (6) has the unique solution in $\mathbb{R}$; see, for example, [5, Lemma A.1], [6, Section 3], and [4]. Denoting the solution of (6) by $\sin_p x$, the set of all eigenvalues $\lambda_k \in \mathbb{R}$ and eigenfunctions $u_k \in W_0^{1,p}(0, \pi_p)$ of (4) can be written as

$$\lambda_k = (p-1) k^p,$$
$$u_k(x) = \sin_p(k \cdot x), \tag{7}$$
$$\text{where } k \in \mathbb{N}.$$

A piecewise construction of the solution of (6) was provided in [4]. At first, one sets

$$\operatorname{arcsin}_p x \stackrel{\text{def}}{=} \int_0^x \frac{1}{(1-s^p)^{1/p}} \mathrm{d}s, \quad x \in [0,1]. \tag{8}$$

Then, the restriction of $\sin_p x$ on $[0, \pi_p/2]$ is the inverse function to $\operatorname{arcsin}_p x$. For $x \in (\pi_p/2, \pi_p]$, $\sin_p x$ satisfies $\sin_p x = \sin_p(\pi_p - x)$, where clearly $\pi_p - x \in [0, \pi_p/2)$, and $\sin_p x = -\sin_p(-x)$ for $x \in [-\pi_p, 0)$. Finally, $\sin_p x$ is a $2\pi_p$-periodic function on $\mathbb{R}$.

We also extend $\operatorname{arcsin}_p x$ from (8) to $[-1,1]$ as an odd function. Then, it is the inverse function to the restriction of $\sin_p x$ to $[-\pi_p/2, \pi_p/2]$, and we have

$$\sin_p\left(\operatorname{arcsin}_p x\right) = x, \quad \forall x \in [-1,1]. \tag{9}$$

Finally, let us define $\cos_p x \stackrel{\text{def}}{=} \sin_p' x$ for all $x \in \mathbb{R}$. Then, the functions $\sin_p x$ and $\cos_p x$ satisfy the so-called p-trigonometric identity

$$\left|\cos_p x\right|^p + \left|\sin_p x\right|^p = 1 \tag{10}$$

for all $x \in \mathbb{R}$; see, for example, [4–6].

Note that there is an alternative definition of "$\cos_p x$" in [7] and/or [8] which leads to different "p-trigonometric" identity. Yet another alternative generalization of trigonometric and hyperbolic functions motivated by geometrical point of view was introduced in [9]. Studies of relations between their respective generalizations of p-trigonometric and p-hyperbolic functions were suggested in [7] and in [9], respectively.

Remark 1. In the paper, we use Gauss' hypergeometric function ${}_2F_1(a,b,c,z)$, where $a,b,c \in \mathbb{C}$ are parameters and $z \in \mathbb{C}$ is variable (for definition see [10, 15.1.1. p. 556]), to express integrals of the type

$$\int_0^x \frac{1}{(1 \pm s^p)^{1/p}} \mathrm{d}s,$$
$$\int_0^z \frac{1}{(1 \pm s^p)^{1/p}} \mathrm{d}s, \tag{11}$$

for $p > 1$, $x \in \mathbb{R}$, and $z \in \mathbb{C}$ (by $z^{1/p}$ we understand the principal branch thereof). Indeed,

$$z\, {}_2F_1\left(\frac{1}{p}, \frac{1}{p}, 1+\frac{1}{p}, \mp z^p\right) = \int_0^z \frac{1}{(1 \pm s^p)^{1/p}} \mathrm{d}s \tag{12}$$

for $|z| < 1$ (by comparing respective series expansions). By the uniqueness of analytic extension, the equation is valid for $z \in \mathbb{C} \setminus \{x + iy : x > 1 \text{ and } y = 0\}$ (for analytic continuation of ${}_2F_1$ see, e.g., [10, 15.3.1, p. 558] and [11, Theorem 2.2.1, p. 65]).

In the definition of π_p (i.e., (5)) and in (8), we need to evaluate integral

$$\int_0^1 \frac{1}{(1-s^p)^{1/p}} \mathrm{d}s. \tag{13}$$

By [11, Theorem 2.2.2, p. 66], this is possible, since $\Re[c-a-b] = 1 + 1/p - 1/p - 1/p = 1 - 1/p > 0$ for $p > 1$.

Notation 1. This paper combines real variable and complex variable approach to the p-trigonometric and p-hyperbolic functions. Each of these approaches has its own natural way of how to define the functions $\sin_p$ and $\sinh_p$. Thus, we need to distinguish between real and complex definitions. By $\sin_p x$ and $\sinh_p x$, we mean functions defined by the real variable approach and by $\sin_p z$ and $\sinh_p z$ we mean functions defined by the complex variable approach, throughout the paper.

2. Real Analyticity Results for $\sin_p x$ and $\cos_p x$

It is well known that the p-trigonometric functions are not real analytic functions in general; see, for example, [12, 13]. Very detailed study of the degree of smoothness of the restriction of $\sin_p x$ to $(-\pi_p/2, \pi_p/2)$ was performed in [2] including the following two results. The first one concerns "generic" $p > 1$.

Proposition 2 (see [2], Theorem 3.2 on p. 105). *Let $p \in \mathbb{R} \setminus \{2m\}$, $m \in \mathbb{N}$, $p > 1$. Then,*

$$\sin_p x \in C^{\lceil p \rceil}\left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right), \tag{14}$$

but

$$\sin_p x \notin C^{\lceil p \rceil + 1}\left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right). \tag{15}$$

Here, $\lceil p \rceil \stackrel{\text{def}}{=} \min\{k \in \mathbb{N}: k \geq p\}$.

The second result treats only the even integers $p > 2$ and differs significantly from the previous case in an unexpected way.

Proposition 3 (see [2], Theorem 3.1 on p. 105). *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then,*

$$\sin_p x \in C^{\infty}\left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right). \tag{16}$$

Thus, the Maclaurin series of $\sin_p x$

$$M_p(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n!} \sin_p^{(n)}(0) \cdot x^n \tag{17}$$

is well defined for $p = 2(m+1)$, $m \in \mathbb{N}$. Moreover, the following result establishes an explicit expression for the radius of convergence of M_p.

Proposition 4 (see [2], Theorem 3.3 on p. 106). *Let $p = 2(m+1)$ for $m \in \mathbb{N}$. Then, the Maclaurin series $M_p(x)$ of $\sin_p x$ converges on $(-\pi_p/2, \pi_p/2)$.*

Thus, for $p = 2(m+1)$, $m \in \mathbb{N}$, we can compute approximate values of $\sin_p x$ using Maclaurin series. It turns out that the most effective method of computing coefficients in (17) is to use formal inversion of the Maclaurin series of

$$\begin{aligned}\operatorname{arcsin}_p x &= \int_0^x \frac{1}{(1-s^p)^{1/p}} ds \\ &= x \cdot {}_2F_1\left(\frac{1}{p}, \frac{1}{p}, 1 + \frac{1}{p}, x^p\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k+1/p)}{(kp+1)\,k!\,\Gamma(1/p)} x^{kp+1},\end{aligned} \tag{18}$$

where $p = 2(m+1)$ for some $m \in \mathbb{N}$. The procedure of inverting power series is well known; see, for example, [14]. This task can be easily performed using computer algebra systems. In Pseudocode 1, we provide an example of computing the partial sum of M_4 up to terms of order 32 in *Mathematica*® v. 9.0. In this way, we can easily get partial sums of M_p and get approximations of $\sin_p$ for any $p = 2(m+1)$, $m \in \mathbb{N}$, up to terms of orders of hundreds. Note that this formal inverse can be applied also for $p = 2m+1$, $m \in \mathbb{N}$. The question is what is the mathematical sense of the resulting formal series. Let

$$T_p(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} a_n \cdot x^n \tag{19}$$

denote the series that is the formal inverse of (18) for $p = 2m+1$, $m \in \mathbb{N}$. For $p = 2m+1$, $m \in \mathbb{N}$, let us also define

$$M_p(x) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n!}\left(\lim_{x\to 0+} \sin_p^{(n)} x\right) \cdot x^n, \tag{20}$$

which is a formal Maclaurin series of some unknown function. It turns out that this unknow function is not $\sin_p x$ as the following result holds.

Proposition 5 (see [2], Theorem 3.4 on p. 106). *Let $p = 2m+1$, $m \in \mathbb{N}$. Then, the formal Maclaurin series M_p converges on $(-\pi_p/2, \pi_p/2)$. Moreover, the formal Maclaurin series M_p converges towards $\sin_p x$ on $[0, \pi_p/2)$ but does not converge towards $\sin_p x$ on $(-\pi_p/2, 0)$.*

In Appendix A, we prove that T_p and M_p are identical.

Theorem 6. *Let $p = 2m+1$, $m \in \mathbb{N}$. Then,*

$$a_n = \frac{1}{n!}\left(\lim_{x\to 0+} \sin_p^{(n)} x\right), \quad \forall n \in \mathbb{N}, \tag{21}$$

and $T_p(x) = M_p(x)$ for all $x \in (-\pi_p/2, \pi_p/2)$.

It turns out that the pattern of zero coefficients of M_p is the same as in the Maclaurin series of $\operatorname{arcsin}_p x$; compare (18).

Theorem 7. *Let $p > 2$ be an integer. Then, $a_i = 0$ for all $i \in \mathbb{N}$ such that $i - 1$ is not divisible by p.*

The proof is technical and thus postponed to Appendix B. It is based on the formal inversion of (18). Note that the structure of powers in (18) does not allow any substitution that will transform it into a power series of new variable without zero coefficients. This makes the proof technically complicated.

Using Theorem 7, we can omit zero entries and rewrite the series M_p:

$$M_p(x) = \sum_{l=0}^{\infty} \alpha_l \cdot x^{lp+1}, \tag{22}$$

where α_l can be obtained by formal inversion of the Maclaurin series of $\operatorname{arcsin}_p x$ in (18). In particular,

$$\begin{aligned}\alpha_0 &= 1, \\ \alpha_1 &= -\frac{1}{p(p+1)}, \\ \alpha_2 &= -\frac{p^2 - 2p - 1}{2p^2(p+1)(2p+1)}, \ldots.\end{aligned} \tag{23}$$

3. Extension of $\sin_p z$ and $\cos_p z$ to the Complex Domain for Integer $p > 1$

The conclusion of this theorem follows from the discussion in [2].

Theorem 8. *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then, the Maclaurin series of $\sin_p x$ converges on the open disc*

$$B_p \overset{\text{def}}{=} \left\{z \in \mathbb{C} : |z| < \frac{\pi_p}{2}\right\}. \tag{24}$$

Proof. In fact, the Maclaurin series $\sum_{l=0}^{\infty} \alpha_l \cdot x^{l \cdot p+1}$ converges towards the values of $\sin_p x$ on $(-\pi_p/2, \pi_p/2)$ absolutely for $p = 2(m+1)$, $m \in \mathbb{N}$. □

For $p = 2(m+1)$, $m \in \mathbb{N}$, the expressions with powers in the initial-value problem (6) can be written without the absolute values. Thus, the resulting initial-value problem

$$\begin{aligned}(u')^{p-2} u'' + u^{p-1} &= 0, \\ u(0) &= 0, \\ u'(0) &= 1\end{aligned} \tag{25}$$

makes sense also in the complex domain (the derivatives are understood in the sense of the derivative with respect to

```
In[1] := Series[s*Hypergeometric2F1[1/4, 1/4, 5/4, s^4],
            {s, 0, 32}]
(* computes the Maclaurin series of arcsin_4 *)
Out[1] = s + s^5/20 + 5s^9/288 + 15s^13/1664 + 195s^17/34816 + 221s^21/57344 + 4641s^25/1638400 + 16575s^29/7602176 + O(s^33)
In[2]  := InverseSeries[%]
(* computes the inverse series *)
Out[2] = s - s^5/20 - 7s^9/1440 - 463s^13/374400 - 211741s^17/509184000 - 104361161s^21/641571840000 - 8978996213s^25/128314368000000 - 7995735867463s^29/248783228928000000 + O(s^33)
(*which is M_4 up to terms of order 32*)
```

Pseudocode 1: *Mathematica* v. 9.0 code.

complex variable). Let us observe that, using the substitution $u' = v$, we get the following first-order system:

$$\begin{aligned} u' &= v, \\ v' &= -\frac{u^{p-1}}{v^{p-2}}, \\ u(0) &= 0, \\ v(0) &= 1. \end{aligned} \tag{26}$$

By [15, Theorem 9.1, p. 76], there exists $\delta_p > 0$ such that problems (26) and hence (25) have the unique solution on the open disc $|z| < \delta_p$.

Now we will consider initial-value problems (25) and (26) also for $p = 2m+1$, $m \in \mathbb{N}$.

Theorem 9. *Let $p = 2m+1$, $m \in \mathbb{N}$. The unique solution $u(z)$ of (25) restricted to open disc B_p is the Maclaurin series M_p.*

Proof. Let $u(z) = \sum_{k=1}^{\infty} b_k z^k$ be the unique solution of (25) in any point of the open disc $|z| < \delta_p$. Observe that the solution $u(z) = \sum_{k=1}^{\infty} b_k z^k$ solves also the real-valued Cauchy problem (6) for $x > 0$ (where there is no need for $|\cdot|$). On the other hand, $\sin_p x$ is the unique solution of the real-valued Cauchy problem (6). Since M_p given by (20) converges towards $\sin_p x$ in $[0, \pi_p/2)$, we find that M_p satisfies (6) in $[0, \pi_p/2)$. Thus, $M_p(x) = u(x + i \cdot y)$ for $x \in [0, \delta_p)$ and $y = 0$. In particular, taking a sequence of points $z_n = \delta_p/(n+1) + 0 \cdot i$, $n \in \mathbb{N}$, we have $M_p(z_n) = u(z_n)$; thus, we infer from Proposition A.2 that

$$b_k = \frac{1}{k!}\left(\lim_{x\to 0+} \sin_p^{(k)} x\right). \tag{27}$$

M_p has radius of convergence $\pi_p/2$ by Proposition 5, and so does $u(z)$. Thus, $\delta_p = \pi_p/2$. □

Theorems 8 and 9 enable us to extend the range of definition of the function $\sin_p x$ to the complex open disc B_p by M_p for $p = 2(m+1)$ and for $p = 2m+1$, $m \in \mathbb{N}$, respectively. Thus, we can consider $p = m+2$ in the following definition. Note that all the powers of z are of positive-integer order $l \cdot p + 1$ and the function $\sin_p z$ is an analytic complex function on B_p and thus is single-valued.

Definition 10. Let $p = m+2$, $m \in \mathbb{N}$, and $z \in B_p$. Then,

$$\begin{aligned} \sin_p z &\overset{\text{def}}{=} \sum_{l=0}^{\infty} \alpha_l \cdot z^{l\cdot p+1}, \\ \cos_p z &\overset{\text{def}}{=} \sin_p' z = \frac{\mathrm{d}}{\mathrm{d}z}\sin_p z, \end{aligned} \tag{28}$$

where the derivative $\mathrm{d}/\mathrm{d}z$ is considered in the sense of complex variables.

The following fundamental results were proved in [3] (providing explicit value for δ_p).

