Approximate symmetries of planar algebraic curves with inexact input

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Abstract

In this paper, we formulate a simple algorithm for an approximate reconstruction of an inexact planar curve which is assumed to be a perturbation of some unknown planar curve with symmetry. The input curve is given by a perturbed polynomial. We use a matrix complex representation of algebraic curves for simple estimation of the potential symmetry. A functionality of the designed reconstruction method is presented on several examples.

Keywords: Planar algebraic curves, matrix complex representation, symmetry detection, approximation

1. Introduction and motivation

Many problems originated not only in pure or applied mathematics but also in technical and natural sciences are closely related to investigating classes of equivalent objects and corresponding transformations. Especially, the Euclidean equivalences (isometries and symmetries) belong among fundamental equivalence relations which have been thoroughly studied for many years. Being symmetric is a potentially very useful feature which many real shapes exhibit and symmetries in the natural world have also significantly inspired people to incorporate symmetry when producing tools, buildings, artwork etc.

The notion of symmetry as invariance under certain geometric transformations is a fundamental concept introduced to geometry by Felix Klein in his Erlangen Program, see [1]. Klein proposed to characterize different classes of geometry based on the underlying symmetry groups. This approach to classifying geometries based on symmetry groups can be transferred also to geometric shapes. Then the goal is to decide whether two given geometric objects are related by some (projective, affine, similar, or isometric) transformation and in the affirmative case to detect all such equivalences. These problems are often addressed in papers coming from applied fields like Computer Aided Geometric Design, Pattern Recognition or Computer Vision, see [2–4] for the exhaustive list of references. In fields like Patter Recognition or Computer Vision especially the problem of detecting similarity is essential because objects must be recognized regardless of their position and scale. In Computer Aided Geometric Design, symmetry is important on its own right, since it is a distinguished feature of the shape of an object. Nonetheless it is also important in terms of storing or managing images, because knowing the symmetries of an image allows the machine to reconstruct the object at a lower computational or memory cost.

This paper is devoted to the symmetries of planar curves. One can find several papers focused on the detection and computation of symmetries and some equivalences of curves, see e.g. [5–8], or a recent sequence of papers [2, 3, 9–11]. Lately, the problem of deterministically computing the symmetries of a given planar algebraic curve, implicitly defined, and the problem of deterministically checking whether or

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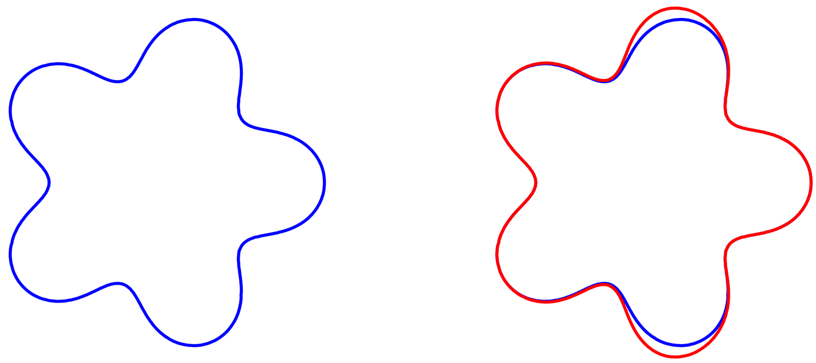

Figure 1: A curve with the rotational symmetry with the center at the origin and the angle $2\pi/5$ (blue) and its perturbation which does not possess any symmetry (red).

not two implicitly given, planar algebraic curves are similar, i.e., equal up to a similarity transformation, was considered in [4]. A problem of computing projective equivalences of special algebraic varieties, including projective (and other) equivalences of rational curves was investigated in [12].

As already stated, many real world shapes exhibit a symmetry. However, in most cases this symmetry is not perfect but only approximate. And this may happen also in situations when some input error (or some error caused by numerical computations) occurs. For instance, consider a planar algebraic curve $\mathcal{C}$ given by the polynomial

$$\begin{aligned} f(x,y) = x^{10} + 5x^8y^2 - 5x^8 + 10x^6y^4 - 20x^6y^2 - 35x^6 - 486x^5 + 10x^4y^6 - 30x^4y^4 \\ - 105x^4y^2 - 280x^4 + 4860x^3y^2 + 5x^2y^8 - 20x^2y^6 - 105x^2y^4 - 560x^2y^2 \\ - 2560x^2 - 2430xy^4 + y^{10} - 5y^8 - 35y^6 - 280y^4 - 2560y^2 - 17232. \end{aligned} \quad (1)$$

This is a curve possessing (among others) the rotational symmetry with the center at the origin and the rotational angle $2\pi/5$, see Fig. 1 (left). Now changing only one coefficient of the defining polynomial $f(x,y)$, e.g. the coefficient of y^{10} from 1 to 0.7, yields a new curve, see Fig. 1 (right). The new perturbed curve *is not symmetric* anymore. This, for instance, causes that all subsequent exact algorithms and techniques formulated for algebraic curves with symmetries may fail. Thus a natural question reads: How can we find a planar symmetric curve "close" to $\mathcal{C}$? By "close" one can mean e.g. the minimum distance of the coefficients of their implicit equations or another suitable criterion.

In this paper we aim at investigating important aspects of geometric modelling stemming from perturbations of symmetric algebraic varieties, which is a problem that falls within the scope of approximate algebraic geometry, see [13]. The problem is still relatively new in geometric modelling and many challenges are to be solved and answered. We focus on approximate reconstruction of an inexact planar curve which is assumed to be a perturbation of some unknown planar curve with symmetry of a regular polygon. The rest of the paper is organized as follows. Section 2 recalls some basic facts concerning algebraic curves and symmetries. Section 3 is devoted to the matrix complex representation of curves possessing a symmetry. We discuss how to easily detect such curves. In Section 4, determining suitable characteristics necessary for the construction of bases of symmetric curves of a given degree and having a symmetry of a regular polygon with a prescribed center is discussed in more detail. Based on this, we design in Section 5 a simple algorithm for an approximate reconstruction of an inexact planar curve which is assumed to be a perturbation of some unknown planar curve with the symmetry of a regular polygon. The method is presented on a number of examples in Section 6. Finally, we conclude the paper in Section 7.

2. Preliminaries

In this section we recall some elementary notions and basic properties whose knowledge is further assumed in the following parts of the paper.

2.1. Complex representation of real algebraic curves

A *planar algebraic curve* $\mathcal{C}$ of degree d is a subset of the Euclidean plane $\mathbb{E}^2_{\mathbb{R}}$ defined as the zeroset of a polynomial of degree d

$$f(x,y) = \sum_{i=0}^{d}\sum_{j=0}^{d-i} a_{i,j} x^i y^j. \tag{2}$$

We will assume that the coefficients $a_{i,j}$ of (2) are real, f is irreducible over $\mathbb{C}$ and $\dim_{\mathbb{R}} \mathcal{C} = 1$. Moreover we use the matrix representation of polynomials, i.e.,

$$f(x,y) = (1, x, x^2, \dots, x^d)\,\mathbf{A}_f \begin{pmatrix} 1 \\ y \\ y^2 \\ \vdots \\ y^d \end{pmatrix}, \quad \mathbf{A}_f = \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,d-1} & a_{0,d} \\ a_{1,0} & a_{1,1} & \dots & a_{1,d-1} & 0 \\ \vdots & \vdots & & & \vdots \\ a_{d-1,0} & a_{d-1,1} & & 0 & 0 \\ a_{d,0} & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{3}$$

Every affine algebraic curve of equation $f(x,y) = 0$ may be completed into the projective curve of equation $F(X,Y,Z) = 0$, where

$$F(X,Y,Z) = Z^d f\left(X/Z, Y/Z\right) = \sum_{i=0}^{d}\sum_{j=0}^{d-i} a_{i,j} X^i Y^j Z^{d-i-j} \tag{4}$$

is the result of the homogenization of f, and $X : Y : Z$ are the homogeneous coordinates in the projective plane. Let us write $\mathbf{c} = (a_{d,0} : a_{d-1,1} : \dots : a_{00})$ and we say that $\mathbf{c}$ represents $\mathcal{C}$. The space of all planar projective curves of degree d can be identified with the projective space $\mathbb{P}^N_{\mathbb{R}}$, where $N = \binom{d+2}{2} - 1$.

