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Generalized modal reduction method in dynamics of mechanical systems

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The rotating mechanical systems (e.g., high-speed gearboxes, bladed disks, rotors, turbochargers) are composed of many flexible and rigid bodies (below subsystems) mutually joined by flexible nonlinear discrete couplings. The mathematical models of these subsystems are non-conservative with nonsymmetrical matrices and after discretization by the finite element method have large number of degrees of freedom (DOF number). The standard numerical methods of dynamic analyses of the rotating systems with nonlinear couplings are very hardly applicable. A suitable and established methods for DOF number reduction of the large multi-body systems is the modal synthesis method [2, 3, 6, 7]. The classical approach of the modal synthesis method is based on the reduction of the natural modes of subsystems conservative models respected in dynamic response. Rotatig mechanical systems contain gyroscopic effects and additional influences of rotation and dissipation [1, 2, 4, 5]. On this account the eigenvalues and right and left eigenvectors of rotating subsystems are complex. The main aim of this contribution is to present the generalized modal reduction method with reduction DOF number of the whole system or individual subsystems for modelling of the multi-body systems with strong gyroscopic effects, damping and friction in couplings.

Let us consider the mechanical system (rotor, blade packet, rings) which can be decomposed into N linearized rotating or nonrotating subsystems. The first step of modelling using modal synthesis method consists in the first-order formulation of the equations of motion [1] in the form

$$\boldsymbol{N}_j\dot{\boldsymbol{u}}_j + \boldsymbol{P}_j\boldsymbol{u}_j = \boldsymbol{p}_j\,,\ \boldsymbol{p}_j=\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{f}_j^C+\boldsymbol{f}_j^E(t) \end{bmatrix}\,,\ j = 1,\ldots,N \tag{1}$$

in the state space defined by the state vector of subsystem j, $\boldsymbol{u}_j = [\dot{\boldsymbol{q}}_j^T, \boldsymbol{q}_j^T]^T$, where $\boldsymbol{q}_j$ is the vector of the generalized coordinates of dimension n_j. Vector $\boldsymbol{p}_j$ expresses effect of internal couplings $\boldsymbol{f}_j^C$ of the subsystem j with surrouding subsystems and excitation forces $\boldsymbol{f}_j^E(t)$.

Let all modal values $\lambda_\nu^{(j)}$, $\nu = 1,\ldots,n_j$, of each mutually uncoupled subsystem j (for $\boldsymbol{f}_j^C = \boldsymbol{0}$) satisfy the stability conditions $\mathrm{Re}[\lambda_\nu^{(j)}] < 0$. Modal properties of subsystem j are expressed by the complex diagonal spectral matrix $\boldsymbol{\Lambda}_j \in C^{2n_j,2n_j}$ and complex couple right and left modal matrices $\boldsymbol{U}_j \in C^{2n_j,2n_j}$, $\boldsymbol{W}_j \in C^{2n_j,2n_j}$ satisfying the biorthonormality conditions [1]

$$\boldsymbol{W}_j^T\boldsymbol{N}_j\boldsymbol{U}_j = \boldsymbol{E}_j\,,\ \boldsymbol{W}_j^T\boldsymbol{P}_j\boldsymbol{U}_j = -\boldsymbol{\Lambda}_j\,,\ j = 1,\ldots,N\,, \tag{2}$$

where $\boldsymbol{E}_j$ is the identity matrix of the $2n_j$-th order. We chose for each subsystem j two sets of $2m_j$ ($m_j \leq n_j$) so called master right and left natural modes corresponding to m_j pairs of complex conjugate eigenvalues (diagonal elements of $\boldsymbol{\Lambda}_j$) $\lambda_\nu^{(j)} = \alpha_\nu^{(j)} + \mathrm{i}\beta_\nu^{(j)}$, $\lambda_\nu^{(j)*} = \alpha_\nu^{(j)} - \mathrm{i}\beta_\nu^{(j)}$

ordered according to the size of the imaginary parts $\beta_1^{(j)} \leq \beta_2^{(j)} \leq \cdots \leq \beta_{m_j}^{(j)}$. Corresponding natural modes are represented by pairs of the complex conjugate right $\boldsymbol{u}_\nu^{(j)}$, $\boldsymbol{u}_\nu^{(j)*}$ and left $\boldsymbol{w}_\nu^{(j)}$, $\boldsymbol{w}_\nu^{(j)*}$ eigenvectors ordered in the master (subscript m) right and left modal submatrices