Proposition 11 (see [3], Theorem 2.1 on p. 226). *Let $p = 2(m+1)$, $m \in \mathbb{N}$; then, the unique solution of the initial-value problem (25) on B_p is the function $\sin_p z$.*

Proposition 12 (see [3], Theorem 3.1 on p. 229). *Let $p = 2m+1$, $m \in \mathbb{N}$. Then, the unique solution $u(z)$ of the complex initial-value problem (25) differs from the solution $\sin_p x$ of the Cauchy problem (6) for $z = x \in (-\pi_p/2, 0)$.*

In [3], it was shown that there is no hope for solutions of (25) to be entire functions for $p = m+2$, $m \in \mathbb{N}$. This result follows from the complex analogy of the p-trigonometric (10).

Lemma 13. *Let $p = m+1$, $m \in \mathbb{N}$, and $r > 0$ be such that the solution u of (25) is holomorphic on a disc $D_r = \{z \in \mathbb{C} : |z| < r\}$. Then, u satisfies the complex p-trigonometric identity*

$$\left(u'(z)\right)^p + (u(z))^p = 1 \tag{29}$$

on the disc D_r.

Proof. Multiplying (25) by u' and integrating from 0 to $z \in D_r$, we obtain

$$\left(u'(z)\right)^p - \left(u'(0)\right)^p + (u(z))^p - (u(0))^p = 0. \tag{30}$$

Now using the initial conditions of (25), we get (29), which is the first integral of (25) and we can think of it as complex p-trigonometric identity for holomorphic solutions of (25) for $p = m+1$, $m \in \mathbb{N}$. □

Now we state the very classical result from complex analysis.

Proposition 14 (see [16], Theorem 12.20 on p. 433). *Let f and g be entire functions and for some positive integer n satisfy identity*

$$f^n + g^n = 1. \tag{31}$$

(i) *If $n = 2$, then there is an entire function h such that $f = \cos \circ h$ and $g = \sin \circ h$.*

(ii) *If $n > 2$, then f and g are each constant.*

The following interesting connection between complex analysis (including the classical reference [16, Theorem 12.20]) and p-trigonometric functions was studied in [3]. We should point out that it was an interesting internet discussion [17] that called our attention towards this connection. It seems to us that this connection was overlooked by the "p-trigonometric community." Thus, we provide its more precise proof here.

Theorem 15. *The solution u of complex initial-value problem (25) cannot be entire function for any $p = m + 2$, $m \in \mathbb{N}$.*

Proof. Assume by contradiction that the solution u of (25) is entire function. Then, we can choose $r > 0$ arbitrarily large in Lemma 13. Thus, u and u' must satisfy (29) at any point $z \in \mathbb{C}$. Note that u' is an entire function too. Thus, by Proposition 14 u and u' are constant which contradicts $u'(0) = 1$. This concludes the proof. □

In particular, the solution of (25) is $u(z) = \sin_p z$ with $u'(z) = \cos_p z$. Thus, (29) becomes

$$\cos_p^p z + \sin_p^p z = 1 \tag{32}$$

and we see that $\sin_p$ and $\cos_p$ cannot be entire functions for $p = m + 2$, $m \in \mathbb{N}$.

4. Generalized Hyperbolometric Function $\operatorname{argsinh}_p x$ and Generalized Hyperbolic Function $\sinh_p x$ in the Real Domain for Real $p > 1$

In analogy to $p = 2$, we define $\sinh_p x$ for $p > 1$ as the solution to the initial-value problem:

$$\begin{gathered} -\left(\left|u'\right|^{p-2} u'\right)' + (p-1)\,|u|^{p-2}\,u = 0, \quad x \in \mathbb{R}, \\ u(0) = 0, \\ u'(0) = 1. \end{gathered} \tag{33}$$

The uniqueness of the solution of this problem can be proved in the same way as in the case of (6) using the first integral (see [4]). Note that the first integral of the real-valued initial-value problem (33) is the real p-hyperbolic identity

$$1 + |u|^p = \left|u'\right|^p, \tag{34}$$

for $p > 1$; compare [4]. Thus, $|u'| \geq 1$ on the domain of definition of solution to (33). Since $u'(0) = 1$ and u' must be absolutely continuous, we find that $u' > 0$ on the domain of definition of solution to (33) and the real p-hyperbolic identity can be rewritten in equivalent form

$$u' = \left(1 + |u|^p\right)^{1/p}, \tag{35}$$

which is a separable first-order ODE in $\mathbb{R}$. By the standard integration procedure, we obtain *inverse function* of the solution u (cf. [4]).

Therefore, it is natural to define

$$\operatorname{argsinh}_p x \stackrel{\text{def}}{=} \int_0^x \frac{1}{\left(1 + |s|^p\right)^{1/p}}\,ds, \quad x \in \mathbb{R}, \tag{36}$$

for any $p > 1$, in the real domain (cf., e.g., [18–21]). Note that the integral on the right-hand side can be evaluated in terms of the analytic extension of Gauss's ${}_2F_1$ hypergeometric function to $\mathbb{C} \setminus \{s + it : s > 1,\ t = 0\}$ (see, e.g., [10, § 15.3.1, p. 558] and [11, Theorem 2.2.1, p. 65]); thus, (taking into account that integrand in (36) is even)

$$\operatorname{argsinh}_p x = \begin{cases} x \cdot {}_2F_1\left(\dfrac{1}{p}, \dfrac{1}{p}, 1 + \dfrac{1}{p}, -x^p\right), & x \in [0, +\infty) \\ -\operatorname{argsinh}_p(-x), & x \in (-\infty, 0). \end{cases} \tag{37}$$

Since $\operatorname{argsinh}_p : \mathbb{R} \to \mathbb{R}$ is strictly increasing function on $\mathbb{R}$, its inverse exists and it is, in fact, $\sinh_p x$ by the same reasoning as in [4] (cf., e.g., [20]).

5. Generalized Hyperbolic Functions $\sinh_p z$ and $\cosh_p z$ in Complex Domain for Integer $p > 1$

In the previous section, we introduced real-valued generalization of $\sinh x$ called $\sinh_p x$. Our aim is to extend this function to complex domain. It is important to observe that, for $p = 2$, the following relations between complex functions $\sin z$ and $\sinh z$ are known:

$$\begin{array}{c|c} \sin z & \sinh z \\ \hline u'' + u = 0 & u'' - u = 0 \\ u(0) = 0 & u(0) = 0 \\ u'(0) = 1 & u'(0) = 1 \end{array} \tag{38}$$

$$\sin z = -i \cdot \sinh(i \cdot z),$$

where $z \in \mathbb{C}$ and the equations are understood in the sense of ordinary differential equations in the complex domain.

Since the function $|\cdot|:\mathbb{C}\to[0,+\infty)$ (complex modulus) is not analytic at $0\in\mathbb{C}$, we cannot work with (6) and (33), but we need to consider (25) and

$$\begin{gathered}\left(u'\right)^{p-2}u''-u^{p-1}=0,\\ u(0)=0,\\ u'(0)=1\end{gathered}\tag{39}$$

in our discussion in the complex domain. Thus, the direct analogy of the classical relations summarized in the table above for $p\neq 2$ is stated in the following table:

$$\begin{array}{c|c}\sin_p z & \sinh_p z\\ \hline \left(u'\right)^{p-2}u''+u^{p-1}=0 & \left(u'\right)^{p-2}u''-u^{p-1}=0\\ u(0)=0 & u(0)=0\\ u'(0)=1 & u'(0)=1\end{array}\tag{40}$$

$$\sin_p z\overset{?}{=}-i\cdot\sinh_p(i\cdot z),$$

where z belongs to some complex disc centred at $0\in\mathbb{C}$ with radius small enough such that both complex initial-value problems are solvable. However, it turns out (see below) that if we define $\sinh_p z$ as the solution (39), then the "p-analogies" of (2)-(3) are satisfied, but the "p-analogy" of the identity (1), that is,

$$\sin_p z=-i\cdot\sinh_p(i\cdot z),\tag{41}$$

is not satisfied in general. Our aim is to provide conditions when (41) holds as well.

Let us formalize the above-stated ideas. Denote

$$D_p\overset{\text{def}}{=}\left\{z\in\mathbb{C}:|z|<\gamma_p\right\}\tag{42}$$

an open disc in $\mathbb{C}$, where $\gamma_p>0$ is given radius. At first we prove unique solvability of (39) in D_p.

Lemma 16. *Let $p=m+2$, $m\in\mathbb{N}$. Then, there exists a complex disc D_p such that the initial-value problem in complex domain (39) has a unique solution on D_p.*

Proof. Using the substitution $u'=v$, we get the following first-order system:

$$\begin{gathered}u'=v,\\ v'=\frac{u^{p-1}}{v^{p-2}},\\ u(0)=0,\\ v(0)=1.\end{gathered}\tag{43}$$

By [15, Theorem 9.1] on page 76, the statement of the lemma follows. □

Now we can define $\sinh_p:D_p\to\mathbb{C}$ for any integer $p>2$.

Definition 17. Let $p=m+2$, $m\in\mathbb{N}$. The complex function $\sinh_p z$ is defined on D_p as the unique solution of the initial-value problem (39) and $\cosh_p z\overset{\text{def}}{=}\sinh_p' z$ for all $z\in D_p$.

Lemma 18. *Let $p=m+1$, $m\in\mathbb{N}$, and $r>0$ be such that the solution u of (25) is holomorphic on a disc $D_r=\{z\in\mathbb{C}:|z|<r\}$. Then, u satisfies the complex "p-hyperbolic" identity*

$$\left(u'(z)\right)^p-(u(z))^p=1\tag{44}$$

on the disc D_r.

The proof of Lemma 18 is analogous to the proof of Lemma 13 and thus it is omitted.

Remark 19. Les us note that the real-valued identity

$$\left(u(x)'\right)^p-(u(x))^p=1\tag{45}$$

for general $p>0$ already appeared in [22], where the formal Maclaurin power series expansion of the solution to this identity was treated. Interesting recurrence formula for the coefficients of the Maclaurin power series can be found there. It will be very interesting to use the following relations between $\sin_p z$ and $\sinh_p z$ to find the analogous recurrence formulas for $\sin_p z$.

Now we are ready to state main results of Section 5.

Theorem 20. *Let $p=4l+2$, $l\in\mathbb{N}$. Then,*

$$\sin_p z=-i\cdot\sinh_p(i\cdot z),\tag{46}$$

$$\cos_p z=\cosh_p(i\cdot z)\tag{47}$$

for all $z\in B_p$. Moreover,

$$\sinh_p z=\sum_{k=0}^{\infty}(-1)^k\cdot\alpha_k\cdot z^{kp+1}.\tag{48}$$

On the other hand, we have also the following surprising result.

Theorem 21. *Let $p=4l$, $l\in\mathbb{N}$. Then,*

$$\begin{gathered}\sin_p z=-i\cdot\sin_p(i\cdot z),\\ \cos_p z=\cos_p(i\cdot z)\end{gathered}\tag{49}$$

for all $z\in B_p$.

The statement of the previous theorem is closely related to similar result for p-hyperbolic functions.

Theorem 22. *Let $p=4l$, $l\in\mathbb{N}$. Then,*

$$\sinh_p z=-i\cdot\sinh_p(i\cdot z),\tag{50}$$

$$\cosh_p z=\cosh_p(i\cdot z)\tag{51}$$

for all $z\in D_p$.

Proof of Theorem 20. Let $p = 4l + 2$, $l \in \mathbb{N}$, and $u(z) = \sinh_p z$ be the unique solution of the initial-value problem (39) on D_p.

We show that $w(z) = -i \cdot u(i \cdot z)$ satisfies (25) on $D_p \cap B_p$. Due to uniqueness of solution of (25), the identity (46) must hold on $D_p \cap B_p$.

Indeed, plugging into the left-hand side of (25), we get

$$\begin{aligned}
&\left(w'(z)\right)^{p-2} w''(z) + w(z)^{p-1} \\
&= \left(\frac{d}{dz} w(z)\right)^{p-2} \frac{d^2}{dz^2} w(z) + w(z)^{p-1} \\
&= \left[\frac{d}{dz}(-i \cdot u(i \cdot z))\right]^{p-2} \frac{d^2}{dz^2}(-i \cdot u(i \cdot z)) \\
&\quad + (-i \cdot u(i \cdot z))^{p-1} \\
&= i \cdot \left[\frac{d}{d(i \cdot z)} u(i \cdot z)\right]^{p-2} \frac{d^2}{d(i \cdot z)^2} u(i \cdot z) \\
&\quad + (-i \cdot u(i \cdot z))^{p-1} \\
&= i \cdot \left[\frac{d}{ds} u(s)\right]^{p-2} \frac{d^2}{ds^2} u(s) - i \cdot (-i)^{p-2} \cdot u(s)^{p-1} \\
&= i \cdot \left(\left[\frac{d}{ds} u(s)\right]^{p-2} \frac{d^2}{ds^2} u(s) - (-i)^{p-2} \cdot u(s)^{p-1}\right) \\
&= i \cdot \left(\left[\frac{d}{ds} u(s)\right]^{p-2} \frac{d^2}{ds^2} u(s) - u(s)^{p-1}\right) = 0.
\end{aligned} \tag{52}$$

Note that for $p = 4l + 2$, $l \in \mathbb{N}$, $(-i)^{p-2} = 1$. The last equality then follows from (39). The right-hand side of (25) is zero. So we have verified that $w(z) = -i \cdot \sinh_p(i \cdot z)$ satisfies the differential equation in (39). The initial conditions of (25) are also satisfied by $w(z) = -i \cdot \sinh_p(i \cdot z)$, since $u(z) = \sinh_p z$ satisfies the initial conditions of (39).

Now it remains to show that $D_p = B_p$ and hence $\sin_p z = -i \cdot \sinh_p(i \cdot z)$ on B_p. To this end, let us write $\sinh_p z = \sum_{n=1}^{\infty} c_n \cdot z^n$ for yet unknown $c_n \in \mathbb{C}$. Then,

$$\sum_{n=1}^{\infty} a_n \cdot z^n = \sin_p z = -i \cdot \sinh_p(i \cdot z) = \sum_{n=1}^{\infty} c_n \cdot i^{n-1} \cdot z^n \tag{53}$$

on $D_p \cap B_p$. From here,

$$c_n = \frac{a_n}{i^{n-1}} = i^{3n+1} \cdot a_n. \tag{54}$$

Since $|i^{3n+1}| = 1$, $D_p = B_p$.

Now taking into account that $a_n = 0$ for $n-1$ not divisible by p, we immediately get $c_n = 0$ for $n-1$ not divisible by p. Now using our notation $\alpha_k = a_{kp+1}$, $k \in \mathbb{N}$, we find that

$$c_{kp+1} = (i)^{3(kp+1)+1} \alpha_k = (-1)^k \alpha_k \tag{55}$$

which establishes (48).

Equation (47) follows directly from $\cosh_p z = \sinh'_p z$ and (46). □

Proof of Theorem 21. Let $p = 4l$, $l \in \mathbb{N}$, and $u(z) = \sin_p z$. Now, plugging $w(z) = -i \cdot u(i \cdot z)$ into the left-hand side of (25), we proceed in the same way as in the proof of Theorem 20. The most important difference is that for $p = 4l$, $l \in \mathbb{N}$, $(-i)^{p-2} = -1$. Then, all following steps are analogous to those in proof of Theorem 20 with several obvious changes. □

Proof of Theorem 22. The proof is almost identical to the proof of Theorem 20 with obvious changes (cf. the proof of Theorem 21). □

6. Real Restrictions of the Complex Valued Solutions of (25) and (39) and Their Maximal Domains of Extension as Real Initial-Value Problems

Let us denote the restriction of the complex valued solution of (25) and (39) to the real axis by $\widehat{s}_p(x)$ and by $\widehat{sh}_p(x)$, respectively. Since the equation in (25) and (39) contains only integer powers of the solution and its derivatives, all coefficients in the equation are real, and the initial conditions in (25) and (39) are real, the value of $\sin_p z$ and $\sinh_p z$ must be a real number for $z = x + iy$ with $-\pi_p/2 < x < \pi_p/2$ and $-\gamma_p < x < \gamma_p$ and $y = 0$, respectively. Hence, $\widehat{s}_p(x)$ and $\widehat{sh}_p(x)$ attain only real values.