As we are dealing with rotations of planar curves, complex representation will provide a significant simplification in the analysis of the centre and the angle, see also [7, 14]. Consider a curve $\mathcal{C}$ defined by polynomial (2). The standard substitution

$$x = \frac{z + \overline{z}}{2} \qquad \text{and} \qquad y = \frac{z - \overline{z}}{2\mathrm{i}} \tag{5}$$

allows to write $f(x,y)$ as a complex function $f(z,\overline{z})$ in the complex variable z in the form

$$f(z,\overline{z}) = \sum_{i=0}^{d}\sum_{j=0}^{d-i} b_{i,j} z^i \overline{z}^j, \tag{6}$$

where $b_{i,j} = \overline{b}_{j,i} \in \mathbb{C}$, and $b_{i,i} = \overline{b}_{i,i} \in \mathbb{R}$, cf. [14]. Then the matrix form of f in the complex representation looks as follows

$$f(z,\overline{z}) = (1, z, z^2, \dots, z^d)\,\mathbf{B}_f \begin{pmatrix} 1 \\ \overline{z} \\ \overline{z}^2 \\ \vdots \\ \overline{z}^d \end{pmatrix}, \quad \mathbf{B}_f = \begin{pmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,d-1} & b_{0,d} \\ b_{1,0} & b_{1,1} & \dots & b_{1,d-1} & 0 \\ \vdots & \vdots & & & \vdots \\ b_{d-1,0} & b_{d-1,1} & & 0 & 0 \\ b_{d,0} & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{7}$$

The coefficients of the Hermitian matrix $\mathbf{B}_f$ are given by

$$b_{k,\ell} = \frac{1}{2^{k+\ell}} \sum_{i=0}^{k+\ell} \sum_{j=0}^{i} \binom{k+\ell-i}{\ell-j} \binom{i}{j} \, \mathrm{i}^{2j-i} \, a_{k+\ell-i,i}, \tag{8}$$

when we set $n! = 0$ whenever $n < 0$, cf. [7].

Let us recall that the complex representation of algebraic curves and its exploitation for pose estimation was first presented in [15] and then extended in [14]. A later paper [16] presented a partial complex representation of algebraic curves for gaining some recognition invariants. Finally, the authors of [7] used the complex representation also for detecting symmetries of algebraic curves. In what follows, we would like to show that new matrix complex representation (7) brings further advantages and significantly simplifies the process of determining symmetries.

2.2. Fundamental facts about symmetric algebraic curves in plane

Any *isometry* $\phi \in \mathbf{Iso}_2$ of $\mathbb{E}^2_{\mathbb{R}}$ possesses the form $\mathbf{x} \mapsto \mathbf{Mx} + \mathbf{h}$, where $\mathbf{M} \in \mathbf{O}(\mathbb{R}, 2)$ and $\mathbf{h} \in \mathbb{R}^2$. For $\det(\mathbf{M}) = 1$, or $= -1$ we speak about *direct*, or *indirect* isometries, respectively.

We write $\mathbf{Sym}(\mathcal{C})$ for the group of symmetries of the curve $\mathcal{C}$, i.e.,

$$\mathbf{Sym}(\mathcal{C}) := \{\phi \in \mathbf{Iso}_2;\ \phi(\mathcal{C}) = \mathcal{C}\}. \tag{9}$$

It is well known that $\mathbf{Sym}(\mathcal{C})$ is finite unless $\mathcal{C}$ is a union of parallel lines or a union of concentric circles. Moreover, if $\mathbf{Sym}(\mathcal{C})$ is finite then it is isomorphic to a subgroup of the group of symmetries of some regular m-gon, $m \leq \deg(\mathcal{C})$. In what follows we are interested solely in curves with a finite group of symmetries. The elements of a finite symmetry group are rotations (all of them with the same center) and reflections (axes of all of them passing through the same point).

Let us recall the following statement, which can be efficiently used to verify whether $\phi \in \mathbf{Sym}(\mathcal{C})$, see [4] for more details:

Proposition 2.1. *An isometry $\phi \in \mathbf{Sym}(\mathcal{C})$ if and only if $f(\mathbf{Mx} + \mathbf{h}) = \lambda f(\mathbf{x})$, where $\lambda = 1$ or $\lambda = -1$.*

Then analogously to $\mathbf{Sym}(\mathcal{C})$ we can write that $\phi \in \mathbf{Sym}(f)$, as well. When there is no danger of confusion we will not distinguish between the symmetries of $\mathcal{C}$ and f.

The main topic of this paper is an identification of a suitable symmetric curve in the neighborhood of the given perturbed curve. Once the approximate symmetric curve is known then we can follow any exact approach for computing symmetries of curves; e.g. a method from [4] which has been formulated recently. This exact method works with the observation that it is not easy, in general, to find symmetries ϕ belonging to $\mathbf{Sym}(\mathcal{C})$ directly and one has to apply a suitable alternative approach – for instance to find some new polynomial $h(x, y)$ such that $\mathbf{Sym}(h)$ is finite, easy to determine (i.e., easier then $\mathbf{Sym}(f)$) and $\mathbf{Sym}(\mathcal{C}) = \mathbf{Sym}(f) \subset \mathbf{Sym}(h)$. In [4], a successive application of the Laplace operator yielding the sequence

$$f \longmapsto \triangle f \longmapsto \triangle^2 f \longmapsto \cdots \longmapsto \triangle^\ell f = h, \tag{10}$$

and followed by the associated chain of groups of symmetries

$$\mathbf{Sym}(f) \subset \mathbf{Sym}(\triangle f) \subset \mathbf{Sym}(\triangle^2 f) \subset \cdots \subset \mathbf{Sym}(\triangle^\ell f) = \mathbf{Sym}(h), \tag{11}$$

was efficiently used for finding such a polynomial h. Application of this technique is justified by the fact that the *Laplace operator* commutes with isometries, i.e., it holds $(\triangle f) \circ \phi = \triangle(f \circ \phi)$. The authors show how to compute the centre and axes of symmetry and for rotational symmetries also the angles from h.

As we will see one of the benefits of the introduced matrix complex representation is that it also enables to simplify some steps of the exact method, e.g. it yields directly the centre of symmetry and the number of vertices of the regular polygon with group of symmetries of the considered curve.

3. Planar algebraic curves with symmetries of a regular polygon

In this section we investigate in detail the matrix complex representation of curves possessing a symmetry of a regular m-gon.

3.1. Symmetric curves with the center of rotations at the origin

The rotation around the origin and with the angle φ is given by

$$z \mapsto e^{\mathrm{i}\varphi} z, \quad \overline{z} \mapsto e^{-\mathrm{i}\varphi} \overline{z}. \tag{12}$$

Hence any curve (6) with the symmetry of m-gon has to satisfy the condition, cf. Proposition 2.1

$$f(z, \overline{z}) = \pm f\left(e^{2\mathrm{i}\pi/m} z, e^{-2\mathrm{i}\pi/m} \overline{z}\right). \tag{13}$$

This leads to two families of linear equations

$$b_{i,j} = \pm e^{2\mathrm{i}\pi(i-j)/m} b_{i,j}, \quad \overline{b}_{i,j} = \pm e^{-2\mathrm{i}\pi(i-j)/m} \overline{b}_{i,j}. \tag{14}$$

The "+" equations imply $b_{i,j} = \overline{b}_{i,j} = 0$ for an arbitrary $m \in \mathbb{N}$ and $(i-j) \neq km$, $k \in \mathbb{Z}$. The "−" equations possess solution only for an even m and $b_{i,j} = \overline{b}_{i,j} = 0$ holds for $(i-j) \neq (2k+1)m/2$. Hence the two bases of all curves of degree d with the symmetry of m-gon and having the center of symmetry at the origin are

$$\left\{ (z\overline{z})^i \left(z^{km} + \overline{z}^{km}\right) \right\}_{i=0,k=0}^{\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d-2i}{m} \rfloor} \tag{15}$$

and

$$\left\{ (z\overline{z})^i \left(z^{\frac{(2k+1)}{2}m} + \overline{z}^{\frac{(2k+1)}{2}m}\right) \right\}_{i=0,k=0}^{\lfloor \frac{d}{2} \rfloor, \lfloor \frac{2d-4i-m}{2m} \rfloor}, \tag{16}$$

where the second one exists only for an even m. In addition, we speak about s^+ or s^- curves if (13) is satisfied w.r.t + or −, respectively, i.e., if the curve is generated by (15), or (16).

Next, respecting the condition $b_{ij} = \overline{b}_{ji}$ any s^+ curve generated by the basis (15) has its equation of the form

$$f(z, \overline{z}) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{d-2i}{2m} \rfloor} \left((\alpha_{ik} + \mathrm{i}\beta_{ik}) (z\overline{z})^i z^{km} + (\alpha_{ik} - \mathrm{i}\beta_{ik}) (z\overline{z})^i \overline{z}^{km} \right), \tag{17}$$

i.e.,

$$f(z, \overline{z}) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{d-2i}{2m} \rfloor} \left[\alpha_{ik} \left((z\overline{z})^i \left(z^{km} + \overline{z}^{km}\right) \right) + \mathrm{i}\beta_{ik} \left((z\overline{z})^i \left(z^{km} - \overline{z}^{km}\right) \right) \right]. \tag{18}$$

And analogously for (16). So in what follows we will work with the bases

$$\mathcal{B}^+_{d,m} = \left\{ (z\overline{z})^i \left(z^{km} + \overline{z}^{km}\right), \mathrm{i} (z\overline{z})^i \left(z^{km} - \overline{z}^{km}\right) \right\}_{i=0,k=0}^{\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d-2i}{m} \rfloor}, \tag{19}$$

and

$$\mathcal{B}^-_{d,m} = \left\{ (z\overline{z})^i \left(z^{\frac{(2k+1)}{2}m} + \overline{z}^{\frac{(2k+1)}{2}m}\right), \mathrm{i} (z\overline{z})^i \left(z^{\frac{(2kj+1)}{2}m} - \overline{z}^{\frac{(2k+1)}{2}m}\right) \right\}_{i=0,k=0}^{\lfloor \frac{d}{2} \rfloor, \lfloor \frac{2d-4i-m}{2m} \rfloor}, \tag{20}$$

which allows us to consider only *real* coefficients α_{ik}, β_{ik} of the linear combination and the condition $b_{ij} = \overline{b}_{ji}$ is then satisfied automatically.