$$
{}^m\boldsymbol{U}_j = [\boldsymbol{u}_1^{(j)}, \ldots, \boldsymbol{u}_{m_j}^{(j)}, \boldsymbol{u}_1^{(j)*}, \ldots, \boldsymbol{u}_{m_j}^{(j)*}] \in C^{2n_j,2m_j},
$$

$$
{}^m\boldsymbol{W}_j = [\boldsymbol{w}_1^{(j)}, \ldots, \boldsymbol{w}_{m_j}^{(j)}, \boldsymbol{w}_1^{(j)*}, \ldots, \boldsymbol{w}_{m_j}^{(j)*}] \in C^{2n_j,2m_j},\ j = 1, \ldots, N\,, \tag{3}
$$

corresponding to master spectral submatrix

$$
{}^m\boldsymbol{\Lambda}_j = \mathrm{diag}[\lambda_1^{(j)}, \ldots, \lambda_{m_j}^{(j)}, \lambda_1^{(j)*}, \ldots, \lambda_{m_j}^{(j)*}] \in C^{2m_j,2m_j},\ j = 1, \ldots, N\,. \tag{4}
$$

State vectors $\boldsymbol{u}_j$ in model (1) are transformed by the master right modal submatrices ${}^m\boldsymbol{U}^{(j)} \in$ $\in C^{2n_j,2m_j}$ mutually uncoupled subsystems into the modal coordinates as

$$
\boldsymbol{u}_j = {}^m\boldsymbol{U}_j\boldsymbol{x}_j = \sum_{\nu=1}^{m_j} \left(\boldsymbol{u}_\nu^{(j)} x_\nu^{(j)} + \boldsymbol{u}_\nu^{(j)*} x_\nu^{(j)*}\right),\ j = 1, \ldots, N\,. \tag{5}
$$

After modal transformation (5) and premultiplaying of Eqs. (1) by the transposed left master modal submatrices ${}^m\boldsymbol{W}_j^T$ with regard to the biorthonormality conditions (2), Eqs. (1) become

$$
\dot{\boldsymbol{x}}_j - {}^m\boldsymbol{\Lambda}_j\boldsymbol{x}_j = {}^m\boldsymbol{W}_j^T\boldsymbol{p}_j,\ j = 1, \ldots, N\,. \tag{6}
$$

Taking into account structure of state vectors $\boldsymbol{u}_j$, the eigenvectors of the subsystems can be written in the form

$$
\boldsymbol{u}_\nu^{(j)} = \begin{bmatrix} \lambda_\nu^{(j)}\boldsymbol{q}_\nu^{(j)} \\ \boldsymbol{q}_\nu^{(j)} \end{bmatrix},\ \boldsymbol{w}_\nu^{(j)} = \begin{bmatrix} \lambda_\nu^{(j)}\boldsymbol{r}_\nu^{(j)} \\ \boldsymbol{r}_\nu^{(j)} \end{bmatrix},\ \nu = 1, \ldots, n_j,\ j = 1, \ldots, N\,. \tag{7}
$$

The modal submatrices defined in (3) can be written as

$$
{}^m\boldsymbol{U}_j = \begin{bmatrix} {}^m\boldsymbol{Q}_j\,{}^m\boldsymbol{\Lambda}_j \\ {}^m\boldsymbol{Q}_j \end{bmatrix},\ {}^m\boldsymbol{W}_j = \begin{bmatrix} {}^m\boldsymbol{R}_j\,{}^m\boldsymbol{\Lambda}_j \\ {}^m\boldsymbol{R}_j \end{bmatrix},\ j = 1, \ldots, N\,, \tag{8}
$$

where

$$
{}^m\boldsymbol{Q}_j = [\boldsymbol{q}_1^{(j)}, \ldots, \boldsymbol{q}_{m_j}^{(j)}, \boldsymbol{q}_1^{(j)*}, \ldots, \boldsymbol{q}_{m_j}^{(j)*}] \in C^{n_j,2m_j}\,,
$$

$$
{}^m\boldsymbol{R}_j = [\boldsymbol{r}_1^{(j)}, \ldots, \boldsymbol{r}_{m_j}^{(j)}, \boldsymbol{r}_1^{(j)*}, \ldots, \boldsymbol{r}_{m_j}^{(j)*}] \in C^{n_j,2m_j} \tag{9}
$$

are the right and left master modal submatrices of uncoupled subsystems in the original configuration space of generalized coordinates $\boldsymbol{q}_j$. Eqs. (6) can be rewritten in the form

$$
\dot{\boldsymbol{x}}_j - {}^m\boldsymbol{\Lambda}_j\boldsymbol{x}_j = {}^m\boldsymbol{R}_j^T(\boldsymbol{f}_j^C + \boldsymbol{f}_j^E(t)),\ j = 1, \ldots, N\,. \tag{10}
$$