We start with the slightly more complicated case, which is $\widehat{sh}_p(x)$. Moreover, since the solution of (39) is an analytic function, it has the series representation

$$\sinh_p z = \sum_{k=1}^{\infty} c_k z^k, \quad z \in D_p, \tag{56}$$

where $c_k \in \mathbb{R}$, $k \in \mathbb{N}$ (note that $\sinh_p z$ must be a real number for any $z = x + iy$ with $-\gamma_p < x < \gamma_p$ and $y = 0$).

Now we show that $\widehat{sh}_p(x)$ solves (33) (in the sense of differential equations in real domain) for $p = 2(m+1)$, $m \in \mathbb{N}$, and does not solve (33) (in the sense of differential equations in real domain) for $p = 2m + 1$, $m \in \mathbb{N}$, and $x < 0$. For this purpose, we use an interesting consequence of omitting of the modulus function.

Theorem 23. *Let $p = 2m+1$, $m \in \mathbb{N}$, and $x \in (-\pi_p/2, \pi_p/2)$. Then,*

$$\widehat{sh}_p(x) = \begin{cases} \sinh_p x, & x \in \left[0, \dfrac{\pi_p}{2}\right), \\ \sin_p x & x \in \left(-\dfrac{\pi_p}{2}, 0\right). \end{cases} \tag{57}$$

Proof. For $x \in [0, \gamma_p)$, the statement of Theorem follows directly from the definition of real function $\sinh_p x$ and the facts that $\sinh_p x \geq 0$ and $\sinh'_p x \geq 0$ on $[0, \gamma_p)$.

By the definition, the function $\sin_p x$ is the unique solution of (6); that is,

$$-\left(\left|u'\right|^{p-2}u'\right)' - (p-1)\left|u\right|^{p-2}u = 0,$$
$$u(0) = 0, \tag{58}$$
$$u'(0) = 1.$$

Assume that $x \in (-\pi_p/2, 0)$. Then, $\sin_p x < 0$ and $\sin'_p x > 0$. Hence, we can rewrite (6) as

$$\left(u'\right)^{p-2}u'' - u^{p-1} = 0,$$
$$u(0) = 0, \tag{59}$$
$$u'(0) = 1,$$

which is formally (39) but here considered in real domain.

By Lemma 16, (39) has the unique solution on D_p. Its restriction to $(-\gamma_p, 0) \cap (-\pi_p/2, 0)$ clearly satisfies (59). Hence,

$$\sin_p x = \sum_{k=1}^{\infty} c_k x^k = \widehat{\mathrm{sh}}_p(x), \quad x \in \left(-\gamma_p, 0\right) \cap \left(-\frac{\pi_p}{2}, 0\right). \tag{60}$$

Moreover, $\sin_p x = \sum_{k=0}^{\infty} \alpha_k \cdot x \cdot |x|^{kp}$, which is generalized Maclaurin series of $\sin_p x$ (see [2, Remark 6.6, p. 125]) convergent on $(-\pi_p/2, \pi_p/2)$. For $(-\pi_p/2, 0)$, we obtain

$$\sum_{k=0}^{\infty} \alpha_k \cdot x \cdot |x|^{kp} = \sum_{k=0}^{\infty} \alpha_k \cdot (-1)^{kp} \cdot x^{kp+1} =: G(x). \tag{61}$$

Hence, the Maclaurin series $G(x)$ converges on $(-\pi_p/2, \pi_p/2)$ (but not towards $\sin_p x$ for $x > 0$). From (60) we get

$$\sum_{k=1}^{\infty} c_k x^k = G(x) \quad \text{on } \left(-\gamma_p, 0\right) \tag{62}$$

and using Proposition A.2 we obtain $\gamma_p = \pi_p/2$. □

Corollary 24. *Let $p = 2m+1$, $m \in \mathbb{N}$. Then, $\widehat{\mathrm{sh}}_p(x)$ does not solve (33) for $x \in (-\pi_p/2, 0)$.*

Proof. Since $\sin_p x \neq \sinh_p x$ for $x \neq 0$, the statement of Corollary follows directly from Theorem 23 and the uniqueness of solution of (33). □

Theorem 25. *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then, $\widehat{\mathrm{sh}}_p(x)$ solves (33) for $x \in (-\gamma_p, \gamma_p)$. In particular, $\gamma_p = \pi_p/2$ for $p = 4m+2$, $m \in \mathbb{N}$.*

Proof. Since p is even, we can drop the absolute values in (33) obtaining (59), which is formally (39) but here considered in real domain. Since $\sinh_p(z)$ solves (39) on D_p, its restriction $\widehat{\mathrm{sh}}_p(x)$ to $(-\gamma_p, \gamma_p)$ must solve (59) on $(-\gamma_p, \gamma_p)$.

For $p = 4m+2$, $m \in \mathbb{N}$, we get $\gamma_p = \pi_p/2$ by (46) in Theorem 20. □

Theorem 26. *Let $p = 2m+1$, $m \in \mathbb{N}$, and $x \in (-\pi_p/2, \pi_p/2)$. Then,*

$$\widehat{\mathrm{s}}_p(x) = \begin{cases} \sin_p x, & x \in \left[0, \dfrac{\pi_p}{2}\right), \\ \sinh_p x, & x \in \left(-\dfrac{\pi_p}{2}, 0\right). \end{cases} \tag{63}$$

Proof. The proof follows the same steps as the proof of Theorem 23 with obvious modifications. □

Now we will consider (25) and (39) as real-valued problems and find their maximal domains of extension. Let $\overset{\leftrightarrow}{\mathrm{s}}_p(x)$ and $\overset{\leftrightarrow}{\mathrm{sh}}_p(x)$ denote solutions with maximal domains of (25) and (39), respectively. We also define $\overset{\leftrightarrow}{\mathrm{c}}_p(x) \overset{\text{def}}{=} (d/dx)\overset{\leftrightarrow}{\mathrm{s}}_p(x)$ and $\overset{\leftrightarrow}{\mathrm{ch}}_p(x) \overset{\text{def}}{=} (d/dx)\overset{\leftrightarrow}{\mathrm{sh}}_p(x)$.

Theorem 27. *Let $p = 2m+1$, $m \in \mathbb{N}$. Then,*

$$\overset{\leftrightarrow}{\mathrm{s}}_p x = \begin{cases} \sinh_p x, & x \in (-\infty, 0), \\ \sin_p x, & x \in \left[0, \dfrac{\pi_p}{2}\right), \end{cases}$$
$$\overset{\leftrightarrow}{\mathrm{sh}}_p x = \begin{cases} \sin_p x, & x \in \left(-\dfrac{\pi_p}{2}, 0\right), \\ \sinh_p x, & x \in [0, +\infty). \end{cases} \tag{64}$$

Theorem 28. *Let $p = 2(m+1)$, $m \in \mathbb{N}$. Then,*

$$\overset{\leftrightarrow}{\mathrm{s}}_p x = \sin_p x, \quad x \in \mathbb{R},$$
$$\overset{\leftrightarrow}{\mathrm{sh}}_p x = \sinh_p x, \quad x \in \mathbb{R}. \tag{65}$$

Proof of Theorems 27 and 28. The solutions with maximal domain of extension are known for (6) and (33). The proof uses uniqueness of the solutions of real initial-value problems (6) and (33) and initial-value problems (25) and (39) considered in real domain and the fact that (6) and (33) can be rewritten as (25) and (39) depending on $u(x) \lesseqgtr 0$, $u'(x) \lesseqgtr 0$, and the parity of the positive integer p. The main ideas of how to combine these ingredients are contained in the proof of Theorem 23. □

It easily follows from Theorems 27 and 28 that $\overset{\leftrightarrow}{\mathrm{c}}_p(x)$ is defined on $(-\infty, \pi_p/2)$ for $p = 2m+1$ and on $\mathbb{R}$ for $p = 2(m+1)$, $m \in \mathbb{N}$. Similarly, $\overset{\leftrightarrow}{\mathrm{ch}}_p(x)$ is defined on $(-\pi_p/2, +\infty)$ for $p = 2m+1$ and on $\mathbb{R}$ for $p = 2(m+1)$, $m \in \mathbb{N}$. Moreover, functions $\overset{\leftrightarrow}{\mathrm{s}}_p(x)$ and $\overset{\leftrightarrow}{\mathrm{c}}_p(x)$ satisfy the complex p-trigonometric identity (29); that is,

$$\left(\overset{\leftrightarrow}{\mathrm{s}}_p(x)\right)^p + \left(\overset{\leftrightarrow}{\mathrm{c}}_p(x)\right)^p = 1. \tag{66}$$

```
In[1] := pip[p_] =
      Integrate[1/(1 - s^p)^(1/p), {s, 0, 1},
          Assumptions -> p > 1]
(* assigns definition to function pip which returns pi_p/2 *)
Out[1] = πcsc(π/p)/p
In[2] := s3[x_] = (u[x]/.
          NDSolve[
          {u'[x] == (v[x])^(1/2), v'[x] == - 2 u[x]^2,
            u[0] == 0, v[0] == 1},
          {u, v}, {x, -5, pip[3]}][[1]])
(* assigns definition to auxiliar function s3 *)
In[3] := sh3[x_] = (u[x]/.
          NDSolve[
          {u'[x] == (v[x])^(1/2), v'[x]== 2 u[x]^2,
            u[0]== 0, v[0] == 1},
          {u, v}, {x, -pip[3], 5}][[1]])
(* assigns definition to auxiliar function sh3 *)
In[4] := sin3[x_] = (u[x]/.
          NDSolve[{u'[x] == (Abs[v[x]])^(1/2) Sign[v[x]],
                    v'[x] == -2 Abs[u[x]]*u[x],
                    u[0] == 0, v[0] == 1},
                    {u,v}, {x, -4 pip[3], 4 pip[3]}][[1]])
(* assigns definition to auxiliar function sin3 *)
```

Pseudocode 2: *Mathematica*, v. 9.0 code. Code for `pip[p_]` computes $\pi_p/2$ for given argument p. Code for `s3[x_]` computes $\overleftrightarrow{s}_3(x)$ for $x \in (-5, \pi_3/2) \subsetneq (-\infty, \pi_3/2)$, code for `sh3[x_]` computes $\overleftrightarrow{\mathrm{sh}}_3(x)$ for $x \in (\pi_3/2, 5) \subsetneq (\pi_3/2, +\infty)$, and code for `sin3[x_]` computes real-valued function $\sin_3(x)$ for $x \in (-2\pi_3, 2\pi_3) \subsetneq \mathbb{R}$.

Analogously, functions $\overleftrightarrow{\mathrm{sh}}_p(x)$ and $\overleftrightarrow{\mathrm{ch}}_p(x)$ satisfy the complex p-hyperbolic identity (44); that is,

$$\left(\overleftrightarrow{\mathrm{ch}}_p(x)\right)^p - \left(\overleftrightarrow{\mathrm{sh}}_p(x)\right)^p = 1. \tag{67}$$

7. Visualizations

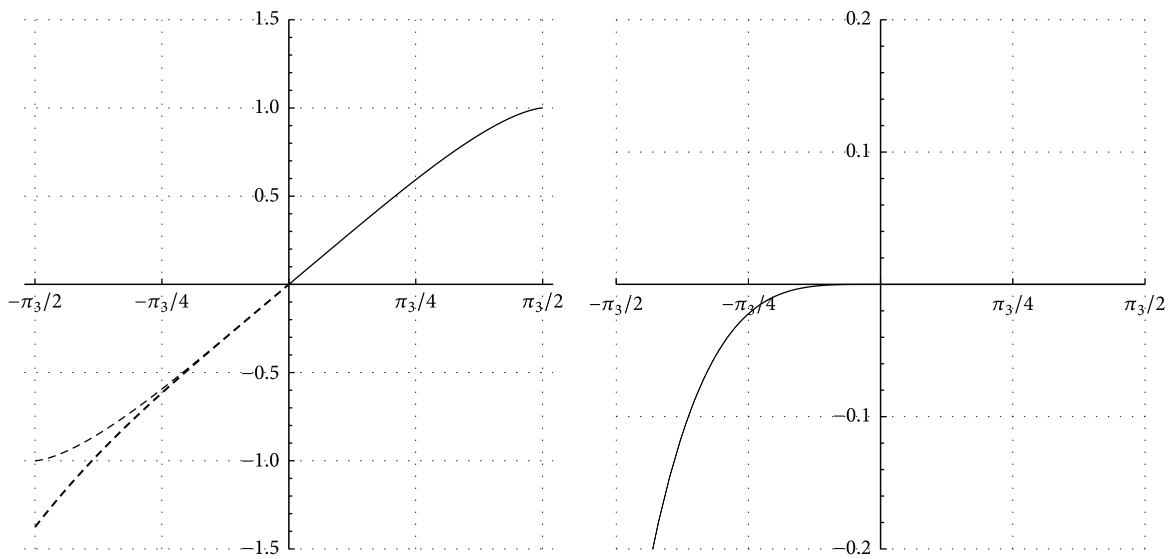
In this section, we provide visualizations of theoretical results from previous sections. To generate graphical output, we need to approximate special functions from previous sections numerically. Note that the standard numerical methods (available in *Mathematica* or *Matlab*®) can handle only initial-value problems on real intervals. Thus, in our numerical calculations we need to consider initial-value problems in real domain. This is not a problem for (6) and (33). For (25) and (39), we calculate either the partial sum of the Maclaurin series of solutions or we calculate functions $\overleftrightarrow{s}_p(x)$ and $\overleftrightarrow{\mathrm{sh}}_p(x)$ which come from real initial-value problems. In our graphical outputs, the solutions of real initial-value problems are numerically approximated by the `NDSolve` command of *Mathematica*, version 9.0. For the convenience of the reader, we provide some source code. In Pseudocode 2, we list source code for approximation of functions $\overleftrightarrow{s}_3(x)$, $\overleftrightarrow{\mathrm{sh}}_3(x)$, and $\sin_3 x$.