In addition, from the discussion presented above a close relation between the degree of studied symmetric curves and the characteristics of associated rotational symmetries (of a regular m-gon) immediately follows:

Lemma 3.1. *Let $\mathcal{C}$ be a planar algebraic curve with the symmetry of a regular m-gon. Then* *(i) for an s^+ curve of even degree, m can be arbitrary;*

(ii) for an s^+ curve of odd degree, m is an odd number;

(iii) for an s^- curve of even degree, m is an even number of the form $m = 4k$, $k \in \{1, 2, \ldots\}$;

(iv) for an s^- curve of odd degree, m is an even number of the form $m = 4k - 2$, $k \in \{1, 2, \ldots\}$.

Proof. For the sake of brevity we sketch the proof only for one particular case, e.g. for (ii). Considering the s^+ curves we have the basis elements $1, z^m, z^{2m}, z^{3m}, \ldots, z\overline{z}, z^{1+m}\overline{z}, z^{1+2m}\overline{z} \ldots, z^2\overline{z}^2, \ldots$ Particular degrees of these terms are $0, m, 2m, 3m, \ldots, 2, m+2, 2m+2, \ldots, 4, \ldots$. As the given curve is of odd degree d and a term of this degree must be among the basis elements, m must be an odd number. We proceed similarly in other cases. The results of Lemma 3.1 are also summarized in Table 1. □

Table 1: Possible types of symmetric curves depending on the degree d and the number m of the vertices of a regular polygon for $n = 1, 2, \ldots$, and $k = 1, \ldots, \lfloor d/4 \rfloor$.

	$m = 4k-2$	$m = 4k-1$	$m = 4k$	$m = 4k+1$
$d = 2n$	s^+	s^+	s^+/s^-	s^+
$d = 2n+1$	s^-	s^+	$\times$	s^+

Example 3.2. Consider now all rotationally symmetric algebraic curves of degree six having a symmetry of square. Using the previous results, we know that for $d = 6$ and $m = 4$ cases (i) and (iii) may occur and thus a particular curve can be either of type s^+, or s^-. So, the bases of all curves of degree 6 having a symmetry of a regular 4-gon (a square) with the center at the origin have the form

$$\mathcal{B}^+_{6,4} = \left\{1, z^4 + \overline{z}^4, \mathrm{i}\left(z^4 - \overline{z}^4\right), z\overline{z}, z\overline{z}\left(z^4 + \overline{z}^4\right), \mathrm{i}z\overline{z}\left(z^4 - \overline{z}^4\right), (z\overline{z})^2, (z\overline{z})^3\right\}, \tag{21}$$

and

$$\mathcal{B}^-_{6,4} = \left\{z^2 + \overline{z}^2, \mathrm{i}\left(z^2 - \overline{z}^2\right), z^6 + \overline{z}^6, \mathrm{i}\left(z^6 - \overline{z}^6\right), z\overline{z}\left(z^2 + \overline{z}^2\right), \mathrm{i}z\overline{z}\left(z^2 - \overline{z}^2\right), (z\overline{z})^2\left(z^2 + \overline{z}^2\right), \mathrm{i}\,(z\overline{z})^2\left(z^2 - \overline{z}^2\right)\right\}. \tag{22}$$

3.2. Symmetric curves with an arbitrary center of rotations

Consider a linear substitution

$$z \mapsto z + p,\ \overline{z} \mapsto \overline{z} + \overline{p}, \quad p \in \mathbb{C}. \tag{23}$$

The coefficients of curve (6) after transformation (23) are

$$c_{k,\ell} = \sum_{i=0}^{d-k} \sum_{j=0}^{d-k-\ell-i} \binom{k+i}{i}\binom{\ell+j}{j} p^i \overline{p}^j b_{k+i,\ell+j}. \tag{24}$$

In Fig. 2 the structures of the non-zero elements of the coefficient matrices of polynomials $\mathrm{i}z\overline{z}\left(z^4 - \overline{z}^4\right)$ and $z^2\overline{z}^2\left(\overline{z}^2 + z^2\right)$ are shown. The non-zero coefficients $b_{k,\ell}$ of these polynomials correspond to the orange cells only, whereas after generic substitution (23) the both blue and orange cells correspond to non-zero coefficients $c_{k,\ell}$, in general.

When $p = p_1 + p_2\mathrm{i}$ and $\overline{p} = p_1 - p_2\mathrm{i}$ then (23) is equivalent to

$$x \mapsto x + p_1, \quad y \mapsto y + p_2, \tag{25}$$

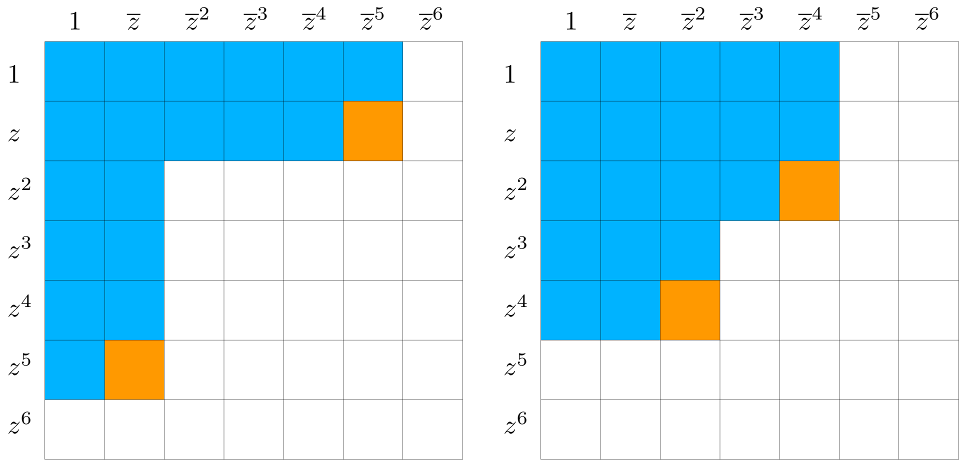

Figure 2: The influence of substitution (23) to the structure of the non-zero elements of the coefficient matrices. Originally, the only non-zero elements are the orange ones, whereas after a generic linear substitution the blue and also orange elements may be non-zero.

i.e., to a translation. In addition, applying the mapping $x \mapsto x - p_1$, $y \mapsto y - p_2$ (i.e., the coefficient transformation differs from (24) only by multiplication of each element with $(-1)^{i+j}$) on $\mathcal{B}^{\pm}_{d,m}$ yields the bases of all algebraic curves of degree d having a symmetry of a regular m-gon with the center $\mathbf{p} = (p_1, p_2)$. We will denote them by $\mathcal{B}^{\pm}_{d,m}(\mathbf{p})$. On the other hand employing (23) the center $\mathbf{p} = (p_1, p_2)$ is sent to the origin.

Suppose that $f(x, y)$ defines a curve $\mathcal{C}$ possessing a symmetry of a regular m-gon with an arbitrary center. Then we can immediately compute the center of symmetry.

Theorem 3.3. *Let $b_{k,\ell}$ be an element in the matrix $\mathbf{B}_f$ such that for $i, j \in \{0, 1, \ldots\}$ all $b_{k+i,\ell+j}$, except at least one of $b_{k+1,\ell}$, $b_{k,\ell+1}$, are zero. Then the center of symmetry is computed as*

$$p = \frac{(\ell+1)b_{k,\ell+1}\overline{b}_{k,\ell} - (k+1)b_{k,\ell}\overline{b}_{k+1,\ell}}{(k+1)^2|b_{k+1,\ell}|^2 - (l+1)^2|b_{k,\ell+1}|^2}. \tag{26}$$

Proof. To reveal the position of the center of symmetry, we want to find a particular substitution (23) which yields as many zero elements in the coefficient matrix as possible. It follows from (24) that the above considered element on the position (k, ℓ) of the coefficient matrix of the translated curve is linear in p and $\overline{p}$, i.e.,

$$c_{k,\ell} = b_{k,\ell} + (k+1)b_{k+1,\ell}p + (\ell+1)b_{k,\ell+1}\overline{p}. \tag{27}$$

Hence solving $c_{k,\ell} = 0$ directly yields (26).

