IMPORTANCE SAMPLING FOR MONTE CARLO SIMULATION TO EVALUATE COLLAR OPTIONS UNDER STOCHASTIC VOLATILITY MODEL

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Abstract: *The collar option is one kind of exotic options which is useful when institutional investors wish to lock in the profit they already have on the underlying asset. Under the constant volatility assumption, the pricing problem of collar options can be solved in the classical Black Scholes framework. However the smile-shaped pattern of the Black Scholes implied volatilities which extracted from options has provided evidence against the constant volatility assumption, so stochastic volatility model is introduced. Because of the complexity of the stochastic volatility model, a closed-form expression for the price of collar options may not be available. In such case, a suitable numerical method is needed for option pricing under stochastic volatility. Since the dimensions of state variable are usually more than two after the introduction of another volatility diffusion process, the classical finite difference method seems inefficient in the stochastic volatility scenario. For its easy and flexible computation, Monte Carlo method is suitable for evaluating option under stochastic volatility. This paper presents a variance reduction method for Monte Carlo computation to estimate collar option under stochastic volatility model. The approximated price of the collar option under fast mean reverting stochastic volatility model is derived from the partial differential equation by singular perturbation technique. The importance sampling method based on the approximation price is used to reduce the variance of the Monte Carlo simulation. Numerical experiments are carried out under the context of different mean reverting rate. Numerical experiment results demonstrate that the importance sampling Monte Carlo simulation achieves better variance reduction efficiency than the basic Monte Carlo simulation.*

Keywords: *Importance sampling, Monte Carlo simulation, collar options, stochastic volatility.*

JEL Classification: *C40.*

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Introduction

The collar option is one kind of exotic options which is useful when institutional investors wish to lock in the profit they already have on the underlying asset. Collar options can be implemented by investors on the stock they have already own. Usually investors will obtain the collar when they have enjoyed a decent gain on their investment but they want to hedge against potential downside in their shares. Collar options are very useful and practical instruments in revenue management and project management. Shan et al. (2010) study the use of collar options to manage revenue risks in real toll public-private partnership transportation projects, in particular how to redistribute the profit and losses in order to improve the effectiveness of risk management and fulfill the stakeholder's needs.

Under the constant volatility assumption, the pricing problem of collar option can be solved in the classical Black Scholes framework. However, the smile-shaped pattern of the Black Scholes implied volatilities which extracted from options has provided evidence against the constant volatility assumption in the Black Scholes model. Numerous methods have been carried out to relax the constant volatility assumption. One of these approaches is dropping the assumption of constant volatility and assumes that the underlying asset is driven by a stochastic volatility process. Stochastic volatility models were first studied by Hull and White (1987), Scott (1987), and Stein and Stein (1991). Other stochastic volatility model, like Heston (1993) has become important because the call price in the Heston model is available in closed form. Because of the complexity of the assumption, a closed-form expression for the option price may not be available. In such case, a suitable numerical method is needed for option pricing under stochastic volatility. Since the dimensions of state variable are usually more than two after the introduction of another volatility diffusion process, the classical finite difference method seems inefficient in the stochastic volatility scenario. For its easy and flexible computation, Monte Carlo method is suitable for evaluating option under stochastic volatility.

The Monte Carlo method has proven particularly useful in the analysis of the risk of large portfolios of financial products. A great strength of Monte Carlo techniques for risk analysis is that they can be easily used to run scenario analysis. The Monte Carlo method is not only used to analyze financial risks, but also plays a critical role in the pricing of financial instruments. Monte Carlo methods have become an increasingly important tool for analyzing financial products, as financial products become more and more complex. The use of Monte Carlo methods in financial derivatives pricing was popularized in Boyle (1977), Broadie and Glasserman (1996). Most complex derivatives are not known to have closed form pricing formula, consequently Monte Carlo simulation are employed to solve the pricing problem of complex derivatives. Longstaff and Schwartz (2001), Rogers (2002), Liu (2010) study Monte Carlo simulation in the application of pricing American options and Bermuda options. The Monte Carlo methods are also effective in solving problems concerning a number of different sources of uncertainty. Giles (2008) uses Monte Carlo methods for stochastic differential equations to model financial time series.