Figure 1 compares graphs of $\sin_3 x$ and $\overleftrightarrow{s}_3(x)$ for $x \in (-\pi_3/2, \pi_3/2)$. Figure 2 compares graphs of $\overleftrightarrow{s}_3(x)$ and $\widehat{M}_{3;28}(x)$ for $x \in (-\pi_3/2, \pi_3/2)$. Here, $\widehat{M}_{3;28}(x)$ is partial sum of $M_3(x)$ up to the order 28, which is

$$\begin{aligned}\widehat{M}_{3;28}(x) = x &- \frac{x^4}{12} - \frac{x^7}{252} - \frac{83\,x^{10}}{90\,720} - \frac{1\,817\,x^{13}}{7\,076\,160} \\ &- \frac{199\,691\,x^{16}}{2\,377\,589\,760} - \frac{12\,324\,719\,x^{19}}{406\,567\,848\,960} \\ &- \frac{22\,008\,573\,061\,x^{22}}{1\,878\,343\,462\,195\,200} \\ &- \frac{107\,355\,387\,043\,x^{25}}{22\,540\,121\,546\,342\,400} \\ &- \frac{89\,152\,153\,354\,993\,x^{28}}{44\,304\,862\,911\,490\,621\,440}.\end{aligned} \tag{68}$$

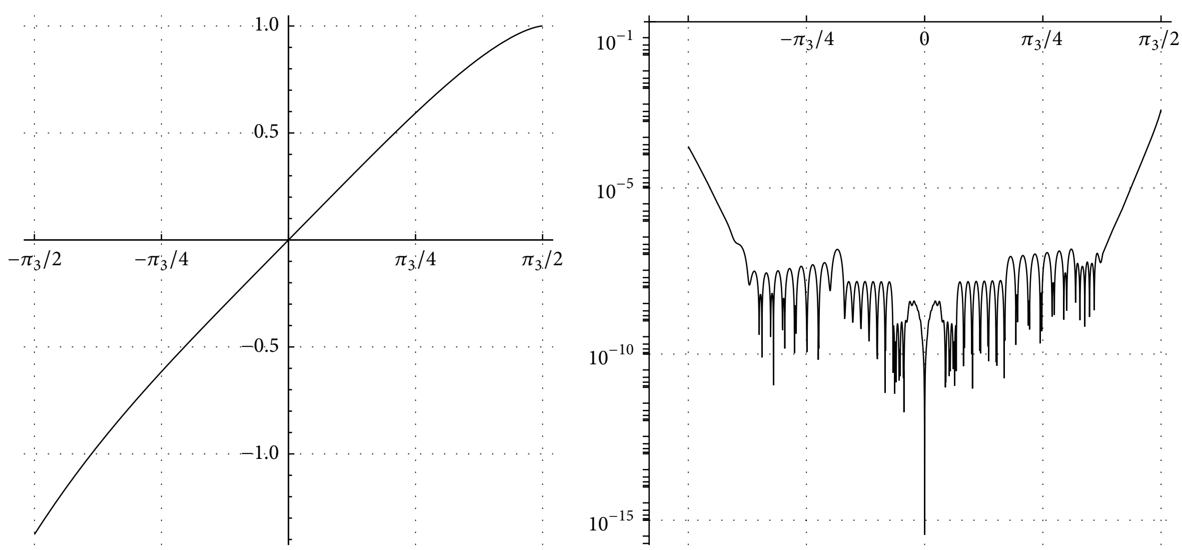
Since the difference $|\overleftrightarrow{s}_3(x) - \widehat{M}_{3;28}(x)|$ varies in order of several magnitudes throughout the radius of convergence $\pi_3/2$, we use logarithmic scale on the vertical axis.

We can also compute the functions $\overleftrightarrow{s}_p(x)$ and $\overleftrightarrow{\mathrm{sh}}_p(x)$ by inverting formulas (8) and (36) for $\arcsin_p x$ and $\mathrm{argsinh}_p x$, respectively. It turns out that this approach provides more precision than solving differential equation and enables computing values of $\overleftrightarrow{c}_p(x)$ and $\overleftrightarrow{\mathrm{ch}}_p(x)$ using identity (66) and identity (67), respectively. We provide sample code for computing $\overleftrightarrow{\mathrm{sh}}_3(x)$ for $x \in (\pi_3/2, 5) \subsetneq (\pi_3/2, +\infty)$ in



(a) (b)

Figure 1: (a) Real function $\sin_3 x$ (short-dashed line) versus $\overset{\leftrightarrow}{\mathrm{s}}_3(x)$ (dashed line). (b) Plot of $\sin_3 x - \overset{\leftrightarrow}{\mathrm{s}}_3(x)$.



(a) (b)

Figure 2: (a) $\overset{\leftrightarrow}{\mathrm{s}}_3(x)$ versus $\widehat{M}_{3;28}$ (the partial sum of $M_3(x)$ up to the order 28, which is (68)). (b) The logarithmic plot of $|\overset{\leftrightarrow}{\mathrm{s}}_3(x) - M_3(x)|$.

Pseudocode 3. Analogously, we wrote a code for computing $\overset{\leftrightarrow}{\mathrm{sh}}_4(x)$, $\overset{\leftrightarrow}{\mathrm{sh}}_{30}(x)$, $\overset{\leftrightarrow}{\mathrm{sh}}_{31}(x)$, $\overset{\leftrightarrow}{\mathrm{s}}_3(x)$, $\overset{\leftrightarrow}{\mathrm{s}}_4(x)$, $\overset{\leftrightarrow}{\mathrm{s}}_{30}(x)$, and $\overset{\leftrightarrow}{\mathrm{s}}_{31}(x)$.

In the same way, as we defined real function $\operatorname{argsinh}_p x$ by (36), we can define complex valued function $\operatorname{argsinh}_p z$ by

$$\operatorname{argsinh}_p z \overset{\text{def}}{=} \int_0^z \frac{1}{(1+s^p)^{1/p}} \mathrm{d}s = z\,{}_2F_1\left(\frac{1}{p}, \frac{1}{p}, 1+\frac{1}{p}, -z^p\right), \quad z \in \mathscr{D} \subsetneq \mathbb{C} \tag{69}$$

for any $p = m+2$. Note that the integrand has poles at z satisfying $1 + z^p = 0$. Thus, the function $\operatorname{argsinh}_p z$ is not an entire function. In particular, for $p = 2m+1$, there is a pole at $z = -1$. In Figure 3, we compare graphs of $\operatorname{argsinh}_p x$ for $p = 2, 3, 31$ and $x \in (-3, 3) \subsetneq \mathbb{R}$, with the restriction of the complex valued function $\operatorname{argsinh}_p z$ for $p = 2, 3, 31$, where $z = x + iy$, where $y = 0$, $x \in (-3, 3)$ for $p = 2$ and $x \in (-1, 3)$ for $p = 3, 31$, and with $\operatorname{argsinh}_p x = \operatorname{argsinh}_p z$ for $p = 2, 4, 30$, $x \in (-3, 3)$, $y = 0$, and $z = x + iy$.

In Figures 4, 5, and 6, we compare graphs of real-valued functions: $\sinh x$, $\sinh_4 x$, $\sinh_{30} x$, $\cosh x$, $\cosh_4 x$, $\cosh_{30} x$

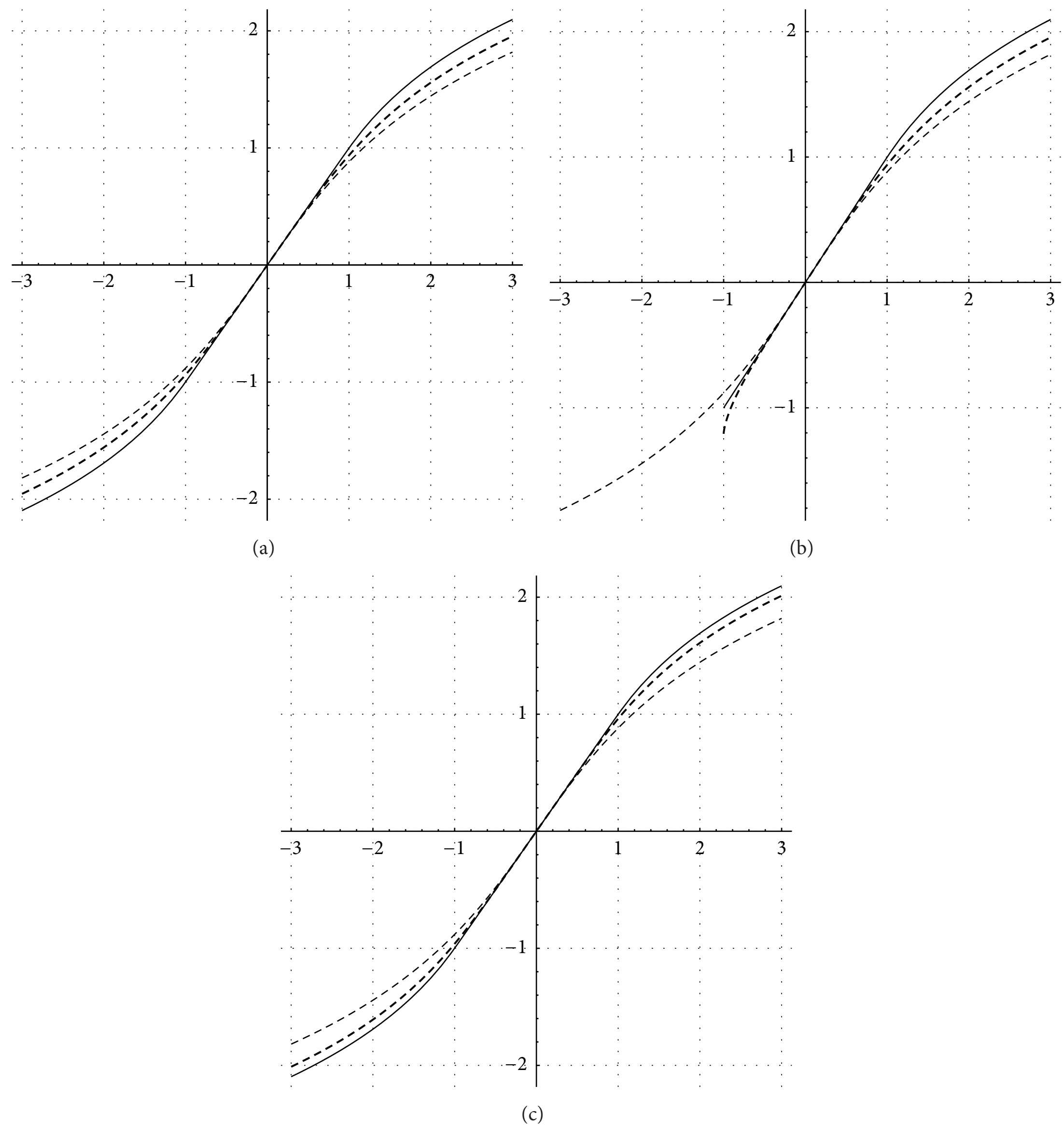
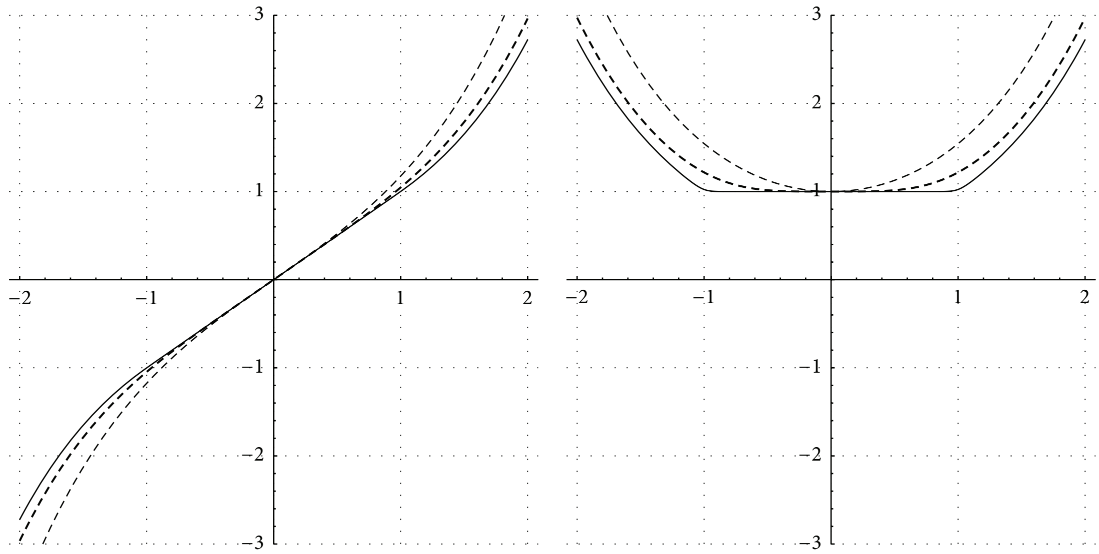


Figure 3: (a) $\operatorname{argsinh}_p x (= x\ {}_2F_1(1/p, 1/p, 1+1/p, -|x|^p))$ for $p = 2, 3, 31$, and $x \in (-3, 3)$. (b) $\operatorname{argsinh}_p z (= z\ {}_2F_1(1/p, 1/p, 1+1/p, -z^p))$ for $p = 2, 3, 31$. Here, $z = x + iy$, where $y = 0$, $x \in (-3, 3)$ for $p = 2$, and $x \in (-1, 3)$ for $p = 3, 31$. (c) $\operatorname{argsinh}_p x = \operatorname{argsinh}_p z$ for $p = 2, 4, 30$, $x \in (-3, 3)$, $y = 0$, and $z = x + iy$.

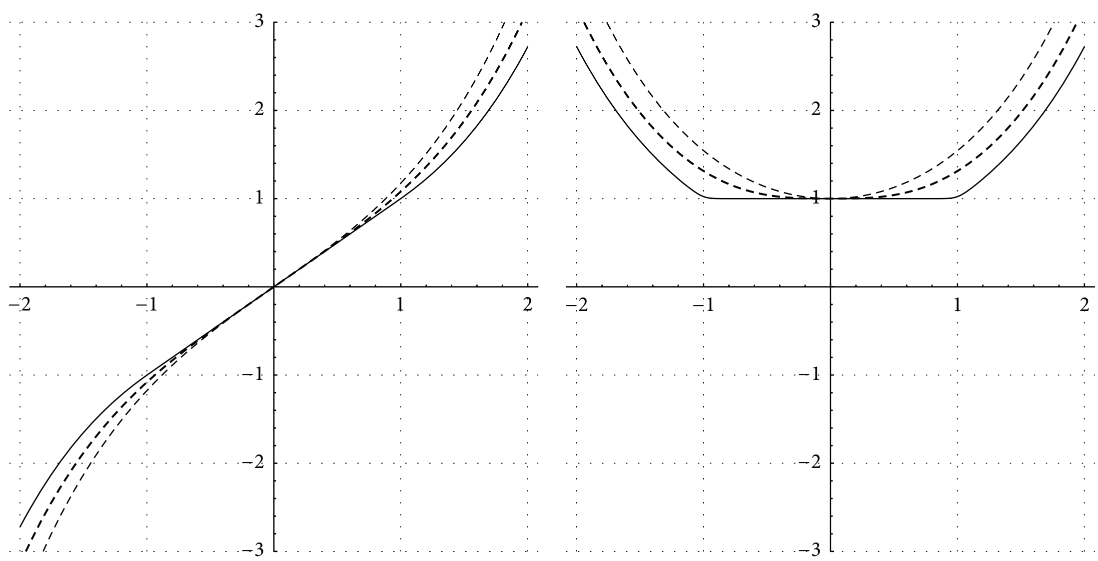
```
In[2] := auxAgSh3[s_?NumberQ] :=
      s Hypergeometric2F1[1/3, 1/3, 1 + 1/3, -s^3]
(* assigns definition to auxiliar function auxAgSh3 *)
In[3] := sh3inv[x_?NumericQ] :=
      (s/. FindRoot[auxAgSh3[s] - x, {s, 0, -1, 20}])
(* assigns definition to function sh3inv *)
In[4] := Plot[
      sh3inv[x], {x, -pip[3], 2},
      PlotStyle -> Thick, PlotRange -> {-2, 7}]
(* plots the graph *)
```

Pseudocode 3: *Mathematica* v. 9.0 code. Function `sh3inv` returns values of $\overset{\leftrightarrow}{\mathrm{sh}}_3(x)$ for $x \in (\pi_3/2, 5) \subsetneq (\pi_3/2, +\infty)$ using inversion of the formula (36) for $\operatorname{argsinh}_3 x$. This approach provides more precision than solving differential equation.