In addition, let us note that (26) is defined for $(\ell+1)^2 b^2_{k,\ell+1} \neq (k+1)^2 b^2_{k+1,\ell}$. The trivial case when the equality occurs is for $k = \ell$. Then $b_{k,\ell}$ is real, $b_{k,\ell+1} = \overline{b}_{k+1,\ell}$ and (26) is not defined since (27) corresponds to one real equation for two real variables. □

From Theorem 3.3 it follows that the description of the structure of the matrix $\mathbf{B}_f$ and identification of some special elements is the key process in investigating symmetric curves and finding the associated regular m-gon. For this purpose we assign to $\mathbf{B}_f$ a non-increasing sequence of integers

$$\Lambda_f = \{\lambda_0, \ldots, \lambda_d\}, \tag{28}$$

where λ_ℓ is the position ($(k+1)$-th row) of the first element $b_{k,\ell}$ (in the $(\ell+1)$-th column) in the matrix $\mathbf{B}_f$ such that for $i, j \in \{0, 1, \ldots\}$ all $b_{k+1+i,\ell+j} = 0$. If $b_{k,\ell} = 0$ for all k we set $\lambda_\ell = -1$, see also Example 3.8

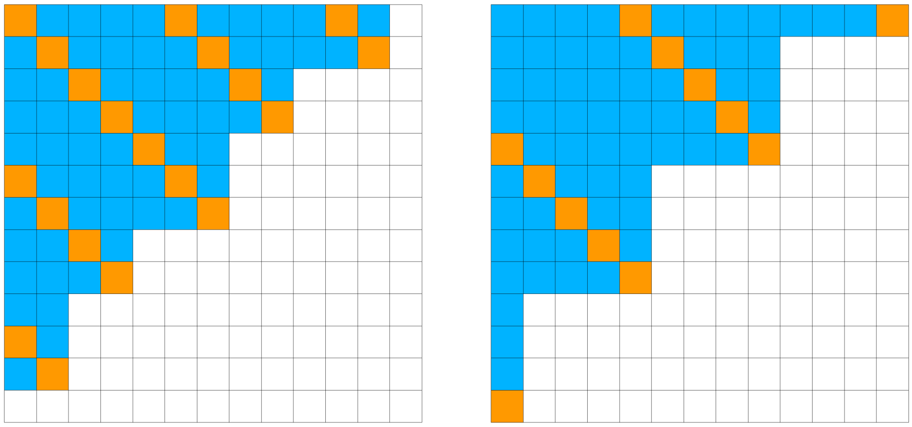

Figure 3: The structure of the coefficient matrix for s^+ curve of degree 12 with 5-gon symmetry (left) and s^- curve of degree 12 with 8-gon symmetry (right). The non-zero elements of the symmetric curve with the center at the origin are the orange ones, whereas the blue and orange cells correspond to non-zero elements of a curve with a generic center of the symmetry.

and Fig. 3 in which the construction of the sequence Λ_f is presented. In other words, Λ_f describes a certain staircase pattern of the coefficient matrix $\mathbf{B}_f$ with stairs of a given step size and height, reflecting the type of the curve.

Theorem 3.4. *Sequence* (28) *of a generic curve of degree d with the symmetry of m-gon has the form*

$$\left\{ r_i + \left\lfloor \frac{d-i-r_i}{2} \right\rfloor \right\}_{i=0}^{d}, \tag{29}$$

where we take

$$r_i = i + m\left\lfloor \frac{d-2i}{m} \right\rfloor, \qquad or \qquad r_i = i + \frac{m}{2} + m\left\lfloor \frac{d-\frac{m}{2}-2i}{m} \right\rfloor \tag{30}$$

for s^+, or s^- curves, respectively. For $\lambda_i < -1$ we set them to $\lambda_i = -1$.

Proof. First, consider s^+ curves generated by $\mathcal{B}^+_{d,m}$. The basis elements corresponding to the first column (and row) of a matrix $\mathbf{B}_f$ are

$$\{1, z^m + \overline{z}^m, \dots, z^{r_0} + \overline{z}^{r_0}\}, \tag{31}$$

where $r_0 = m\lfloor d/m \rfloor$, cf. (19). Next, there are $\lfloor (d-r_0)/2 \rfloor$ columns (and rows) right (bellow) with the same length as the first one and each such column (and row) longer the first column (and row) by one. Hence we have

$$\lambda_0 = r_0 + \left\lfloor \frac{d-r_0}{2} \right\rfloor, \quad r_0 = m\left\lfloor \frac{d}{m} \right\rfloor. \tag{32}$$

Now we derive the value for λ_i, $i = 1, \dots, d$. Consider the $(i+1)$-th row/column. If we omit the first i rows and columns we have a curve of degree $d-2i$. However, we have to add i to the value of r_i to preserve the true degree of the original element. Hence we obtain $r_i = i + m\lfloor (d-2i)/m \rfloor$, cf. (30), left. Furthermore there are exactly $\lfloor (d-i-r_i)/2 \rfloor$ columns (rows) right (bellow) which contribute to the $(i+1)$-th column (row) by one. This yields exactly (29).

The s^- curves can be treated analogously. The only difference is that the basis elements of s^- curves are translated by $m/2$, which affects r_i and we obtain (30), right. □

Theorem 3.5. *The coefficient matrix $\mathbf{B}_f$ of a generic curve with m-gon symmetry contains the steps of length $m/2$ for even m, and $(m-1)/2$ and $(m+1)/2$ for odd m.*

Proof. The curve of degree d with the symmetry of m-gon must contain an element $b_{k,\ell}$ such that $k+\ell=d$. It follows from the properties of Λ_f that this element contributes to the sequence as $\lambda_k=\ell$. Based on the structure of the basis elements, cf. (19) and (20), we can determine which next element $b_{i,j}$ such that $i+j=d$ contributes again to Λ_f. From $k+i+\ell-m+i=d$ we immediately have $2i=m$. If this element exists then it is $b_{k+m/2,\ell-m/2}$ for even m, and $b_{k+m,\ell-m}$ for odd m. However for odd m already after $(m-1)/2$ steps we obtain another element contributing to Λ_f despite being of degree $d-1$, in particular $b_{k+(m-1)/2,\ell-(m+1)/2}$.

Of course, it may happen that the next element $b_{i,j}$ such that $i+j=d$ used in the previous considerations does not exist. Then we can go back in sequence Λ_f to the element $b_{k-m/2,\ell+m/2}$ for even m and to the element $b_{k-(m+1)/2,\ell+(m-1)/2}$ for odd m.

To sum up, for even m we obtain the length of steps $m/2$; and for odd m we obtain steps of two lengths $(m-1)/2$ and $(m+1)/2$. □

Corollary 3.6. *Sequence of integers* (28) *of a generic curve with the symmetry of m-gon contains a subsequence of the form*

$$\Lambda_f=\left\{\dots\lambda,\overbrace{\lambda-a,\dots,\lambda-a}^{b},\overbrace{\lambda-a-b,\dots,\lambda-a-b}^{a},\dots\right\}, \tag{33}$$

where $a+b=m$.

Proof. It follows from Theorem 3.5 that for even m, the three consecutive elements contributing to Λ_f are $b_{i,j}$, $b_{i+m/2,j-m/2}$, $b_{i+m,j-m}$ (when $i+j=d$), and for odd m, the three consecutive elements contributing to Λ_f are $b_{i,j}$, $b_{i+(m-1)/2,j-(m+1)/2}$, $b_{i+m,j-m}$ or $b_{i,j}$, $b_{i+(m+1)/2,j-(m-1)/2}$, $b_{i+m,j-m}$ (when $i+j=d$ or $i+j=d-1$). The second indices give the values in the sequence and the difference between the first indices yields the height of the step, i.e., how many times the value is repeated in the sequence. Hence it easily follows that $a=b=m/2$ for even m, and $a=(m+1)/2$, $b=(m-1)/2$ or $a=(m-1)/2$, $b=(m+1)/2$ for odd m. □

Remark 3.7. As mentioned above the coefficient matrix $\mathbf{B}_f$ of a generic curve contains steps of a certain size, see Fig. 3. Hence whenever $b_{k,\ell+1}=0$ (in Theorem 3.3), then center of symmetry (26) becomes

$$p=-\frac{b_{k,\ell}}{(k+1)\,b_{k+1,\ell}}. \tag{34}$$

Analogously whenever $b_{k+1,\ell}=0$, we arrive at

$$p=-\frac{\bar{b}_{k,\ell}}{(\ell+1)\,\bar{b}_{k,\ell+1}}=-\frac{b_{\ell,k}}{(\ell+1)\,b_{\ell+1,k}}. \tag{35}$$

Example 3.8. Consider the coefficient matrix for s^+ curve of degree 12 with 5-gon symmetry, see Fig. 3 (left). Then sequence (28) looks like

$$\Lambda=\{11,11,8,8,6,6,6,3,3,1,1,1,-1\}, \tag{36}$$

which describes the structure of the matrix $\mathbf{B}_f$ (considering the blue and orange rightmost cells in Fig. 3, left). Furthermore, the steps in the matrix have length/height 3 and 2, which confirms $m=3+2=5$.