One of the main advantages of the Monte Carlo method is that it is efficient in pricing financial instruments with high dimensions. It is widely used in the case that the numbers of state variables are greater than two such as the stochastic volatility models. Because Monte Carlo simulation method is crucial in option pricing, there is an important need for a numerical approach to provide variance reduction. Typical methods for increasing the efficiency of Monte Carlo simulation by reducing the variance include control variate method and importance sampling method. Glasserman et al. (1999) develop a variance reduction technique for Monte Carlo simulation of path-dependent options driven by high-dimensional Gaussian vectors. Su and Fu (2000) formulate the importance sampling problem by a combination of infinitesimal perturbation analysis and stochastic approximation to minimize the variance of the price estimation. Fu et al. (2001) empirically test some Monte Carlo simulation based algorithms on the pricing of American derivatives and introduce a simultaneous perturbation stochastic approximation algorithm. By using an approximation of the option price, Fouque and Tullie (2002) proposed an importance sampling method to reduce variance in Monte Carlo computation of option price under stochastic volatility. Fouque and Han (2004) present a variance reduction method for Monte Carlo simulation to evaluate option prices under multi-factor stochastic volatility based on importance sampling. Fouque and Han (2007) propose a control variate method to price options under stochastic volatility by Monte Carlo simulations. Ma and Xu (2010) propose an efficient control variate method when the volatility follows the log-normal process, and they studied the pricing problem of variance swap option under stochastic volatility by the control variate technique. By constructing the control variate method with the order moment of the stochastic volatility, Du et al. (2013) study the pricing problem of Asian options under the stochastic volatility. Lai et al. (2015) present a control variate method with applications to Asian and basket options pricing under exponential jump diffusion model. Kassim et al. (2015) extend the adaptive importance sampling method to jump process and proved the efficiency of their method on the valuation of derivatives in several jump models. Agarwal et al. (2016) developed an efficient control variate method to price American put under stochastic volatility model via Monte Carlo simulation.

The method of importance sampling is one of the widely used variance reduction approaches. Unlike the other variance reduction methods, importance sampling is based on the idea of changing the underlying probability measure from which paths are generated. In this paper, we consider the importance sampling method developed by Fouque et al. (2002) for accelerating the Monte Carlo simulation to the pricing problem of the collar option under fast mean-reverting stochastic volatility. The main idea of this method is using the singular perturbation technique to derive the approximated formula of the collar option price, and then this approximation formula can be applied to the importance sampling method. The rest of this paper is organized as follows. A class of stochastic volatility models is introduced in section 1. Section 2 includes a general description of the importance sampling method and its application in the Monte Carlo simulation for collar option pricing. Numerical experiments comparing the basic Monte Carlo and importance sampling Monte Carlo simulation are given in Section 3. And the final section concludes the paper.

1. Stochastic Volatility Model Setting

Denoting S_t as the underlying asset price at time t. The mean-reverting process Y_t evolves as an Ornstein-Uhlenbeck (OU) process. Denoting W_t and Z_t as two independent standard Brownian motions and ρ is the correlation coefficient between these two Brownian motions. Under the risk-neutral world probability measure P^*, the model can be written as:

$$dS_t = rS_t dt + f(Y_t) S_t dW_t^* \quad (1)$$

$$\left.\begin{aligned} dY_t &= [\alpha(m - Y_t) - v\sqrt{2\alpha}\Lambda(Y_t)]dt + v\sqrt{2\alpha}(\rho dW_t^* + \sqrt{1-\rho^2}\, dZ_t^*) \\ \Lambda(y) &= \rho\frac{\mu - r}{f(y)} + \gamma_t\sqrt{1-\rho^2} \end{aligned}\right\} \quad (2)$$