(a) (b)

FIGURE 4: (a) $\sinh x$ (short-dashed line), $\sinh_4 x$ (dashed line), and $\sinh_{30} x$ (solid line). (b) $\cosh x$ (short-dashed line), $\cosh_4 x$ (dashed line), and $\cosh_{30} x$ (solid line).



(a) (b)

FIGURE 5: (a) $\sinh x$ (short-dashed line), $\sinh_3 x$ (dashed line), and $\sinh_{31} x$ (solid line). (b) $\cosh x$ (short-dashed line), $\cosh_3 x$ (dashed line), and $\cosh_{31} x$ (solid line).

(see Figure 4), $\sinh x$, $\sinh_3 x$, $\sinh_{31} x$, $\cosh x$, $\cosh_3 x$, $\cosh_{31} x$ (see Figure 5), $\overleftrightarrow{\mathrm{sh}}_3(x)$, $\overleftrightarrow{\mathrm{sh}}_{31}(x)$, $\overleftrightarrow{\mathrm{ch}}_3(x)$, and $\overleftrightarrow{\mathrm{ch}}_{31}(x)$ (see Figure 6).

Figure 7 illustrates the relation between $\sin_3 x$, $\sinh_3 x$, $\overleftrightarrow{\mathrm{s}}_3(x)$, and $\overleftrightarrow{\mathrm{sh}}_3(x)$, which are due to Theorem 27.

It follows from identity (66) and identity (67) that the pairs of functions $(\overleftrightarrow{\mathrm{s}}_p(x), \overleftrightarrow{\mathrm{c}}_p(x))$ and $(\overleftrightarrow{\mathrm{sh}}_p(x), \overleftrightarrow{\mathrm{ch}}_p(x))$, respectively, are parametrizations of Lamé curves restricted to the first and fourth quadrant, see [23, Book V, Chapter V, pp. 384–407] and [24]. Since the Lamé curves are frequently used in geometrical modeling, we provide graphical comparison of the Lamé curves and phase portraits of initial-value problems (25) and (39) in real domain on Figures 8 and 9, respectively.

8. Conclusion

We have discussed real and complex approaches of how to define generalized trigonometric and hyperbolic functions.

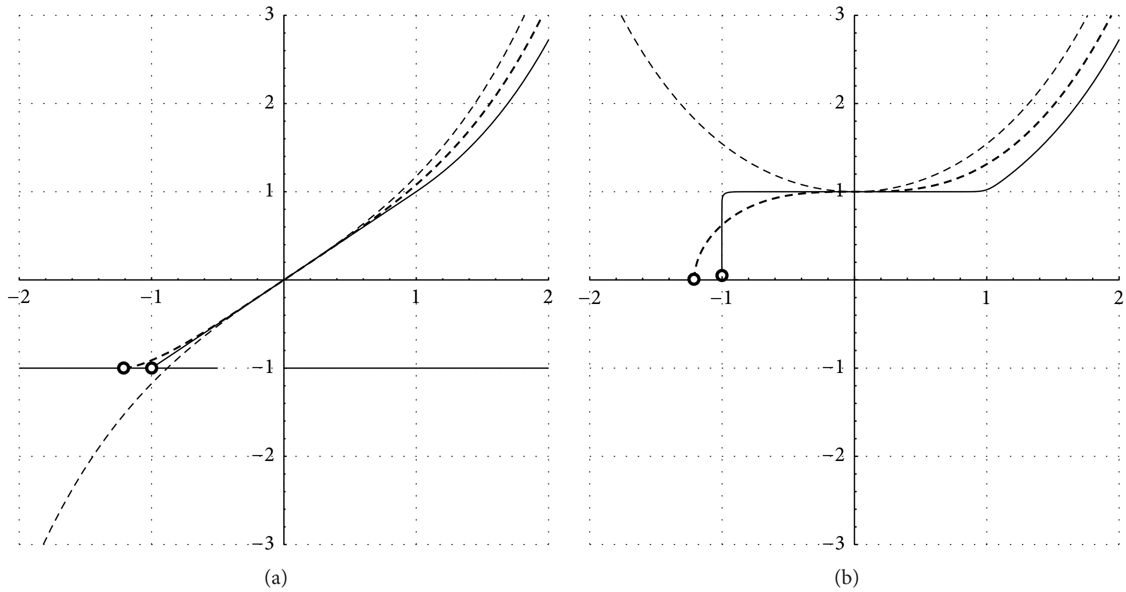


FIGURE 6: (a) $\sinh x$ (short-dashed line), $\overset{\leftrightarrow}{\mathrm{sh}}_3(x)$ (dashed line), and $\overset{\leftrightarrow}{\mathrm{sh}}_{31}(x)$ (solid line). (b) $\cosh x$ (short-dashed line), $\overset{\leftrightarrow}{\mathrm{ch}}_3(x)$ (dashed line), and $\overset{\leftrightarrow}{\mathrm{ch}}_{31}(x)$ (solid line). Note that $\overset{\leftrightarrow}{\mathrm{sh}}_p(x)$ and $\overset{\leftrightarrow}{\mathrm{ch}}_p(x)$ are defined on $(-\pi_p/2, +\infty)$ for integer $p > 1$ odd.

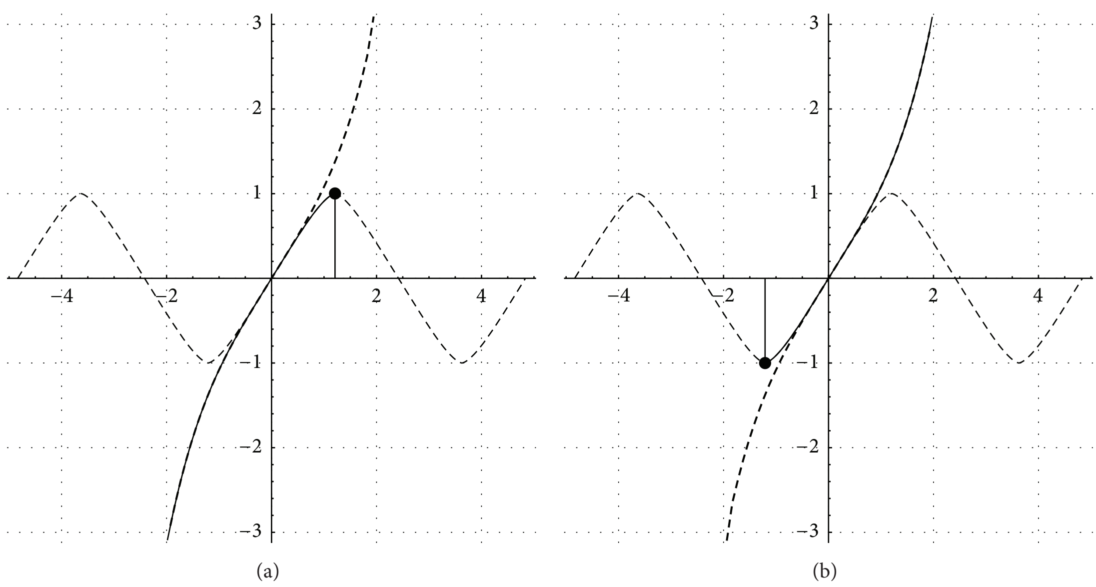


FIGURE 7: (a) $\sin_3 x$, $\sinh_3 x$, and $\overset{\leftrightarrow}{\mathrm{s}}_3(x)$. (b) $\sin_3 x$, $\sinh_3 x$, and $\overset{\leftrightarrow}{\mathrm{sh}}_3(x)$.

The real approach is motivated by minimization of Rayleigh quotient (see, e.g., [1, Equation (3.4), p. 51] and references therein):

$$\frac{\int_0^{\pi_p} \left|u'\right|^p \mathrm{d}x}{\int_0^{\pi_p} |u|^p \mathrm{d}x} \tag{70}$$

in $W_0^{1,p}(0,\pi_p)$. This leads to (4) with $\lambda = p-1$ and to initial-value problem (6) in turn. Thus, from this point of view, the solution of (6) and its derivative can be seen as natural generalizations of the functions sine and cosine. Unfortunately, presence of absolute value in (6) does not allow for extension to complex domain for general $p > 1$.

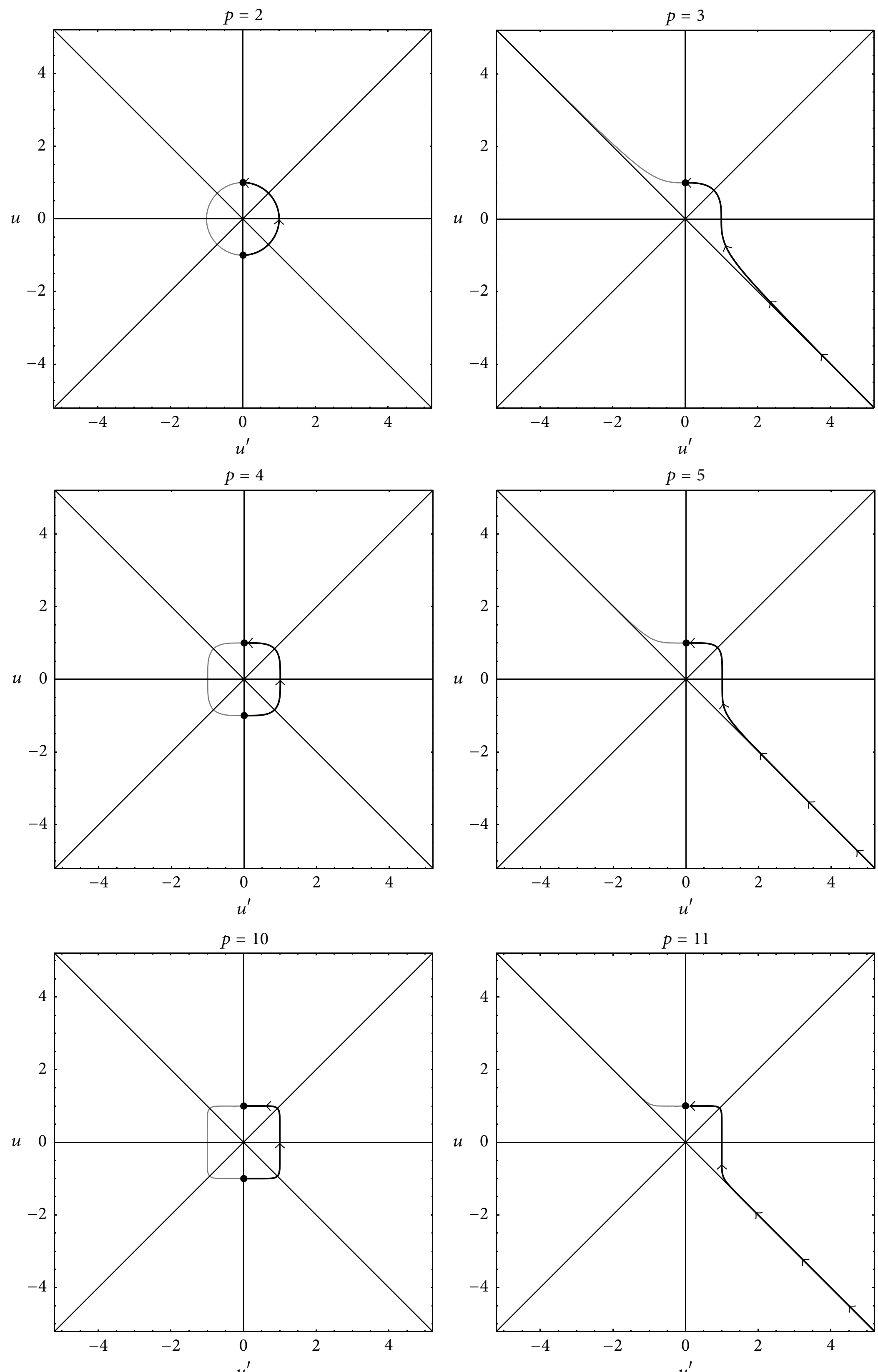


FIGURE 8: Bold lines: dependence of $u = \overset{\leftrightarrow}{\mathrm{s}}_p(x)$ on $u' = \overset{\leftrightarrow}{\mathrm{c}}_p(x)$. Both bold and thin lines: dependence of restriction to real axes of derivative of solution of (29) on the restriction to real axes of the solution itself (Lamé curves).

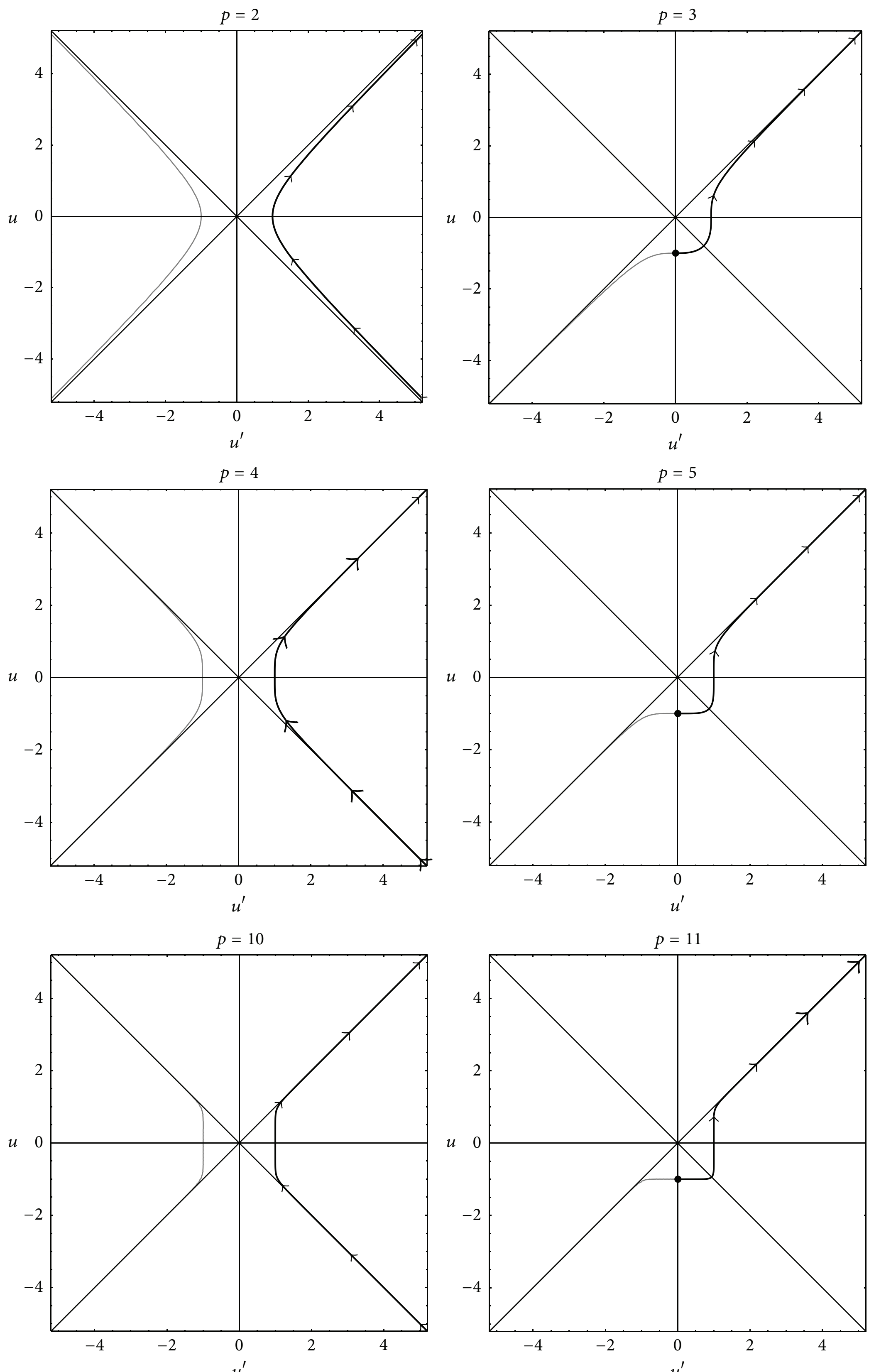


FIGURE 9: Bold lines: dependence of $u = \overleftrightarrow{\mathrm{sh}}_p(x)$ on $u' = \overleftrightarrow{\mathrm{ch}}_p(x)$. Both bold and thin lines: dependence of restriction to real axes of derivative of solution of (44) on the restriction to real axes of the solution itself (Lamé curves).