4. Determining suitable characteristics of symmetric algebraic curves

The results presented in this section will be exploited in the next section when formulating the algorithm for computing approximate symmetries of planar algebraic curves with inexact input.

Consider a *generic* algebraic s^+/s^- curve of degree d with the symmetry of m-gon and the center at $\mathbf{p}$. By such a curve we mean again a curve generated as a generic linear combination of $\mathcal{B}^+_{d,m}(\mathbf{p})/\mathcal{B}^-_{d,m}(\mathbf{p})$,

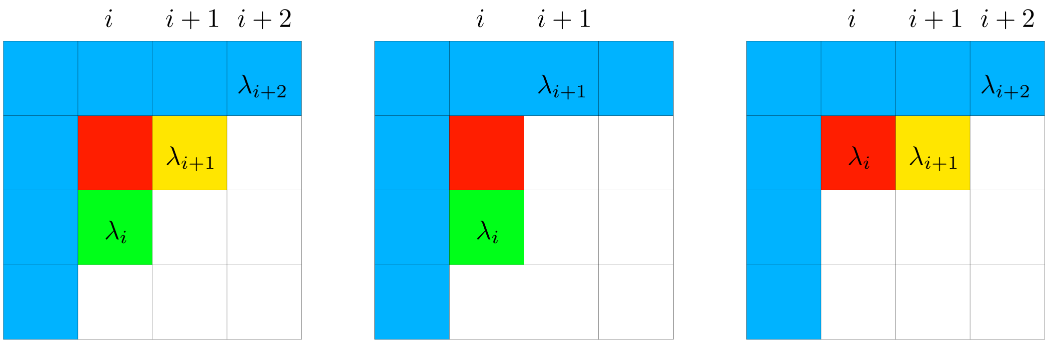

Figure 4: To obtain the center of the symmetry we have to identify the following sub-structures in the coefficient matrix. The center is consequently obtained by the computation changing the red element to zero. In all three cases the center can be obtained from (26). Of course, for the special cases in the middle or in the right one can use (34) or (35), respectively.

where $\mathbf{p}$ is a generic point. The goal of this section is to determine the center of symmetry $\mathbf{p}$, the angle of rotation $2\pi/m$ (i.e., the type of an associated regular polygon) and also the type (i.e., s^+, or s^-) of the basis of the given curve. For these purposes sequence (28) can be efficiently used.

First, we employ sequence (28) for identifying (k, ℓ) and hence determining the center of symmetry, cf. Fig. 4 for particular situations mentioned in the following lemma.

Lemma 4.1. *Consider the sequence Λ (additionally setting $\lambda_{d+1} = \lambda_{d+2} = -1$) associated to a given symmetric curve. Then for each $i \in \{0, \dots, d\}$ we proceed as follows:*

(i) When $\lambda_i = \lambda_{i+1} + 1 > \lambda_{i+2} + 1$ we set

$$k = \lambda_i - 1, \quad \ell = i, \tag{37}$$

and the centre $\mathbf{p}$ can be computed using (26).

(ii) When $\lambda_i > \lambda_{i+1} + 1$ we also set (37) and the centre $\mathbf{p}$ can be computed using simplified formula (34).

(iii) When $\lambda_i = \lambda_{i+1} > \lambda_{i+2}$ we set

$$k = \lambda_i, \quad \ell = i, \tag{38}$$

and the centre $\mathbf{p}$ can be computed using simplified formula (35).

Proof. This lemma immediately follows from the construction of (28) and Theorem 3.3. □

Remark 4.2. Let us emphasize, that for the sake of Hermite symmetry when (k, ℓ) provides center of symmetry (26), then exactly the same ratio (26) is given also by (ℓ, k). Hence it is sufficient to find such λ_i satisfying $\lambda_i = \lambda_{i+1} + 1 > \lambda_{i+2} + 1$ or $\lambda_i > \lambda_{i+1} + 1$ and set (37), see Fig. 3 (left and middle). Situation in Fig. 3 (right) corresponds to Fig. 3 (middle) when swapping (k, ℓ) to (ℓ, k). The two cases both satisfy

$$\lambda_{i+1} < \lambda_i > \lambda_{i+2} + 1, \tag{39}$$

which can be used for identifying the positions (k, ℓ) and hence (ℓ, k).

The following lemma shows how to easily recognize a particular m-gon just from the elements of sequence (28).

Lemma 4.3. *Consider the sequence Λ associated to a generic symmetric curve. Denote by Δ_j the differences between all consecutive non-negative elements. Then it holds*

$$m = \frac{2\sum_{j=1}^{n-1} \Delta_j}{n-1}, \tag{40}$$

where n is the number of different non-negative elements in Λ.

Proof. The coefficient matrix $\mathbf{B}_f$ (and thus also the sequence Λ) of a generic curve with m-gon symmetry contains the steps of length $m/2$ for even m, and $(m-1)/2$ and $(m+1)/2$ for odd m, cf. Theorem 3.5. Hence, by computing the arithmetic mean of the differences between the consecutive non-negative elements of (28) we obtain $\frac{m}{2}$, which concludes the proof. $\square$

Finally, suppose we have computed $\mathbf{p}$ and m and it remains to determine the type (i.e., s^+, or s^-) of the basis of the given curve. As we know from Lemma 3.1, s^- curves occur only for even m whereas for odd m the curves are always of s^+ type. Moreover when $m = 4k-2$, $k \in \{1, 2, \ldots\}$, it follows from Lemma 3.1 that even/odd d implies s^+/s^- curves. Hence we have to investigate in more detail only the case when $m = 4k$. For this the following lemma significantly helps.

Lemma 4.4. *Consider a symmetric curve of degree d with the symmetry of m-gon, where $m = 4k$, $k \in \{1, \ldots, \lfloor d/4 \rfloor\}$. Then this curve is s^+, or s^- if*

$$2\lambda_0 - d \equiv 0 \,(\operatorname{mod} m), \quad \text{or} \quad 2\lambda_0 - d \equiv \frac{m}{2} \,(\operatorname{mod} m), \tag{41}$$

Proof. The particular expressions can be easily obtained from (29) and (30) for $i = 0$ when setting $m = 4k$ and $d = 2n$. $\square$

Example 4.5. Consider a curve $\mathcal{C}$ defined by the coefficient matrix

$$\mathbf{B}_f = \begin{pmatrix} 790 & 837-792\mathrm{i} & -120-949\mathrm{i} & -426-252\mathrm{i} & -145+75\mathrm{i} & \boxed{30\mathrm{i}} & 2+\mathrm{i} \\ 837+792\mathrm{i} & 802 & 110-364\mathrm{i} & -48-32\mathrm{i} & 1+2\mathrm{i} & 0 & 0 \\ -120+949\mathrm{i} & 110+364\mathrm{i} & 178 & \boxed{12-24\mathrm{i}} & 0 & 0 & 0 \\ -426+252\mathrm{i} & -48+32\mathrm{i} & \boxed{12+24\mathrm{i}} & 4 & 0 & 0 & 0 \\ -145-75\mathrm{i} & 1-2\mathrm{i} & 0 & 0 & 0 & 0 & 0 \\ \boxed{-30\mathrm{i}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2-\mathrm{i} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{42}$$

Then sequence (28) looks like

$$\{6, 4, 3, 3, 1, 0, 0\}, \tag{43}$$

and we obtain four possible pairs (k, ℓ), i.e.,

$$(5, 0), \quad (3, 2), \quad (2, 3), \quad (0, 5), \tag{44}$$

see the boxed elements in (42). Then employing (26) for all these values yields $p = -1 + 2\mathrm{i}$ and hence the center of symmetry is $\mathbf{p} = (-1, 2)$. In particular, it suffices to use formula (34) for the values $k = 5$, $\ell = 0$ and $k = 2$, $\ell = 3$, which is equivalent to using (35) with $k = 0$, $\ell = 5$ and $k = 3$, $\ell = 2$. Moreover from Lemma 4.3, we have

$$m = \frac{2(2+1+2+1)}{4} = 3, \tag{45}$$

i.e., $\mathcal{C}$ has the rotational symmetry of an equilateral triangle and from Lemma 3.1 it is of s^+ type.

Remark 4.6. To conclude this section, let us note that in special (singular) situations, some (generically non-zero) coefficients in $\mathbf{B}_f$ (or even whole rows and columns) may vanish. This can subsequently violate derived results, stated for generic curves. Such singular situations have to be treated separately.

Anyway, the center of symmetry can always be found by identifying suitable blocks in $\mathbf{B}_f$ using Theorem 3.3. Next (in the worst scenario) it is always possible to consider all integers $m \in \{2, \ldots, d\}$ (and for $m = 4k$ also both bases) and set as the result of the algorithm (see the next section) the curve with the minimal deviation from the input curve.