The global form is

$$
\dot{\boldsymbol{x}} - {}^m\boldsymbol{\Lambda}\boldsymbol{x} = {}^m\boldsymbol{R}^T[\boldsymbol{f}_C(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{f}_E(t)]\,, \tag{11}
$$

where

$$
\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_N \end{bmatrix},\ \boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_N \end{bmatrix},\ \boldsymbol{f}_C(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} \boldsymbol{f}_1^C \\ \vdots \\ \boldsymbol{f}_N^C \end{bmatrix},\ \boldsymbol{f}_E(t) = \begin{bmatrix} \boldsymbol{f}_1^E(t) \\ \vdots \\ \boldsymbol{f}_N^E(t) \end{bmatrix},
$$

$$ {}^m\mathbf{\Lambda} = \mathrm{diag}[{}^m\mathbf{\Lambda}_1, \dots, {}^m\mathbf{\Lambda}_N] \in C^{2m,2m},\ {}^m\boldsymbol{R}^T = \mathrm{diag}[{}^m\boldsymbol{R}_1^T, \dots, {}^m\boldsymbol{R}_N^T] \in C^{2m,n}\,. \tag{12} $$

Matrices ${}^m\mathbf{\Lambda}$, ${}^m\boldsymbol{R}$ and vector $\boldsymbol{x}$ in (11) can be rewritten in the form

$$ {}^m\mathbf{\Lambda} = \mathrm{diag}[{}^m\overline{\mathbf{\Lambda}}, {}^m\overline{\mathbf{\Lambda}}^*],\ {}^m\boldsymbol{R} = [{}^m\overline{\boldsymbol{R}}, {}^m\overline{\boldsymbol{R}}^*],\ \boldsymbol{x} = \begin{bmatrix} \overline{\boldsymbol{x}} \\ \overline{\boldsymbol{x}}^* \end{bmatrix}, \tag{13} $$

where spectral submatrix ${}^m\overline{\mathbf{\Lambda}}$ includes the chosen eigenvalues $\lambda_\nu^{(j)} = \alpha_\nu^{(j)} + \mathrm{i}\beta_\nu^{(j)}$ of the all subsystems with positive imaginary part and left master modal submatrix ${}^m\overline{\boldsymbol{R}}$ includes corresponding eigenvectors $\boldsymbol{r}_\nu^{(j)}$. The complex conjugate eigenvalues are arranged in matrices ${}^m\overline{\mathbf{\Lambda}}^*$, ${}^m\overline{\boldsymbol{R}}^*$ and the complex conjugate modal coordinates are arranged in vector $\overline{\boldsymbol{x}}^*$. We can use the MATLAB built in `ode45` solver for the integration of the submodel

$$ \dot{\overline{\boldsymbol{x}}} - {}^m\overline{\mathbf{\Lambda}}\overline{\boldsymbol{x}} = {}^m\overline{\boldsymbol{R}}^T[\boldsymbol{f}_C(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{f}_E(t)]\,, \tag{14} $$

where

$$ {}^m\overline{\mathbf{\Lambda}} = \mathrm{diag}[{}^m\overline{\mathbf{\Lambda}}_1, \dots, {}^m\overline{\mathbf{\Lambda}}_N] \in C^{m,m},\ {}^m\overline{\boldsymbol{R}}^T = [{}^m\overline{\boldsymbol{R}}_1^T, \dots, {}^m\overline{\boldsymbol{R}}_N^T] \in C^{m,n}\,. $$

According to (5) and (8) the vector $\boldsymbol{q}$ of generalized coordinates can be expressed as

$$ \boldsymbol{q} = {}^m\overline{\boldsymbol{Q}}\overline{\boldsymbol{x}} + {}^m\overline{\boldsymbol{Q}}^*\overline{\boldsymbol{x}}^*,\ \dot{\boldsymbol{q}} = {}^m\overline{\boldsymbol{Q}}\,{}^m\overline{\mathbf{\Lambda}}\overline{\boldsymbol{x}} + {}^m\overline{\boldsymbol{Q}}^*\,{}^m\overline{\mathbf{\Lambda}}^*\overline{\boldsymbol{x}}^*\,, \tag{15} $$

where right master modal submatrix ${}^m\overline{\boldsymbol{Q}}$ corresponds to ${}^m\overline{\mathbf{\Lambda}}$. Model (14) of the coupled system in space of modal coordinates $x_\nu^{(j)}$ uncoupled subsystems has $m = \sum_{j=1}^{N} m_j$ DOF number and for $\sum_{j=1}^{N} m_j < \sum_{j=1}^{N} n_j$ is reduced.