Table 1: Summary of results according to discussed functions and their domain.

$p \in$	Function	Domain	IVP	Results
$(1, +\infty)$	$\sin_p x$	$\mathbb{R}$	(6)	Propositions 2–5; Theorems 6, 7, 8, 23, and 26–28
$(1, +\infty)$	$\sinh_p x$	$\mathbb{R}$	(33)	Theorems 23 and 26–28
$\mathbb{N}\setminus\{1\}$	$\sin_p z$	B_p	(25)	Propositions 11 and 12; Lemma 13; Theorems 6, 7, 8, 9, 15, 20, and 21
$\mathbb{N}\setminus\{1\}$	$\sinh_p z$	D_p	(39)	Lemmas 16 and 18; Theorems 20 and 22

Table 2: Interrelations between p-trigonometric and p-hyperbolic functions. Note that the relations must be symmetric.

	$\sin_p x$	$\sinh_p x$	$\sin_p z$	$\sinh_p z$
$\sin_p x$	—	(57), (63)–(64)	—	—
$\sinh_p x$	sym.	—	—	—
$\sin_p z$	sym.	sym.	(49)	(46)
$\sinh_p z$	sym.	sym.	sym.	(50)

It was shown in [2] that functions $\sin_p x$ are real analytic functions for *any even integer* $p > 2$. Moreover, there is no need to write absolute value in (6) for $x \in [-\pi_p/2, \pi_p/2]$ provided $p > 2$ is an *even integer*.

It turns out that the relation between the real and complex approach is not as smooth as in the classical case $p = 2$. Thus, we summarize our results in Tables 1 and 2.

We also discussed the Lamé curves, which are important curves in geometrical modeling. We hope this will stimulate interest in p-trigonometric and p-hyperbolic functions among the geometric-modeling community.

Appendix

A. Proof of Theorem 6

We will use the following result to prove Theorem 6.

Proposition A.1 (see [15], Theorem 2.4b on p. 97). *Let the formal power series* $F \overset{\text{def}}{=} a_1x + a_2x^2 + \cdots$, $a_1 \neq 0$, *have a positive radius of convergence. The inversion* F^{-1} *of* F *then also has positive radius of convergence.*

Let us note that the term reversion of series is used in [15] instead of inversion of series (see [15], p. 46).

Proposition A.2 (see [25], Theorem 16.6 on p. 352). *If the sum of two power series in the variable* $z - z_0$ *coincides on a set of points* E *for which* z_0 *is a limit point and* $z_0 \notin E$, *then identical powers of* $z - z_0$ *have identical coefficients; that is, there is a unique power series in the variable* $z - z_0$ *which has given sum on the set* E.

Proof of Theorem 6. Let us remember that $T_p(x)$ is given by (19), which is the formal inverse of $\arcsin_p x$ and $M_p(x)$ is given by formula (20). The idea is to prove that there exists $\delta_p > 0$ small enough such that both series $T_p(x)$ and $M_p(x)$ have the sum equal to uniquely defined value $\sin_p x$ at any $x \in [0, \delta_p)$. Then, $T_p(x) = M_p(x)$ on $[0, \delta_p)$ and the assumptions of Proposition A.2 are satisfied on $z_n = \delta_p/(n+1)$. It follows that $T_p(x)$ has identical coefficients as $M_p(x)$ has and so $T_p(x)$ also converges on $(-\pi_p/2, \pi_p/2)$.

By Propositions 4 and 5, $M_p(x)$ converges to $\sin_p x$ for $p > 2$ even on $(-\pi_p/2, \pi_p/2)$ and for $p > 1$ odd on $[0, \pi_p/2)$, respectively. It remains to show that there exists $\sigma_p > 0$ such that

$$T_p(x) = \sin_p x \tag{A.1}$$

on $[0, \sigma_p)$. Since $T_p(x)$ is defined as formal inverse of $\arcsin_p x$, (A.1) holds on domain of convergence of $T_p(x)$. Since for $x \in [0, 1]$

$$\begin{aligned} \arcsin_p x &= x \cdot {}_2F_1\left(\frac{1}{p}, \frac{1}{p}, 1 + \frac{1}{p}, x^p\right) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(k + 1/p)}{(k \cdot p + 1) \cdot k! \cdot \Gamma(1/p)} x^{kp+1}, \end{aligned} \tag{A.2}$$

where right-hand side series has radius of convergence equal to 1, hence the existence of $\sigma_p \leq \pi_p/2$ is provided by Proposition A.1 and $\delta_p = \sigma_p$. □

B. Proof of Theorem 7

By Theorem 6, $M_p = T_p$. Hence, we can prove the statement of Theorem 7 for T_p instead of M_p.

Assume by contradiction that there exists $a_n \neq 0$ for some n such that $n - 1$ is *not* divisible by p. For this purpose, let us denote by b_j the jth coefficient of the Maclaurin series of $\arcsin_p$ corresponding to jth power. From (18), we get

$$b_j = \begin{cases} \dfrac{\Gamma(l + 1/p)}{(lp + 1) \cdot l! \cdot \Gamma(1/p)} & \text{if } j = l \cdot p + 1 \text{ for some } l \in \mathbb{N} \cup \{0\}, \\ 0 & \text{otherwise.} \end{cases} \tag{B.1}$$

Since T_p is the formal inverse series of

$$\arcsin_p x = \sum_{j=1}^{\infty} b_j \cdot x^j, \tag{B.2}$$

the coefficients a_n can be computed from the formula

$$a_n = \frac{1}{n \cdot b_1^n} \sum_{m_1, m_2, \ldots, m_{n-1}} (-1)^{m_1+m_2+\cdots+m_{n-1}} \cdot \frac{n(n+1)\cdot \ldots \cdot (n-1+m_1+m_2+m_3+\cdots+m_i+\cdots+m_{n-1})}{m_1! m_2! m_3! \ldots} \cdot \left(\frac{b_2}{b_1}\right)^{m_1} \cdot \left(\frac{b_3}{b_1}\right)^{m_2} \left(\frac{b_4}{b_1}\right)^{m_3} \cdots \left(\frac{b_{i+1}}{b_1}\right)^{m_i} \cdots \left(\frac{b_n}{b_1}\right)^{m_{n-1}}, \tag{B.3}$$

where the summation is taken over all $m_1, m_2, m_3, \ldots \in \mathbb{N} \cup \{0\}$ such that

$$m_1 + 2m_2 + 3m_3 + \cdots + im_i \cdots (n-1)\, m_{n-1} = n - 1, \tag{B.4}$$

and if $m_i = 0$, then the corresponding term $(b_{i+1}/b_1)^{m_i}$ is dropped from the product on the last line of (B.3).

Let us note that this procedure is fully described in [14], p. 411–413 and it requires that $b_1 \neq 0$. Note that

$$b_1 = \frac{\Gamma(1/p)}{1 \cdot 1! \cdot \Gamma(1/p)} = 1 \tag{B.5}$$

by (B.1).

Now, let us fix $m_1, m_2, m_3, \ldots, m_{n-1}$ satisfying (B.4). If $b_i = 0$ and $m_i \neq 0$ for at least one $i = 1, 2, 3, \ldots, n-1$, the summand of sum (B.3) corresponding to $m_1, m_2, m_3, \ldots$ equals 0. Taking into account (B.1), $b_i = 0$ whenever $i \neq lp+1$ for any $l \in \mathbb{N} \cup \{0\}$. This leads us to conclusion that nonzero terms in (B.3) can be formed only from m_i's where i is divisible by p. Thus, (B.4) implies that the following equation must be satisfied:

$$p \cdot m_p + 2p \cdot m_{2p} + 3p \cdot m_{3p} + \cdots + l \cdot p \cdot m_{l \cdot p} + \cdots + k \cdot p \cdot m_{k \cdot p} = n - 1, \tag{B.6}$$

where

$$k = \left\lfloor \frac{n-1}{p} \right\rfloor. \tag{B.7}$$

But here the left-hand side is a multiple of p while the right-hand side $n-1$ is not divisible by p by our assumption. This is a contradiction.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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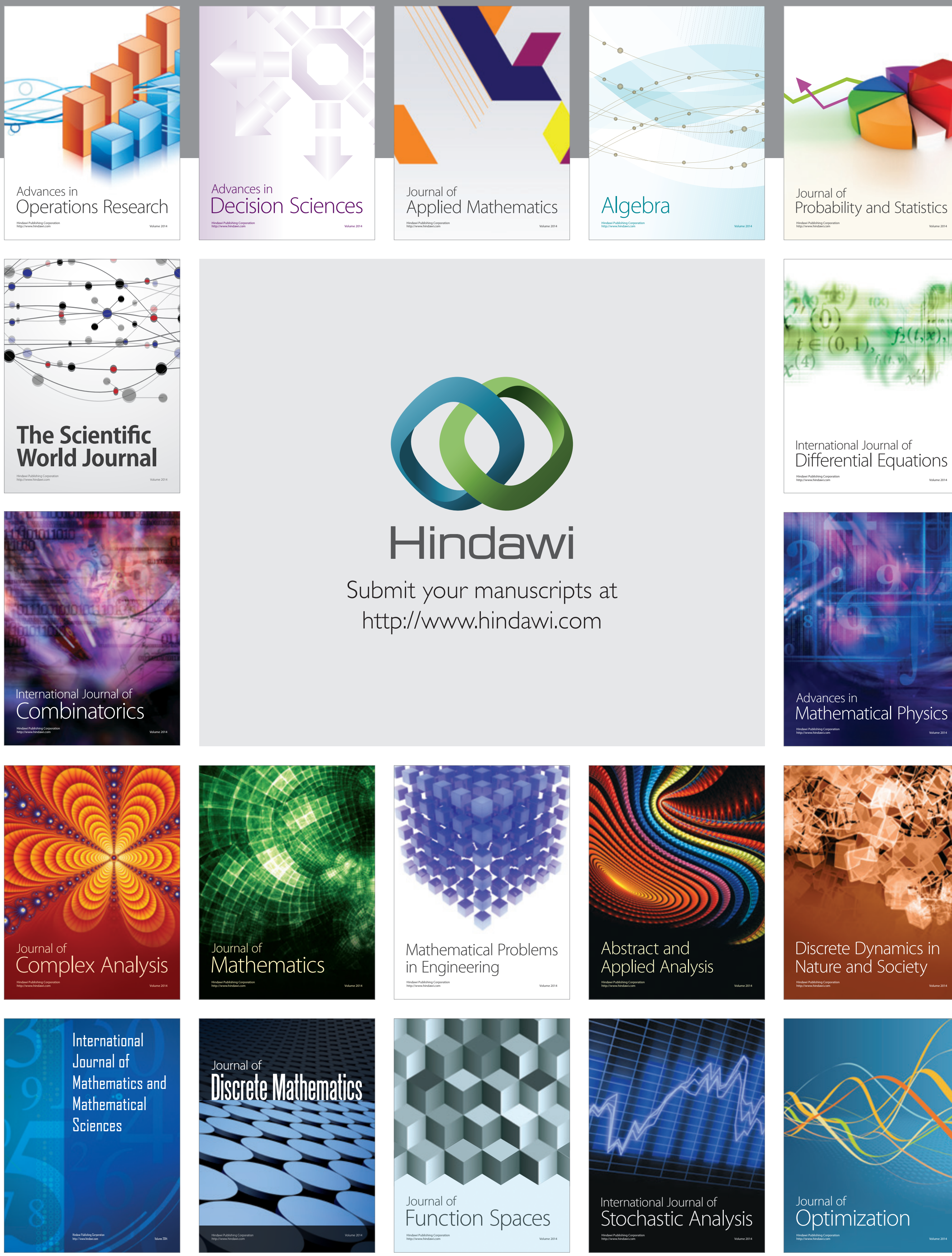
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Appendix A8

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MACLAURIN SERIES FOR $\sin_p$ WITH p AN INTEGER GREATER THAN 2

LUKÁŠ KOTRLA

Abstract. We find an explicit formula for the coefficients of the generalized Maclaurin series for $\sin_p$ provided $p > 2$ is an integer. Our method is based on an expression of the n-th derivative of $\sin_p$ in the form

$$\sum_{k=0}^{2^{n-2}-1} a_{k,n} \sin_p^{p-1}(x) \cos_p^{2-p}(x)\,, \quad x \in (0, \frac{\pi_p}{2}),$$

where $\cos_p$ stands for the first derivative of $\sin_p$. The formula allows us to compute the nonzero coefficients

$$\alpha_n = \frac{\lim_{x\to 0+} \sin_p^{(np+1)}(x)}{(np+1)!}\,.$$

1. Introduction

Let us consider initial value problem

$$\begin{gathered} -(|u'|^{p-2}u')' - (p-1)|u|^{p-2}u = 0\,, \\ u(0) = 0\,, \quad u'(0) = 1\,, \end{gathered} \tag{1.1}$$

where $p > 1$ is a given parameter and $u\colon \mathbb{R} \to \mathbb{R}$ is a function such that $u \in C^1(\mathbb{R})$ and $|u'|^{p-2}u' \in C^1(\mathbb{R})$. It is known that the solution of (1.1) exists and is unique (see Elbert [9]). Since the pioneering work of del Pino, Elgueta and Manásevich [8], this solution is usually denoted by $\sin_p$. Note that it generalizes the *sine* function which is the unique solution of (1.1) for $p = 2$. Moreover, the function $\sin_p$ also satisfies the generalized trigonometric identity

$$|\sin_p(x)|^p + |\cos_p(x)|^p = 1\,, \quad x \in \mathbb{R}\,, \tag{1.2}$$

where $\cos_p(x) := \frac{\mathrm{d}}{\mathrm{d}x}\sin_p(x)$, which resembles the classical trigonometric identity for $p = 2$. We also define

$$\pi_p := 2\int_0^1 \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s = \frac{2\pi}{p\sin(\pi/p)}.$$

Let us note that the function $\sin_p$ is odd, $2\pi_p$-periodic, and $\sin_p(x) = \sin_p(\pi_p - x)$ (see, e.g., [9]). These properties are frequently used when the function $\sin_p$ is

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evaluated numerically. In fact, any evaluation of $\sin_p$ at an arbitrary point $x \in \mathbb{R}$ can be reduced to an evaluation of $\sin_p$ at a point in the interval $[0, \pi_p/2]$.