5. Computing approximate symmetries of algebraic curves with inexact coefficients

The input to our reconstruction algorithm is a planar algebraic curve $\mathcal{C}$ which is a perturbation of some unknown symmetric planar algebraic curve $\mathcal{C}_0$. This perturbed curve is described by a polynomial $f(x,y)$ of degree d.

Let us emphasize that in most of real situations the original symmetric curve $\mathcal{C}_0$ is not known and we have only its perturbed version. This makes it impossible to measure the exact error of a constructed approximate symmetric curve $\widetilde{\mathcal{C}}$ from the original unknown object. Then it is also, for instance, difficult to state which approximate approach is the best one as in the considered neighbourhood one can find more potential candidates, some of them possibly closer to the perturbed object than to the original one.

The presented method is composed of three main steps:

(I) We determine a point $\widetilde{\mathbf{p}}$ (the approximate center) and an integer m (the number of vertices of a regular polygon) from the known perturbed curve $\mathcal{C}$;

(II) We construct a new curve $\widetilde{\mathcal{C}}$ having the symmetry of an m-gon with the center at $\widetilde{\mathbf{p}}$ and being as close as possible to the given perturbed curve $\mathcal{C}$.

(III) We determine all the symmetries of the computed exact symmetric curve $\widetilde{\mathcal{C}}$ to obtain the approximate symmetries of the perturbed curve $\mathcal{C}$ (in particular, it is enough to compute only the mirror symmetries as the rotational symmetries can be determined already from $\widetilde{\mathbf{p}}$ and m computed before).

5.1. Computing the approximate center of the symmetry

Consider a perturbed curve of degree d given by a polynomial

$$f(x,y) = f_0(x,y) + \epsilon(x,y), \tag{46}$$

where f_0 defines a symmetric curve and ϵ is a polynomial (perturbation) having all its coefficients in $[-\varepsilon,\varepsilon]$, i.e.,

$$\epsilon(x,y) = \sum_{i=0}^{d}\sum_{j=0}^{d-i} \varepsilon_{i,j} x^i y^j, \quad |\varepsilon_{i,j}| \le \varepsilon. \tag{47}$$

The initial and crucial step of the reconstruction algorithm is to find a suitable approximate center of symmetry $\widetilde{\mathbf{p}}$ of the resulting curve $\widetilde{\mathcal{C}}$. First we construct a coefficient matrix $\mathbf{B}_f = \mathbf{B}_{f_0} + \mathbf{B}_\epsilon$ of $f(z,\overline{z})$, cf. (8). The key part is to identify the structure (zeroes) of the coefficient matrix.

Lemma 5.1. *The Taxicab (ℓ_1) norms of all elements of $\mathbf{B}_\epsilon$ are less than or equal to $K_d\,\varepsilon$, where*

$$K_d = \frac{(2n+1)}{4^n}\binom{2n}{n}, \qquad n = \left\lfloor \frac{d}{2} \right\rfloor. \tag{48}$$

Proof. From

$$\left\| \sum b_{i,j} \right\|_1 = \left|\mathrm{Re}\left(\sum b_{i,j}\right)\right| + \left|\mathrm{Im}\left(\sum b_{i,j}\right)\right| \le \sum |\mathrm{Re}\,(b_{i,j})| + \sum |\mathrm{Im}\,(b_{i,j})\,| \tag{49}$$

it follows

$$\left\| \frac{1}{2^{k+\ell}} \sum_{i=0}^{k+\ell}\sum_{j=0}^{i} \binom{k+\ell-i}{\ell-j}\binom{i}{j}\, \mathrm{i}^{2j-i}\, \varepsilon_{k+\ell-i,i} \right\|_1 \le \frac{\varepsilon}{2^{k+\ell}} \sum_{i=0}^{k+\ell}\sum_{j=0}^{i} \binom{k+\ell-i}{\ell-j}\binom{i}{j}. \tag{50}$$

First, suppose that $\epsilon(x,y)$ has an even degree $d = 2n$. Then it is easy to see that the right-hand side of (50) is maximal for $k = \ell = n$, i.e.,

$$\max_{\substack{k\,\in\,\{0,\,\dots,\,d\}\\ \ell\,\in\,\{0,\,\dots,\,d-k\}}} \left(\frac{1}{2^{k+\ell}} \sum_{i=0}^{k+\ell}\sum_{j=0}^{i} \binom{k+\ell-i}{\ell-j}\binom{i}{j} \right) = \frac{1}{2^{2n}} \sum_{i=0}^{2n}\sum_{j=0}^{i} \binom{2n-i}{n-j}\binom{i}{j}. \tag{51}$$

Then employing the well-known formula

$$\sum_{j=0}^{q}\binom{m}{j}\binom{p-m}{q-j}=\binom{p}{q} \tag{52}$$

yields

$$\frac{1}{2^{2n}}\sum_{i=0}^{2n}\sum_{j=0}^{i}\binom{2n-i}{n-j}\binom{i}{j}=\frac{1}{4^n}\sum_{i=0}^{2n}\binom{2n}{n}=\frac{(2n+1)}{4^n}\binom{2n}{n}. \tag{53}$$

The case for an odd degree $d=2n+1$ can be treated analogously, i.e, the maximal coefficient of the right-hand side of (50) occurs for $k=n$ and $\ell=n+1$ (or vice versa) and we arrive at

$$\frac{1}{2^{2n+1}}\sum_{i=0}^{2n+1}\sum_{j=0}^{i}\binom{2n+1-i}{n+1-j}\binom{i}{j}=\frac{1}{2^{2n+1}}\sum_{i=0}^{2n+1}\binom{2n+1}{n+1}=\frac{(2n+2)}{2^{2n+2}}\binom{2n+1}{n+1}=\frac{(2n+1)}{4^n}\binom{2n}{n} \tag{54}$$

for $n=\frac{d-1}{2}$, which concludes the proof.

□

Lemma 5.1 allows us to identify the "almost zero" elements of $\mathbf{B}_f$. Hence we can create a new matrix $\widetilde{\mathbf{B}}_f$ omitting these elements, i.e.,

$$\widetilde{b}_{i,j}=\begin{cases} b_{i,j}, & \text{if } \|b_{i,j}\|_1 > K_d\,\varepsilon,\\ 0, & \text{otherwise.}\end{cases} \tag{55}$$

Now, we identify the elements of $\widetilde{\mathbf{B}}_f$ which can yield the candidates to the set of potential centers of symmetry. More precisely we create sequence (28) and detect from it all possible points $\widetilde{\mathbf{p}}_1,\ldots,\widetilde{\mathbf{p}}_n$. Let us emphasize, that for the sake of symmetry we have $\widetilde{\mathbf{p}}_1=\widetilde{\mathbf{p}}_n, \widetilde{\mathbf{p}}_2=\widetilde{\mathbf{p}}_{n-1},\ldots$. From all of these candidates, we choose the one which is with the highest probability closest to the original center of symmetry. Since the generic perturbation affects the matrix $\mathbf{B}_f$ in a way that the error in each coefficient (8) is $\varepsilon\, c_{k,\ell}$, where

$$c_{k,\ell}=\frac{1}{2^{k+\ell}}\left\|\sum_{i=0}^{k+\ell}\sum_{j=0}^{i}\binom{k+\ell-i}{\ell-j}\binom{i}{j}\,\mathrm{i}^{2j-i}\right\|_1, \tag{56}$$

we choose as the approximate center $\widetilde{\mathbf{p}}=\widetilde{\mathbf{p}}_i$ (computed for (k,ℓ)) corresponding to the minimal value of $c_{k,\ell}$.

Example 5.2. Consider a symmetric curve $\mathcal{C}_0$ with the center of symmetry at $\mathbf{p}=(-2,1)$ given by

$$\begin{aligned} f_0(x,y)=\;&6x^6+75x^5+18x^4y^2-66x^4y+444x^4+114x^3y^2-468x^3y+1482x^3\\ &+18x^2y^4-12x^2y^3+192x^2y^2-1140x^2y+2764x^2+87xy^4-108xy^3+66xy^2\\ &-1116xy+2647x+6y^6-42y^5+228y^4-372y^3+172y^2-446y+1018. \end{aligned} \tag{57}$$

Next, we create a perturbed curve $\mathcal{C}$, see Fig. 5 (left)

$$f(x,y)=f_0(x,y)+\epsilon(x,y), \tag{58}$$

where $\epsilon(x,y)$ has all its coefficients in $[-\varepsilon,\varepsilon]$ and $\varepsilon=10^{-2}$, e.g.,

$$\begin{aligned} \epsilon(x,y)=\;&-0.009x^6+0.006x^5y+0.009x^5-0.001x^4y^2-0.009x^4y-0.01x^4-0.004x^3y^3\\ &-0.001x^3y^2+0.004x^3y-0.004x^3+0.01x^2y^4-0.001x^2y^3+0.002x^2y^2+0.006x^2y\\ &+0.005x^2-0.005xy^5+0.004xy^4-0.007xy^3-0.003xy^2-0.01xy+0.002x\\ &+0.004y^6-0.001y^5-0.007y^4-0.004y^3-0.002y^2+0.004y+0.003. \end{aligned} \tag{59}$$