If the linear part of elastic and viscous forces in couplings between subsystems can be excluded from the nonlinear couplings, vector $\boldsymbol{f}_C(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in (11) can be written in the form

$$ \boldsymbol{f}_C(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\boldsymbol{K}_C\boldsymbol{q} - \boldsymbol{B}_C\dot{\boldsymbol{q}} + \boldsymbol{f}_N(\boldsymbol{q}, \dot{\boldsymbol{q}})\,, \tag{16} $$

where $\boldsymbol{K}_C$ and $\boldsymbol{B}_C$ are the global stiffness and damping matrices corresponding to the linearized forces in couplings, vector $\boldsymbol{f}_N(\boldsymbol{q}, \dot{\boldsymbol{q}})$ expresses the nonlinear coupling forces and $\boldsymbol{q} = [\boldsymbol{q}_1^T, \dots, \boldsymbol{q}_N^T]^T$ is the global vector of the generalized coordinates. All equations of motion (1) can be expressed in the state space $\boldsymbol{u} = [\dot{\boldsymbol{q}}^T, \boldsymbol{q}^T]^T$ in the global form

$$ \boldsymbol{N}\dot{\boldsymbol{u}} + \boldsymbol{P}\boldsymbol{u} = \boldsymbol{p}\,, \quad \boldsymbol{p} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{f}_N(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{f}_E(t) \end{bmatrix}, \tag{17} $$

where linearized forces in couplings are included in matrices $\boldsymbol{N}$ and $\boldsymbol{P}$. Similarly as in the modal synthesis method, only the following global submodel with m DOF number can be integrated

$$ \dot{\overline{\boldsymbol{x}}} - {}^m\overline{\mathbf{\Lambda}}\overline{\boldsymbol{x}} = {}^m\overline{\boldsymbol{R}}^T[\boldsymbol{f}_N(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{f}_E(t)]\,, \tag{18} $$

where $\overline{\boldsymbol{x}}$ and ${}^m\overline{\boldsymbol{R}}$ correspond to spectral submatrix ${}^m\overline{\mathbf{\Lambda}} = \mathrm{diag}[\lambda_1, \dots, \lambda_m]$ including chosen eigenvalues $\lambda_\nu = \alpha_\nu + \mathrm{i}\beta_\nu$ of model (17) for $\boldsymbol{p} = \boldsymbol{0}$ with the positive imaginary part.

From the high computational costs point of view, an application of this method is suitable especially for dynamic analysis of the large rotating systems with nonlinear couplings. The method enables dynamic analyses of the damped rotating systems including all rotation effects and nonlinear contact forces in internal couplings between subsystems. Contrary on the classical approach characterized by transformation of the generalized coordinates using the real modal submatrix of the linear part of the undamped and nonrotating system, the new approach is based on the transformation by the complex modal submatrix of the nonconservative linear part of the rotating system including all rotating and dissipative effects. The dynamic response in master modal coordinates is investigated by integration of the first order nonlinear equations, whose number corresponds to identical number of the second order nonlinear equations, using a classical approach. Consideration of the chosen master complex mode shapes improves approximation of the damped gyroscopic structures behaviour in comparison with classical modal reduction in the basis of the real mode shapes of the undamped and nonrotating structures.

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References

[1] Byrtus, M., Hajžman, M., Zeman, V., Dynamics of rotating systems, University of West Bohemia, Pilsen, 2010. (in Czech)

[2] Genta, G., Dynamics of rotating systems, Springer Science and Business Media, New York, 2005.

[3] Irretier, H., Modal synthesis method with free interfaces and residual flexibility matrices for frame structures, Building Journal 37 (9) (1989) 601-610.

[4] Sui, Y., Zhong, W., Eigenvalue problem of a large scale indefinite gyroscopic dynamic system, Applied Mathematics and Mechanics 27 (1) (2006) 15-22.

[5] Yamamoto, T., Ishida, Y., Linear and nonlinear rotordynamics: A modern treatment with applications, John Wiley and Sons, Inc., New York, 2001.

[6] Zeman, V., Vibration of mechanical systems by the modal synthesis method, ZAMM – Journal of applied mathematics and mechanics: Zeitschrift für angewandte Mathematik und Mechanik. 74 (4) (1994) 99-101.

[7] Zeman, V., Hlaváč, Z., Condensed dynamical models in optimization of mechanical systems, ZAMM – Journal of applied mathematics and mechanics: Zeitschrift für angewandte Mathematik und Mechanik 75 (1995) 71-72.