It turns out that the system of functions $\{\sin_p(k\pi_p\, x)\}_{k=1}^{+\infty}$ has applications in approximation theory, see Binding et al. [4] for pioneering work in this direction. Indeed, there exists $p_0 > 1$ such that, for $p > p_0$, $\{\sin_p(k\pi_p\, x)\}_{k=1}^{+\infty}$ forms a Riesz basis of $L^2(0,1)$ and a Schauder basis of $L^r(0,1)$ for any $1 < r < +\infty$. The approach from [4] was corrected and improved by Bushell and Edmunds [7] where the value p_0 was established as the solution of the transcendental equation

$$\frac{2\pi}{p_0 \sin(\pi/p_0)} = \frac{2\pi^2}{\pi^2 - 8}\,.$$

Boulton and Lord [5] use the basis $\{\sin_p(k\pi_p\, x)\}_{k=1}^{+\infty}$ in their numerical implementation of the Galerking method for finding an approximate solution to the boundary-initial value problem

$$\begin{gathered} \frac{\partial u}{\partial t}(x,t) - \frac{\partial}{\partial x}\Big(\big|\frac{\partial u}{\partial x}(x,t)\big|^{p-2}\frac{\partial u}{\partial x}(x,t)\Big) = g(x) \\ u(x,0) = 0\,, \quad x \in (0,1)\,, \\ u(0,t) = u(1,t) = 0 \quad t > 0\,, \end{gathered} \tag{1.3}$$

where $g \in L^2(0,1)$. It appears that this choice of basis leads to very accurate results using only few terms of this basis. However, a main drawback of the Galerkin method in [5] is the evaluation of the values of the function $\sin_p$ on $[0, \pi_p/2]$. In [5], the inverse function of $\sin_p$,

$$\operatorname{arcsin}_p(x) := \int_0^x \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s\,, \quad x \in [0,1]\,, \tag{1.4}$$

is used for that purpose. The function $\sin_p$ on $[0, \pi_p/2]$ is then evaluated using numerical inverse of the function arcsin_p, which is a very time consuming process. Since the problem (1.3) and its generalizations appear in various applications, see e.g. Smreker [23] (bulding of wells), Leibenson [15] (extraction of oil and natural gas), Wilkins [24] (bulding of rock-fill dams), Aronsson et al. [1], Evans et al. [10] (sandpile growth), Kuijper [13] (image analysis), and Bermejo et al. [3] (climatology), it is important to find a more efficient numerical implementations of $\sin_p$. Last but not least, the generalized Prüfer transform using $\sin_p$ and its derivative appears to be a very efficient theoretical tool in studying various initial and/or boundary value problems for quasilinears equation of the type (or some of its generalization)

$$-(|u'|^{p-2}u')' - q(x)|u|^{p-2}u = f(x)$$

(under various conditions on q and f) see, e.g., [9], Reichel and Walter [21], and/or Benedikt and Girg [2]. In Brown and Reichel [6], a numerical method based on the Prüfer transform was proposed. Again the main drawback the method was the lack of an efficient numerical implementation of $\sin_p$. To address the issue in this paper we obtain explicit formulas for coefficients of the Maclaurin series of $\sin_p$. This is very difficult task in general and we are not able to deal with this problem for all $p > 1$. As a starting point for further research in this direction, we provide such formulas for any integer p bigger than 2. Let us note that even this partial result can already be used in practical applications, since (1.3) with $p \to +\infty$ is considered as a model for sandpile growth (see [1] and [10] for more details).

More precisely, our goal is to find Maclaurin series for $\sin_p$ provided p is even and generalized Maclaurin series for $\sin_p$ provided p is odd. Generalized Maclaurin series is defined as

$$\sum_{n=0}^{+\infty} \alpha_n x|x|^{rn}, \quad r \geq 1\,.$$

Peetre [20] conjectured that the radius of convergence of generalized Maclaurin series for $\sin_p$ is $\pi_p/2$ for any $p > 1$. Local convergence of generalized Maclaurin series was studied in Paredes and Uchiyama [19]. Peetre's conjecture [20] was proved in Girg and Kotrla [11] for when $p > 2$ is an integer. It remains to find the coefficients of the (generalized) Maclaurin series. One can employ (1.4) and follow the ideas presented in Lang and Edmunds [14]. Since

$$\operatorname{arcsin}_p(x) = \int_0^x \frac{1}{(1-s^p)^{1/p}}\,\mathrm{d}s = x\,{}_2F_1\Big(\frac{1}{p}, \frac{1}{p}, 1+\frac{1}{p}; x^p\Big), \quad x \in [0,1)\,,$$

where ${}_2F_1(a,b,c;z)$ is Gauss's hypergeometric function,

$$\operatorname{arcsin}_p(x) = \sum_{k=0}^{+\infty} \frac{\Gamma(k+\frac{1}{p})}{(kp+1)\Gamma(\frac{1}{p})} \frac{x^{kp+1}}{k!}\,, \tag{1.5}$$

where Γ stands for the gamma function. We can obtain desired coefficients using the well-known procedure for inverting power series (see, e.g., Morse and Feshbach [18, p. 411 - 413]). Our aim is to derive the coefficients independently of the inverse function. It was shown in Girg and Kotrla [12] that the nonzero coefficients correspond only to the monomials x^{kp+1}, $k \in \mathbb{N}$. Then

$$\sin_p(x) = \sum_{n=0}^{+\infty} \frac{\sin_p^{(np+1)}(0)}{(np+1)!} x^{np+1} \quad x \in \Big(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\Big)\,,$$

for p even. In addition, it was proved in [12] that the series

$$\sum_{n=0}^{+\infty} \frac{\lim_{x\to 0+} \sin_p^{(np+1)}(x)}{(np+1)!} x^{np+1}$$

coincides on $[0, \pi_p/2)$ with the series obtained by formal inversion of (1.5) provided p odd. Hence, by the oddness of $\sin_p$,

$$\sin_p(x) = \sum_{n=0}^{+\infty} \frac{\lim_{x\to 0+} \sin_p^{(np+1)}(x)}{(np+1)!} x|x|^{np}\,, \quad x \in \Big(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\Big).$$

It remains then to find an explicit formula for

$$\alpha_n := \frac{1}{(np+1)!} \lim_{x\to 0+} \sin_p^{(np+1)}(x), \quad p \in \mathbb{N},\ p > 2\,.$$

Notation: In the presented paper, the symbol $\prod$ represents the product of a (possibly finite) sequence of terms as usual. In addition, we define

$$\prod_{i=j_1}^{j_2} b_i = 1$$

for any sequence b_i provided $j_1 = j_2 + 1$.

Theorem 1.1. *Let $p > 2$ be an integer and*

$$\sin_p(x) = \sum_{n=0}^{+\infty} \alpha_n x|x|^{np}, \quad x \in (-\frac{\pi_p}{2}, \frac{\pi_p}{2}). \tag{1.6}$$

Then $\alpha_0 = 1$, $\alpha_1 = -\frac{1}{p(p+1)}$, and for $n \geq 2$,

$$\begin{aligned}
\alpha_n = {} & \frac{(-1)^n}{(np+1)!} \sum_{\substack{i_1=1 \\ i_1 \neq p-1}}^{p} \sum_{\substack{i_2=i_1+1 \\ i_2 \neq 2p-1}}^{2p} \dots \\
& \sum_{\substack{i_{n-1}=i_{n-2}+1 \\ i_{n-1} \neq (n-1)p-1}}^{(n-1)p} \Big[\prod_{m_1=1}^{i_1-1} (p-1-(m_1-1))\Big]\big(1-(p-1-(i_1-1))\big) \\
& \times \Big[\prod_{m_2=i_1+1}^{i_2-1} (2(p-1)-(m_2-2))\Big](1-(2(p-1)-(i_2-2)))\dots \\
& \times \Big[\prod_{m_{n-1}=i_{n-2}+1}^{i_{n-1}-1} ((n-1)(p-1)-(m_{n-1}-(n-1)))\Big]\big(1-((n-1) \\
& \times (p-1)-(i_{n-1}-(n-1)))\big)[n(p-1)-(i_{n-1}-n+1)]!
\end{aligned} \tag{1.7}$$

The proof of Theorem 1.1 is based on a method of rewriting higher derivatives of $\sin_p$ introduced in [11]. The method is described again in Section 2 for the convenience of the reader. Theorem 1.1 is proved in Section 3.

Let us note that the above-mentioned definitions of $\sin_p$ and $\cos_p$ are not the only ones found in the literature (see, e.g., Lindqvist [16]).

2. Higher order derivatives

Let us state some basic notation from formal languages.

Definition 2.1. (Salomaa and Soittola [22], I.2, p. 4, and/or Manna [17], p. 2–3, p. 47, and p. 78) An *alphabet* (denoted by V) is a finite nonempty set of letters. A *word* (denoted by w) over an alphabet V is a finite string of zero or more letters from the alphabet V. The word consisting of zero letters is called the *empty word*. The set of all words over an alphabet V is denoted by V^* and the set of all nonempty words over an alphabet V is denoted by V^+. For strings w_1 and w_2 over V, their juxtaposition w_1w_2 is called *catenation* of w_1 and w_2, in operator notation $\mathrm{cat} : V^* \times V^* \to V^*$ and $\mathrm{cat}(w_1, w_2) = w_1w_2$. We also define the length of the word w, in operator notation $\mathrm{len} : V^* \to \mathbb{N} \cup \{0\}$, which for a given word w yields the number of letters in w when each letter is counted as many times as it occurs in w. We also use *reverse function* $\mathrm{rev} : V^* \to V^*$ which reverses the order of the letters in any word w (see [17, p. 47, p. 78]).

We consider the alphabet $V = \{0, 1\}$ and the set of all nonempty words V^+. Thus words in V^+ are, e.g.,

$$\text{“}\mathtt{0}\text{”}, \text{“}\mathtt{1}\text{”}, \text{“}\mathtt{01}\text{”}, \text{“}\mathtt{10}\text{”}, \text{“}\mathtt{11}\text{”} \cdots .$$

For instance, cat(“`1110`”, “`011`”) = “`1110011`”, and

$$\mathrm{rev}(\text{“}\mathtt{010011000}\text{”}) = \text{“}\mathtt{000110010}\text{”}, \ \mathrm{len}(\text{“}\mathtt{010011000}\text{”}) = 9 .$$

TABLE 1. Differentiability of $\sin_p(x)$

p, k	x in $(0, \pi_p/2)$	$(-\pi_p/2, \pi_p/2)$	$\mathbb{R}$
$p = 2$	C^∞	C^∞	C^∞
$p = 2k$, $k \in \mathbb{N} \setminus \{1\}$	C^∞	C^∞	C^1
$p = 2k+1$, $k \in \mathbb{N}$	C^∞	C^p	C^1
$p \in \mathbb{R} \setminus \mathbb{N}$, $p > 2$	C^∞	$C^{\lceil p \rceil}$	C^1
$p \in (1, 2)$	C^∞	C^2	C^2

Let $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$, $0 \le k \le 2^{m-2} - 1$ and $(k)_{2,n-2}$ be the string of bits of length $m - 2$ which represents binary expansion of k (it means, e.g., for $k = 3$ and $m = 5$, $(3)_{2,5-2} =$ "`011`").

The differentiability of $\sin_p(x)$ at $x = 0$ was studied in [11] leading to the results in Table 1.

In particular, $\sin_p(\cdot) \in C^\infty(0, \pi_p/2)$. Let

$$T := \{a \sin_p^q(\cdot) \cos_p^{1-q}(\cdot) : \ a, q \in \mathbb{R}\},$$

and $\mathrm{D_s} \colon T \to T$ and $\mathrm{D_c} \colon T \to T$ be defined as follows:

$$\mathrm{D_s}\, a \sin_p^q(\cdot) \cos_p^{1-q}(\cdot) = \begin{cases} aq \sin_p^{q-1}(\cdot) \cos_p^{1-(q-1)}(\cdot), & q \neq 0, \\ 0, & q = 0, \end{cases} \tag{2.1}$$

and

$$\mathrm{D_c}\, a \sin_p^q(\cdot) \cos_p^{1-q}(\cdot) = \begin{cases} -a(1-q) \sin_p^{q+p-1}(\cdot) \cos_p^{1-(q+p-1)}(\cdot), & q \neq 1, \\ 0, & q = 1. \end{cases} \tag{2.2}$$

Finally, we define $\mathrm{D}_{k,m}$ in two steps.

Step 1 We create an ordered $(m-2)$-tuple $d_{k,m-2} \in \{\mathrm{D_s}, \mathrm{D_c}\}^{m-2}$ (cartesian product of sets $\{\mathrm{D_s}, \mathrm{D_c}\}$ of length $m - 2$) from $\mathrm{rev}((k)_{2,m-2})$ such that for $1 \le i \le m - 2$, $d_{k,m-2}$ contains $\mathrm{D_s}$ on the i-th position if $\mathrm{rev}((k)_{2,n-2})$ contains "`0`" on the i-th position, and $d_{k,m}$ contains $\mathrm{D_c}$ on the i-th position if $\mathrm{rev}((k)_{2,m-2})$ contains "`1`" on the i-th position (it means, e.g., for $k = 3$, and $m = 5$, we obtain $d_{3,5-2} = (\mathrm{D_c}, \mathrm{D_c}, \mathrm{D_s})$).

Step 2 We define $\mathrm{D}_{k,m}$ as the composition of operators $\mathrm{D_s}, \mathrm{D_c}$ in the order they appear in the ordered $m - 2$-tuple $d_{k,m-2}$ (it means, e.g., for $k = 3$, and $m = 5$, we obtain $\mathrm{D}_{3,5} = (\mathrm{D_c} \circ \mathrm{D_c} \circ \mathrm{D_s})$).

Let us point out that it is possible to recover the index k from the positions of $\mathrm{D_c}$ in $\mathrm{D}_{k,m}$. We will denote by $j(k) \ge 0$ the number of $\mathrm{D_c}$ in $\mathrm{D}_{k,m}$ and, if $j(k) \neq 0$, we denote by $i_1, i_2, \dots, i_{j(k)}$ its positions counted from back (i.e., in the order of application of $\mathrm{D_s}$ and/or $\mathrm{D_c}$). Then

$$k = 2^{m-2-(i_1-1)} + 2^{m-2-(i_2-1)} + \dots + 2^{m-2-(i_{j(k)}-1)}. \tag{2.3}$$

If $j(k) = 0$, $k = 0$.

Definition 2.1 and the definition of $\mathrm{D}_{k,m}$ are taken from [11] in almost unchanged form for the convenience of the reader who is not familiar with our previous work. However, the rewriting diagrams in [11], where the construction of $\mathrm{D}_{k,m}$ is visualized, are not included here.