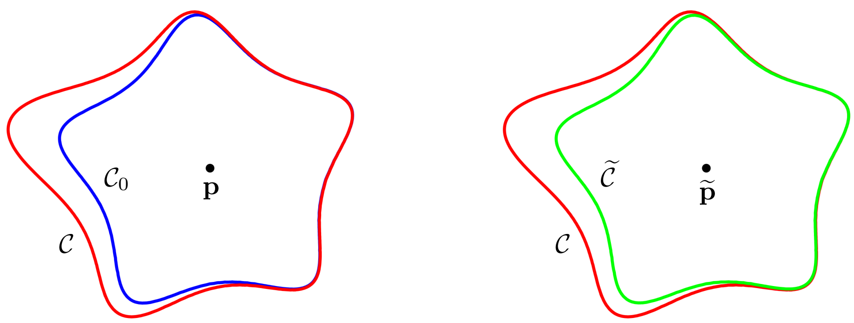

Figure 5: Left: A curve from Example 5.2 with the rotational symmetry with the center at the point $\mathbf{p}$ and the angle $2\pi/6$ (blue) and its perturbation which does not possess any symmetry (red), right: The projection (symmetric curve) $\widetilde{\mathcal{C}}$ (green) of $\mathcal{C}$ (red) to the subspace of all curves of degree 6 with the symmetry of 5-gon having the center of the symmetry at $\widetilde{\mathbf{p}}$.

Setting to zero all the elements of $\mathbf{B}_f$ with their absolute value less or equal to $K_d\,\varepsilon$, where $K_d = 36/15$, and computing (28) we arrive at

$$\{5, 3, 3, 3, 0, 0, -1\}\,. \tag{60}$$

Hence we obtain four possible pairs of (k, ℓ), i.e.,

$$(0, 4), \quad (2, 3), \quad (3, 2), \quad (4, 0). \tag{61}$$

The pairs $(0, 4)$ and $(4, 0)$ provide an approximate center of symmetry

$$\widetilde{\mathbf{p}}_1 = \left(-\frac{144007679537}{72019201310}, \frac{35997179963}{36009600655}\right) \doteq (-1.99957, 0.999655)\,, \tag{62}$$

and the pairs $(2, 3)$ and $(3, 2)$ yield

$$\widetilde{\mathbf{p}}_2 = \left(-\frac{9764166087}{4877430107}, \frac{4874772462}{4877430107}\right) \doteq (-2.00191, 0.999455)\,. \tag{63}$$

Since $c_{0,4} = c_{4,0} = 1/16 < 7/8 = c_{2,3} = c_{3,2}$, cf. (56), we set $\widetilde{\mathbf{p}} = \widetilde{\mathbf{p}}_1$. This choice is indeed better since the distances of $\widetilde{\mathbf{p}}_1$ and $\widetilde{\mathbf{p}}_2$ from $\mathbf{p}$ are approximately equal to 0.000548597 and 0.00198423, respectively. Moreover employing Lemma 4.3, we arrive at

$$m = \frac{2(2+3)}{2} = 5, \tag{64}$$

and hence (Lemma 3.1) $\mathcal{C}$ it is of s^+ type.

5.2. Reconstruction of symmetric curves

The second step of the designed reconstruction algorithm is to find a suitable symmetric curve $\widetilde{\mathcal{C}}$ sufficiently "close" to the given perturbed curve $\mathcal{C}$ when the center $\widetilde{\mathbf{p}}$ and the number of vertices m of the regular polygon are prescribed. For this purpose we introduce a metric on the projective space $\mathbb{P}^N_{\mathbb{R}}$. In particular, we consider the standard metric originated in the ray model and we will measure the angle between the lines through the origin which represent the points in the projective space, i.e., we have

$$\delta(\mathbf{x}, \mathbf{y}) = \arccos\left(\frac{|\mathbf{x}\cdot\mathbf{y}|}{\|\mathbf{x}\|\|\mathbf{y}\|}\right), \tag{65}$$

Algorithm 1 Reconstruction of a symmetric curve with inexact coefficients.

Input: A perturbed symmetric curve $\mathcal{C}$ given by the real polynomial $f(x, y)$ of degree d and a bound estimation of the perturbation ε.

1: Compute the coefficients of the matrix $\mathbf{B}_f$, see (8);
2: From $\mathbf{B}_f$ construct the matrix $\widetilde{\mathbf{B}}_f$, see (55);
3: Compute sequence (28), find the λ_i satisfying (39) and compute the corresponding (k_i, ℓ_i), see (37). From all (k_i, ℓ_i), choose the one corresponding to the minimal value of (56);
4: Compute the approximate center of symmetry $\widetilde{\mathbf{p}}$, cf. (26), and the particular m for the regular m-gon with the angle of rotation $2\pi/m$, cf. Lemma 4.3;
5: For odd m and for even $m = 4k-2$ with even d construct the basis $\mathcal{B}^+_{d,m}$, cf. (19), for even $m = 4k-2$ with odd d construct the basis $\mathcal{B}^-_{d,m}$, cf. (20), and for even $m = 4k$ decide first which basis (either $\mathcal{B}^+_{d,m}$, or $\mathcal{B}^-_{d,m}$, see Lemma 4.4) is correct for the particular d and then construct it;
6: Transform $\mathcal{B}^+_{d,m}$, resp. $\mathcal{B}^-_{d,m}$ to its real counterpart and by employing substitution (25) obtain the basis $\mathcal{B}^+_{d,m}(\mathbf{p})$, resp. $\mathcal{B}^-_{d,m}(\mathbf{p})$;
7: Compute projection (67), where S is given by $\mathcal{B}^+_{d,m}(\mathbf{p})$, resp. $\mathcal{B}^-_{d,m}(\mathbf{p})$. It yields the vector $\widetilde{\mathbf{c}}$;
8: Compute the deviation δ between $\widetilde{\mathbf{c}}$ and $\mathbf{c}$, cf. (65).

Output: A symmetric curve $\widetilde{\mathcal{C}}$ approximating $\mathcal{C}$ with the symmetry of the regular m-gon having the center of symmetry in $\widetilde{\mathbf{p}}$ and the deviation δ between $\mathcal{C}$ and $\widetilde{\mathcal{C}}$.

where '$\cdot$' and $\|\ \|$ denote the standard inner product and the standard norm in the corresponding vector space. The angle δ is real-valued, and runs from 0 to $\pi/2$.

The (standard) inner product enables us to work with orthogonal vectors and orthogonal subspaces. Then an orthogonal projection is a projection for which the range and the null space are orthogonal subspaces. This approach induces a projection in the corresponding projective space. Moreover, it holds

Lemma 5.3. *Let H be a subspace of $\mathbb{P}^N_{\mathbb{R}}$. Consider a point $\mathbf{c} \in \mathbb{P}^N_{\mathbb{R}}$ and denote by $\mathbf{c}^\perp$ the orthogonal projection of $\mathbf{c}$ to H. Then for all $\mathbf{d} \in H$ it holds $\delta(\mathbf{c}, \mathbf{d}) \geq \delta(\mathbf{c}, \mathbf{c}^\perp)$ and the equality occurs iff $\mathbf{d} = \mathbf{c}^\perp$.*

Proof. Consider the vector space $\mathbb{R}^{N+1}$ inducing the projective space $\mathbb{P}^N_{\mathbb{R}}$ and its subspace $S \subset \mathbb{R}^{N+1}$ inducing $H \subset \mathbb{P}^N_{\mathbb{R}}$. Let $\langle \mathbf{c} \rangle$ be the direction given by a representative of the point $\mathbf{c}$. The angle between the direction $\langle \mathbf{c} \rangle$ and the subspace S is defined as the angle between $\mathbf{c}$ and its orthogonal projection $\mathbf{c}^\perp$ onto S, and it is a minimum of the angles between $\langle \mathbf{c} \rangle$ and any direction given by vectors from S. $\square$

Furthermore, let $\mathbf{S}$ be the matrix whose rows are the basis vectors of the subspace S then the orthogonal projection onto the subspace S is given by the matrix

$$\mathbf{T} = \mathbf{S}^\top \left(\mathbf{S}\mathbf{S}^\top\right)^{-1} \mathbf{S}. \tag{66}$$

So, we construct the space S of all curves given by the basis $\mathcal{B}^+_{d,m}(\widetilde{\mathbf{p}})$, resp. $\mathcal{B}^-_{d,m}(\widetilde{\mathbf{p}})$. For the purpose of further calculations, i.e., for projections, we transform $\mathcal{B}^\pm_{d,m}(\widetilde{\mathbf{p}})$ into their real equivalents – if there is no danger of confusion we preserve their notation $\mathcal{B}^\pm_{d,m}(\widetilde{\mathbf{p}})$. Using (66) we compute the orthogonal projection of the point $\mathbf{c}$ representing the (perturbed) curve $\mathcal{C}$ in $\mathbb{P}^N_{\mathbb{R}}$ to the considered space. We obtain

$$\widetilde{\mathbf{c}} = \mathbf{c}\mathbf{T} \tag{67}$$

which represents the sought approximate symmetric curve $\widetilde{\mathcal{C}}$. Finally, using (65), we measure the distance between $\mathbf{c}$ and $\widetilde{\mathbf{c}}$. The whole process of the reconstruction of a symmetric curve with inexact coefficients is summarized in Algorithm 1.

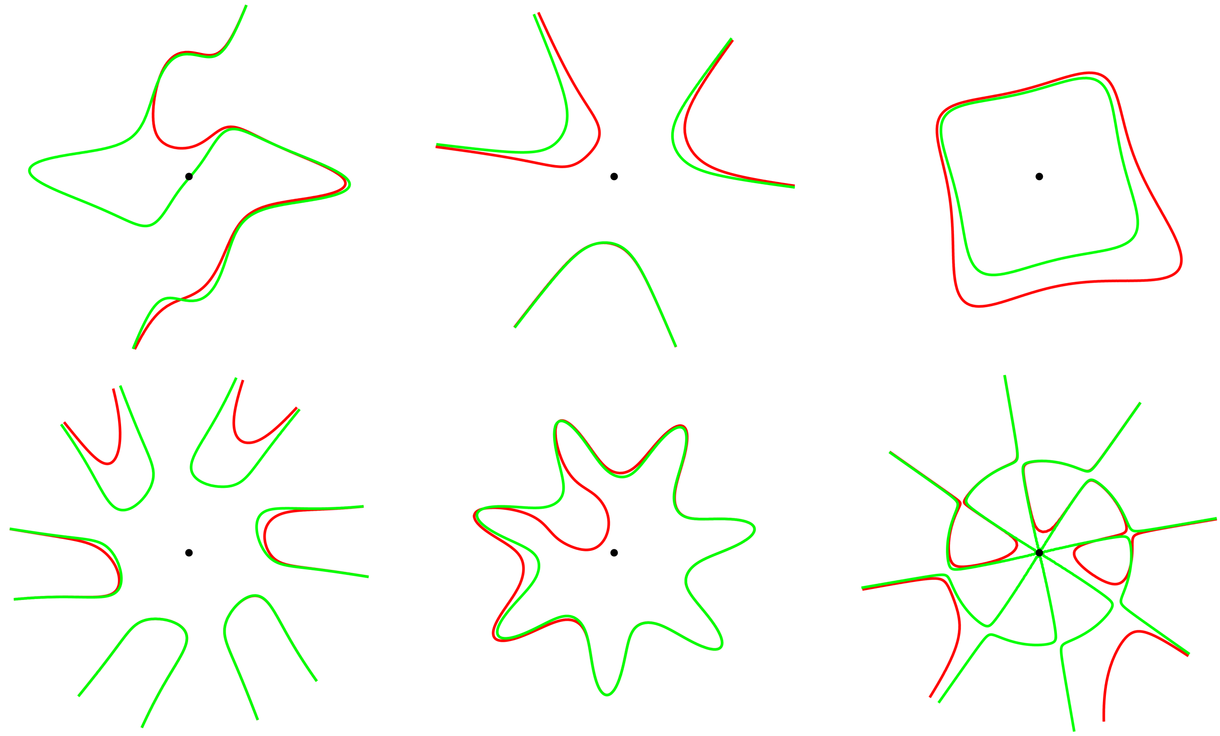

Figure 6: Some algebraic curves (red) obtained by perturbing symmetric curves for various degrees and rotations, and the corresponding approximate symmetric curves (green) computed by the presented reconstruction algorithm.

Example 5.4. We continue with Example 5.2. Thus the input is the curve given by (58) and with the computed approximate center of symmetry (62). And we are looking for a curve of s^+ type with the symmetry of a regular 5-gon. So we construct the basis

$$\mathcal{B}^+_{6,5} = \left\{1, z^5 + \overline{z}^5, \mathrm{i}\left(z^5 - \overline{z}^5\right), z\overline{z}, (z\overline{z})^2, (z\overline{z})^3\right\}. \tag{68}$$

We translate $\mathcal{B}^+_{6,5}$ to $\widetilde{\mathbf{p}}$ and compute projection (67), see Fig. 5 (right). Using (65), the distance between $\mathbf{c}$ and $\widetilde{\mathbf{c}}$, i.e., between $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, is approximately equal to $0.00399705 < \varepsilon$. Finally, applying any standard method formulated for exact symmetric curves (see e.g. [4]) we arrive at exact symmetries of $\widetilde{\mathcal{C}}$ which are the approximate symmetries of the perturbed curve $\mathcal{C}$. Thus we can obtain not only the expected rotational symmetries with the center $\tilde{\mathbf{p}}$ and the angles $\frac{2\pi\ell}{5}$, $\ell = 1, \ldots, 4$, but also all 5 mirror symmetries.

6. Computed examples and tests

In this section we present several examples and results obtained by applying the steps of the designed reconstruction algorithm, see Algorithm 1.

Example 6.1. We show our approach on several particular situations. More precisely, we employ bases (19) and (20) to create symmetric algebraic curves with randomly given centers and coefficients in the interval $[-1, 1]$. The curves in Fig. 6 were generated from $\mathcal{B}^-_{5,2}$, $\mathcal{B}^+_{5,3}$, $\mathcal{B}^+_{5,4}$, $\mathcal{B}^+_{8,6}$, $\mathcal{B}^+_{10,7}$ and $\mathcal{B}^-_{9,8}$, respectively. Then we perturb the coefficients of the defining polynomials of these curves by $\varepsilon = 10^{-2}$.

These perturbed non-symmetric curves then serve as the input to our method, see the red curves in Fig. 6. Employing the approach described in Section 5.1 we find the approximate centers of the symmetries

Table 2: The mean values of distances of the original and approximate center of symmetry and the mean values of the angles between the original, perturbed and reconstructed curves for randomly given inputs.

ϵ	$\|\mathbf{p}-\widetilde{\mathbf{p}}\|$	$\delta(\mathbf{c}_0,\mathbf{c})$	$\delta(\mathbf{c}_0,\widetilde{\mathbf{c}})$	$\delta(\mathbf{c},\widetilde{\mathbf{c}})$
10^{-1}	1.011×10^{-2}	4.904×10^{-1}	4.977×10^{-1}	7.967×10^{-1}
10^{-2}	9.165×10^{-4}	7.527×10^{-2}	6.158×10^{-2}	7.062×10^{-2}
10^{-3}	6.932×10^{-5}	3.023×10^{-3}	4.205×10^{-3}	3.57×10^{-3}
10^{-4}	5.703×10^{-6}	6.514×10^{-4}	6.512×10^{-4}	7.266×10^{-4}
10^{-5}	3.896×10^{-7}	1.886×10^{-5}	2.708×10^{-5}	2.196×10^{-5}
10^{-6}	5.774×10^{-8}	6.45×10^{-6}	6.709×10^{-6}	8.743×10^{-6}
10^{-7}	8.033×10^{-9}	4.122×10^{-7}	1.023×10^{-6}	1.175×10^{-6}
10^{-8}	3.599×10^{-9}	5.123×10^{-7}	4.122×10^{-7}	5.83×10^{-7}

and particular m-gons. Next using the projection described in Section 5.2 we arrive at the symmetric curves which are “close” to the inputs, see the green curves in Fig. 6.

Example 6.2. Analogously as in Example 6.1, we create randomly a symmetric curve $\mathcal{C}_0$ with the center of the symmetry $\mathbf{p}$ and perturb it to obtain a non-symmetric curve $\mathcal{C}$. Then we compute the approximate center of symmetry $\widetilde{\mathbf{p}}$ and the projected (reconstructed) symmetric curve $\widetilde{\mathcal{C}}$.

We are interested in the distance $\|\mathbf{p}-\widetilde{\mathbf{p}}\|$ and deviations (65) between all pairs of the curves $\mathcal{C}_0, \mathcal{C}$ and $\widetilde{\mathcal{C}}$. In Table 2 the means of these distances for 1000 different input curves for individual values of ε are listed.

7. Conclusion

In this paper, we designed a simple algorithm for an approximate reconstruction of an inexact planar curve which is assumed to be a perturbation of some unknown planar curve with symmetries of a regular polygon. As the input perturbed curve is non-symmetric, it does not possess a centre of symmetry. Nonetheless, the original curve was by assumption symmetric. So the initial step of the reconstruction algorithm is to find a suitable approximate centre of symmetry and a particular regular m-gon to whose the group of symmetries of the curve is isomorphic. In this part we use a new matrix complex representation of algebraic curves for simple estimation of the potential symmetry. Next, it was shown how to choose a suitable symmetric curve sufficiently “close” to the given perturbed curve from the subspace of all algebraic curves with the prescribed centre of symmetry and with the given associated m-gon. Finally, by computing the symmetries of the reconstructed exact symmetric curve one can obtain the approximate symmetries of the perturbed curve.

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