It follows from the first derivative of the p-trigonometric identity (1.2) that

$$\sin_p^{(2)}(x) = -\sin_p^{p-1}(x)\cos_p^{2-p}(x)\,, \quad x \in (0, \frac{\pi_p}{2})\,. \tag{2.4}$$

Note that $\sin_p(x) > 0$ and $\cos_p(x) > 0$ for $x \in (0, \pi_p/2)$. Hence, we can use $\mathrm{D}_{k,n}$ to express

$$\begin{aligned} \sin_p^{(m)}(x) &= \sum_{k=0}^{2^{m-2}-1} \mathrm{D}_{k,m} \sin_p^{(2)}(x) \\ &= \sum_{k=0}^{2^{m-2}-1} \mathrm{D}_{k,m}(-1)\sin_p^{p-1}(x)\cos_p^{2-p}(x)\,, \quad x \in (0, \frac{\pi_p}{2})\,, \end{aligned} \tag{2.5}$$

for $m > 2$ be a positive integer. Let us explain the procedure for $m = 3$ at first. In that case

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}(-1)\sin_p^{p-1}(x)\cos_p^{2-p}(x) \\ &= (-1)(p-1)\sin_p^{p-2}(x)\cos_p^{3-p}(x) \\ &\quad + (-1)(2-p)\sin_p^{p-1}(x)\cos_p^{1-p}(x)\sin_p^{(2)}(x) \\ &= (-1)(p-1)\sin_p^{p-2}(x)\cos_p^{3-p}(x) \\ &\quad + (-1)(1-(p-1))\sin_p^{p-1+p-1}(x)\cos_p^{1-(p-1+p-1)}(x) \\ &= \mathrm{D_S}\sin_p^{(2)}(x) + \mathrm{D_c}\sin_p^{(2)}(x) \end{aligned}$$

for any $x \in (0, \pi_p/2)$ by the definition of $\mathrm{D_S}$ and $\mathrm{D_c}$. The proof of (2.5), which proceeds by induction, can be found in [11, Lemma 4.5, p. 110].

There are two special cases in composing the symbolic operators for $p \in \mathbb{N}, p > 2$, which can be used for reducing of terms in (2.5).

Case 1 Assume that there exists $k \in \mathbb{N} \cup \{0\}$, $k \le 2^{m-2} - 1$ such that

$$\mathrm{D}_{k,m}\sin_p^{(2)}(\cdot) = a\sin_p(\cdot)\cos_p^0(\cdot)\,. \tag{2.6}$$

The further application of $\mathrm{D_c}$ is meaningless since it produce 0 by (2.2). The situation (2.6) occurs, e.g., after $p-2$ applications of $\mathrm{D_S}$ on $\sin_p^{(2)}(\cdot)$.

Case 2 If there exists $k \in \mathbb{N}$, $k \le 2^{m-2} - 1$, such that

$$\mathrm{D}_{k,m}\sin_p^{(2)}(\cdot) = a\sin_p^0(\cdot)\cos_p^1(\cdot)\,, \tag{2.7}$$

then the application of $\mathrm{D_s}$ produces 0, see (2.1). The situation (2.7) occurs, e.g., after $p-1$ applications of $\mathrm{D_S}$ on $\sin_p^{(2)}(\cdot)$. This is the essential argument in the proof that the exponent q is always nonnegative, see [11, Lemma 4.6, p.113] for more details.

3. Proof of main result

Proof of Theorem 1.1. It follows from [12, Theorem 6, p. 3] that

$$\alpha_n = \frac{1}{(np+1)!}\lim_{x\to 0+}\sin_p^{(np+1)}(x) \tag{3.1}$$

for p odd, and it is obvious that (3.1) is valid for p even, since $\sin_p(\cdot)$ belongs to $C^\infty(-\pi_p/2, \pi_p/2)$ in this case. We obtain $\alpha_0 = \lim_{x\to 0+}\cos_p(x) = 1$ for $p \in \mathbb{N}$,

$p > 2$. Let $n \in \mathbb{N}$ and $x \in (0, \pi_p/2)$. By [11, Lemma 4.5, p. 110]

$$\begin{aligned}\sin_p^{(np+1)}(x) &= \sum_{k=0}^{2^{np-1}-1} -\mathrm{D}_{k,np+1}\sin_p^{p-1}(x)\cos_p^{2-p}(x)\\ &= \sum_{k=0}^{2^{np-1}-1} a_{k,np+1}\sin_p^{q_{k,np+1}}(x)\cos_p^{1-q_{k,np+1}}(x)\,,\end{aligned}$$

where $a_{k,np+1} \in \mathbb{R}$ and $q_{k,np+1} \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned}\lim_{x\to 0+}\sin_p^{(np+1)}(x) &= \sum_{k=0}^{2^{np-1}-1} a_{k,np+1}\lim_{x\to 0+}\sin_p^{q_{k,np+1}}(x)\cos_p^{1-q_{k,np+1}}(x)\\ &= \sum_{\substack{k=0\\ q_{k,np+1}=0}}^{2^{np-1}-1} a_{k,np+1}\,.\end{aligned} \tag{3.2}$$

Our first aim is to describe $k \in \mathbb{N} \cup \{0\}$, $0 \le k \le 2^{np-1} - 1$ such that $q_{k,n} = 0$. We use the alphabet $V = \{0, 1\}$ introduced in Definition 2.1 for this purpose and we employ the formula

$$q_{k,np+1} = j(k)(p-1) + (np-1-j(k))(-1) + p - 1 \tag{3.3}$$

proved in [11, Lemma 4.5, p. 11)]. Let us recall that $j(k)$ is the number of occurrences of $\mathrm{D_c}$ in $\mathrm{D}_{k,np+1}$. It follows from the condition $q_{k,n} = 0$ that $j(k) = n - 1$. Then $k = 0$ for $n = 1$ which implies

$$\begin{aligned}\lim_{x\to 0+}\sin_p^{(p+1)}(x) &= -\lim_{x\to 0+}\mathrm{D}_{0,p+1}\sin_p^{p-1}(x)\cos_p^{2-p}(x)\\ &= -\lim_{x\to 0+}(p-1)!\sin_p^{0}(x)\cos_p^{1}(x) = -(p-1)!\end{aligned} \tag{3.4}$$

by (2.1), the definition of $\mathrm{D_s}$. Substituting (3.4) into (3.1) we obtain

$$\alpha_1 = -\frac{1}{p(p+1)}\,.$$

We will assume $n \ge 2$ in the rest of the proof. Then

$$k = 2^{np-1-(i_1-1)} + 2^{np-1-(i_2-1)} + \dots + 2^{np-1-(i_{n-1}-1)}$$

by (2.3). Moreover,

$$\forall s \in \mathbb{N},\ 1 \le s \le n-1\colon\ i_s \le sp\,. \tag{3.5}$$

Indeed, let there exist $s_0 \in \mathbb{N}$, $1 \le s_0 \le n - 1$: $i_{s_0} > s_0 p$ and let

$$k_1 := \begin{cases}0 & \text{for } s_0 = 1\,,\\ 2^{np-1-(i_1-1)} + 2^{np-1-(i_2-1)} + \dots + 2^{np-1-(i_{s_0-1}-1)} & \text{for } s_0 \ge 2\,.\end{cases}$$

The binary expansion $(k_1)_{2,i_{s_0}-1}$ of k_1 defines $\mathrm{D}_{k_1,i_{s_0}+1}$ by the composition of the symbolic operators $\mathrm{D_s}$ and/or $\mathrm{D_c}$ taking the first $i_{s_0} - 1$ operators from $\mathrm{D}_{k,np+1}$ (in the order of its application). The exponent $q_{k_1,i_{s_0}+1}$ in $\mathrm{D}_{k_1,i_{s_0}+1}\sin_p^{(2)}(\cdot)$ satisfies

$$q_{k_1,i_{s_0}+1} = (s_0-1)(p-1) + (i_{s_0} - 1 - s_0 + 1)(-1) + p - 1 = s_0 p - i_{s_0} < 0$$

by (3.3) and the assumption $i_{s_0} > s_0 p$. Since $q_{k,np+1} \geq 0$ for any $n \in \mathbb{N} \cup \{0\}$ and all $0 \leq k \leq 2^{np-1} - 1$ provided $p > 1$ be an integer, we get the contradiction. Hence,

$$\alpha_n = \frac{1}{(np+1)!} \sum_{i_1=1}^{p} \sum_{i_2=i_1+1}^{2p} \cdots \sum_{i_{n-1}=i_{n-2}+1}^{(n-1)p} a_{k_0,np+1}\,, \tag{3.6}$$

where $k_0 = 2^{np-1-(i_1-1)} + 2^{np-1-(i_2-1)} + \ldots + 2^{np-1-(i_{n-1}-1)}$.

It remains to express $a_{k_0,np+1}$ as the polynomial in p. We will apply $\mathrm{D_s}$ and/or $\mathrm{D_c}$ on $\sin_p^{(2)}(\cdot)$ recursively. Let us denote by a_i the coefficient and q_i the exponent obtained by i steps of recursion. The base cases are $a_0 = -1$ and $q_0 = p - 1$ by (2.4) and inductive clauses are given by (2.1) and (2.2), i.e.,

$$a_{i+1} = \begin{cases} q_i \cdot a_i & \text{if } \mathrm{D_s} \text{ is applied}\,, \\ -(1-q_i)a_i & \text{if } \mathrm{D_c} \text{ is applied}\,, \end{cases} \tag{3.7}$$

and

$$q_{i+1} = \begin{cases} q_i - 1 & \text{if } \mathrm{D_s} \text{ is applied}\,, \\ q_i + p - 1 & \text{if } \mathrm{D_c} \text{ is applied}\,. \end{cases} \tag{3.8}$$

It follows from the definition of $\mathrm{D}_{k_0,np+1}$ that the operator $\mathrm{D_s}$ is applied in the first $i_1 - 1$ steps of recursion. It means that

$$a_{i_1-1} = -(p-1)(p-2)\cdots(p-1-(i_1-2)) \quad \text{and} \quad q_{i_1-1} = p-1-(i_1-1)\,.$$

by (2.1). Applying the operator $\mathrm{D_c}$ on the next position we have

$$\begin{gathered} a_{i_1} = -(p-1)(p-2)\cdots(p-1-(i_1-2))(-1)(1-(p-1-(i_1-1))), \\ q_{i_1} = 2(p-1)-(i_1-1)\,. \end{gathered}$$

Applying $i_2 - 1 - i_1$ times the operator $\mathrm{D_s}$ and we obtain

$$\begin{aligned} a_{i_2-1} = {} & -(p-1)(p-2)\cdots(p-1-(i_1-2))(-1)(1-(p-1-(i_1-1))) \\ & \times (2(p-1)-(i_1-1))\cdots(2(p-1)-(i_2-3)) \end{aligned}$$

and

$$q_{i_2-1} = 2(p-1)-(i_2-2)$$

(provided $i_2 > i_1 + 1$). The application of $\mathrm{D_c}$ leads to

$$\begin{aligned} a_{i_2} = {} & -(p-1)(p-2)\cdots(p-1-(i_1-2))(-1)(1-(p-1-(i_1-1))) \\ & \times (2(p-1)-(i_1-1))\cdots(2(p-1)-(i_2-3))(-1)(1-(2(p-1)-(i_2-2))) \end{aligned}$$

and

$$q_{i_2} = 3(p-1)-(i_2-2)\,.$$

It follows by the recursive application of $\mathrm{D_s}$ and/or $\mathrm{D_c}$ that

$$\begin{aligned} a_{i_{n-1}} = (-1)\Big[& \prod_{m_1=1}^{i_1-1} (p-1-(m_1-1))\Big](-1)\big(1-(p-1-(i_1-1))\big) \\ & \times \Big[\prod_{m_2=i_1+1}^{i_2-1} (2(p-1)-(m_2-2))\Big](-1)(1-(2(p-1)-(i_2-2)))\cdots \\ & \times \Big[\prod_{m_{n-1}=i_{n-2}+1}^{i_{n-1}-1} ((n-1)(p-1)-(m_{n-1}-(n-1)))\Big] \end{aligned}$$

$$\times (-1)(1-((n-1)(p-1)-(i_{n-1}-(n-1))))$$

and

$$q_{i_{n-1}} = n(p-1)-(i_{n-1}-n+1)\,,$$

where i_{n-1} is the last position of $\mathrm{D_c}$. Since the remaining symbolic operators in $\mathrm{D}_{k_0,np+1}$ are $\mathrm{D_s}$ and $q_{k_0,np+1}=0$ by (3.2), we finally get

$$\begin{aligned}
&a_{k_0,np+1}\\
&= (-1)\Big[\prod_{m_1=1}^{i_1-1}(p-1-(m_1-1))\Big](-1)\big(1-(p-1-(i_1-1))\big)\\
&\times\Big[\prod_{m_2=i_1+1}^{i_2-1}(2(p-1)-(m_2-2))\Big](-1)(1-(2(p-1)-(i_2-2)))\cdots\\
&\times\Big[\prod_{m_{n-1}=i_{n-2}+1}^{i_{n-1}-1}((n-1)(p-1)-(m_{n-1}-(n-1)))\Big]\\
&\times(-1)(1-((n-1)(p-1)-(i_{n-1}-(n-1))))\\
&\times\Big[n(p-1)-(i_{n-1}-n+1)\Big]!
\end{aligned}\tag{3.9}$$

Substituting (3.9) into (3.6) we obtain desired formula (1.7). The positions $i_s = sp-1$ are excluded in (1.7) since it produce zero due to the terms $1-(s(p-1)-(i_s-s))$ in product (3.9) (see Case 1 in Section 2). $\square$

4. Concluding remarks

Remark 4.1. The proof of Theorem 1.1 provides a procedure to generate any coefficient α_n, $n\ge 2$ of Maclaurin series (1.6) for $\sin_p$, when $p>2$ is an integer. It is convenient to generate all vectors $v\in\{0,1\}^{np-1}$ with exactly $n-1$ occurrences of "1"s, which satisfy condition (3.5), i.e.,

$$\forall s\in\mathbb{N},\ 1\le s\le n-1\ :\ i_s\le sp\,.$$

Let us note that i_s is the position of "1" in v. Then the recursions (3.7) with $a_0=-1$ and (3.8) with $q_0=p-1$ can to applied by all possible vectors v to obtain the coefficient $a_v\in\mathbb{R}$. Let us remind that zero and one means that $\mathrm{D_s}$ and $\mathrm{D_c}$ is applied, respectively, and the order of application $\mathrm{D_s}$ and/or $\mathrm{D_c}$ is reversed. Finally, the resulting coefficient α_n is given as sum of all a_v which is divided by $(np+1)!$.

Remark 4.2. The coefficients α_n, $n\ge 2$, can be also computed recursively by the formula

$$\begin{aligned}
\alpha_{n+1} &= (-1)\Big[\frac{(p-1)!}{((n+1)p+1)((n+1)p)\cdots(np+2)}\Big]\alpha_n\\
&\quad+\frac{(-1)^{n+1}}{((n+1)p+1)!}\sum_{\substack{i_1=1\\ i_1\ne p-1}}^{p}\ \sum_{\substack{i_2=i_1+1\\ i_2\ne 2p-1}}^{2p}\cdots\\
&\quad\sum_{i_{n-1}=i_{n-2}+1}^{np-\mathbf{2}}\Big[\prod_{m_1=1}^{i_1-1}(p-1-(m_1-1))\Big]\big(1-(p-1-(i_1-1))\big)
\end{aligned}$$

$$\times \Big[\prod_{m_2=i_1+1}^{i_2-1} (2(p-1)-(m_2-2)) \Big] (1-(2(p-1)-(i_2-2))) \cdots$$
$$\times \Big[\prod_{m_n=i_{n-1}+1}^{i_n-1} (n(p-1)-(m_n-n)) \Big]$$
$$\times (1-(n(p-1)-(i_n-n))))[n(p-1)-(i_n-n)]!$$

with $\alpha_1 = -1/(p(p+1))$.

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