The motivation of model (1) and (2) is to reflect some observed features of the underlying asset's volatility. One feature of volatility is bounded and mean reversion. In (1) we denote the volatility of the underlying asset as $\sigma_t = f(Y_t)$, where $f(y)$ is some is some positive and bounded function, because in reality the volatility is range-bound. For instance, the 30-days realized volatility for the S&P 500 from 2005 through 2014 was never below 5% or above 82%. It is often noted in empirical studies of stock prices that volatility is fluctuating fast and mean reverting. From the financial perspective, mean reverting refers to a linear pull back term in the drift of the volatility process, hence the OU process in (2) is s used to describe the mean-reverting stochastic variables. The driving volatility Y_t is a mean-reverting process with a rate of mean reversion α, the mean level of its invariant distribution and the "volatility of the volatility" $v\sqrt{2\alpha}$ corresponding to a long run standard deviation v. The invariant distribution of Y_t is the normal distribution $\mathrm{N}\left(m\ v\right)$. The drift term pulls Y_t toward m and σ_t is expected to be pulled toward the mean value of $f(Y_t)$. The rate of mean-reversion is governed by the parameter α, the greater the α is the stronger the mean reversion. As noted by Fouque et al. (2000) the empirical evidence from S&P 500 shows that the parameter is large and that v^2 is a stable $\mathrm{O}(1)$ constant. In the following we will be interested in the scenario where α is large, hence Y_t is a fast mean-reverting process on a short time scale $1/\alpha$, and we will compute the price of the collar option by Monte Carlo simulation for finite values of α.

Another feature considered is the volatility shocks are often negatively correlated with asset price shocks. From common experience and empirical studies, when volatility goes up, asset prices tend to go down and vice-versa. This is often referred to as leverage effect and there are economic arguments for a negative correlation between asset price and volatility shocks, hence the instantaneous correlation coefficient $\rho < 0$ between two shocks is considered. The skewed distribution for historical stock price is documented in empirical studies by Bates (1991) and the leverage effect can partially account for skewed distribution for the asset price that zero-correlation stochastic volatility models do not exhibit. The process γ_t which is assumed to be adapted and suitably regular is called the market price of volatility risk or volatility risk premium from the second source of random shock. The function Λ in (2) can be considered as the total risk premium because it is a linear combination of the stochastic Sharpe ratio $(\mu - r)/f(y)$ and the volatility risk premium γ weighted by the correlation ρ and $\sqrt{1-\rho^2}$, where μ represents the constant mean return rate and r represents a constant instantaneous interest rate.

Under the risk-neutral probability measure P^*, the process (S_t, Y_t) is a Markov process. The no arbitrage price of the collar option at time t is the conditional expectation of the discounted payoff at time T given that the present value of the underlying asset $S_t = s$ and the present value of the process driving the volatility $Y_t = y$. Denoting the price of the collar option at time $t < T$ as $V(t,s,y)$, the price of the collar option with stochastic volatility is given as, the expectation $E^*(\bullet)$ is computed under risk-neutral measure P^*:

$$\begin{aligned} V(t,s,y) &= E^*\{e^{-r(T-t)} \min\left(\max\left(S_T, K_1\right), K_2\right) \mid S_t = s, Y_t = y\} \\ &= E^*\{e^{-r(T-t)} \phi\left(S_T\right) \mid S_t = s, Y_t = y\} \end{aligned} \quad (3)$$

where we define $\phi(s) = \min\left(\max\left(s, K_1\right), K_2\right)$. $\phi(s)$ is the payoff of a collar option at expiry time T, where $K_2 > K_1 > 0$ and S_T is the underlying asset price at expiration time. By the Feynman-Kac formula, the pricing function given by (3) can be obtained as the solution of the partial differential equation below:

$$\begin{aligned} &\frac{\partial V}{\partial t} + r(s\frac{\partial V}{\partial s} - V) + \frac{1}{2} f^2(y) s^2 \frac{\partial^2 V}{\partial s^2} + \alpha[(m-y)\frac{\partial V}{\partial y} + v^2 \frac{\partial^2 V}{\partial y^2}] \\ &+ v\sqrt{2\alpha}\rho f(y) s \frac{\partial^2 V}{\partial s \partial y} - v\sqrt{2\alpha}\Lambda(y)\frac{\partial V}{\partial y} = 0 \end{aligned} \quad (4)$$

with the terminal condition as:

$$V(T,s,y) = \phi(s) \quad (5)$$

Considering a small quantity denoted as $\varepsilon = 1/\alpha$ and $0 < \quad < 1$. By introducing the following operators:

$$\mathsf{L}_0 = (m-y)\frac{\partial}{\partial y} + v^2 \frac{\partial^2}{\partial y^2} \quad (6)$$

$$\mathsf{L}_1 = v\sqrt{2}[\rho f(y) s \frac{\partial^2}{\partial s \partial y} - \Lambda(y)\frac{\partial}{\partial y}] \quad (7)$$

$$\mathsf{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2(y) s^2 \frac{\partial^2}{\partial s^2} + rs\frac{\partial}{\partial s} - r \quad (8)$$

the partial differential equation (4) involves terms of order $1/\varepsilon, 1/\sqrt{\varepsilon}$ and 1 becomes:

$$(\frac{1}{\varepsilon}\mathsf{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathsf{L}_1 + \mathsf{L}_2)V = 0 \quad (9)$$

The problem (9) is called a singular perturbation problem owing to the diverging terms when ε goes zero, keeping the time derivatives in L_2 of order 1. The solution V has a limit as ε goes to zero and it is able to characterize the first correction for small but nonzero ε. By the method of singular perturbation of the partial differential equation, the approximated price of the collar option can be obtained. The details of this singular perturbation method can be referred to Fouque et al. (2003).

The approximated price of the collar option is $\tilde{V}(t,s,y) = V_0(t,s) + \sqrt{\varepsilon}V_1(t,s)$, where $\varepsilon = 1/\alpha$ and $0 < \varepsilon \le 1$. In particularly, $V_0(t,s)$ does not depend on y and it is given by:

$$\begin{aligned} V_0(t,s) = K_1 e^{-r(T-t)} + sN\left(d_1^{K_1}\right) - K_1 e^{-r(T-t)} N\left(d_2^{K_1}\right) \\ - sN\left(d_1^{K_2}\right) + K_2 e^{-r(T-t)} N\left(d_2^{K_2}\right) \end{aligned} \quad (10)$$

where $d_1^{K_i} = \dfrac{\ln\left(s/K_i\right) + (r + \bar{\sigma}^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}}$, $d_2^{K_i} = d_1^{K_i} - \bar{\sigma}\sqrt{T-t}$. $(i = 1, 2)$ and $N(\bullet)$ is the cumulative standard normal distribution and $\bar{\sigma}$ is a constant effective volatility which is the average with respect to the invariant distribution of Y. $V_1(t,s)$ is also independent of y and $\sqrt{\varepsilon}V_1(t,s)$ can be expressed as:

$$\sqrt{\varepsilon}V_1(t,s) = -(T-t)(C_1 s^2 \frac{\partial^2 V_0}{\partial s^2} + C_2 s^3 \frac{\partial^3 V_0}{\partial s^3}) \quad (11)$$

where C_1 and C_2 are two parameters which can be calibrated from implied volatility surface. From the results of (10) and (11), by direct calculation, we can easily obtain the approximated price:

$$\begin{aligned} \tilde{V}(t,s,y) &= V_0(t,s) + \sqrt{\varepsilon}V_1(t,s) \\ &= K_1 e^{-r(T-t)} + sN\left(d_1^{K_1}\right) - K_1 e^{-r(T-t)} N\left(d_2^{K_1}\right) \\ &\quad - sN\left(d_1^{K_2}\right) + K_2 e^{-r(T-t)} N\left(d_2^{K_2}\right) \\ &\quad + \frac{sn\left(d_1^{K_1}\right)}{\bar{\sigma}}\left(C_2 \frac{d_1^{K_1}}{\bar{\sigma}} + (C_2 - C_1)\sqrt{T-t}\right) \\ &\quad + \frac{sn\left(d_1^{K_2}\right)}{\bar{\sigma}}\left(C_2 \frac{d_1^{K_2}}{\bar{\sigma}} + (C_2 - C_1)\sqrt{T-t}\right) \end{aligned} \quad (12)$$

where $N(\bullet)$ is the standard normal probability density function. The details of the proof are provided in the Appendix A. This approximated price can be used to implement the importance sampling variance reduction technique.

2. Importance Sampling for Collar Option

According to (1) and (2) the evolution of (S_t, Y_t) under the risk-neutral measure P^* can be presented in the matrix form as following:

$$dX_i = \mu(t, Xt)dt + \sum(t,X_t)dB_t \tag{13}$$

we set the following vectors:

$$x = \begin{pmatrix} s \\ y \end{pmatrix},\ X_t = \begin{pmatrix} S_t \\ Y_t \end{pmatrix},\ B_t = \begin{pmatrix} W_t^* \\ Z_t^* \end{pmatrix}$$

and define the drift vector:

$$\mu(t,x) = \begin{pmatrix} rs \\ \alpha(m-y) - v\sqrt{2\alpha}\Lambda(y) \end{pmatrix}$$

and define the diffusion matrix:

$$\Sigma(t,x) = \begin{pmatrix} f(y)s & 0 \\ v\rho\sqrt{2\alpha} & v\sqrt{2\alpha(1-\rho^2)} \end{pmatrix}$$

where B_t is a standard 2-dimensional Brownian motion under P*, $\mu(t,x) \in \mathrm{R}^2$ and $\Sigma(t,x) \in \mathrm{R}^{2\times 2}$ are regular enough to ensure the existence and uniqueness of the solution. Denoting the value of the collar option at expiration as V_T, the price $V(t,s,y)$ of the collar option with stochastic volatility at time t can be rewritten as:

$$\begin{aligned} V(t,x) &= \mathrm{E}^*\left\{e^{-r(T-t)}\phi(S_T) \mid X_t = x\right\} = \\ &= x\Big\} = \mathrm{E}^*\left\{e^{-r(T-t)}V_T \mid X_t = x\right\} \end{aligned} \tag{14}$$

Hence the basic Monte Carlo simulation for (14) can be approximated by calculating the sample mean in the following way:

$$\begin{aligned} V(t,x) &\approx e^{-r(T-t)}\frac{1}{N}\sum_{k=1}^{N}\phi\left(S_T^{(k)}\right) \approx \\ &\approx e^{-r(T-t)}\frac{1}{N}\sum_{k=1}^{N}\min\left(\max\left(S_T^{(k)},\ K_1\right),\ K_2\right) \end{aligned} \tag{15}$$

where $S_T^{(k)} (k = 1, 2, \cdots, N)$ are independent realizations of the underlying asset price at time T, and N is the total number of independent realizations of the underlying asset price process. Given $\gamma(t, X_t)$ is an adapted R^2 valued process, we consider the following process:

$$\eta_t = \exp\left(\int_0^t \gamma(u, X_u)\cdot dB_u + \frac{1}{2}\int_0^t \left\|\gamma(u, X_u)\right\|^2 du\right)$$

and suppose that $\mathrm{E}^*(\eta_T) = 1$. Defining a probability measure $\overline{\mathrm{P}}^*$ equivalent to P^* by means of the Radon-Nikodym derivative:

$$\frac{d\overline{\mathrm{P}}^*}{d\mathrm{P}^*} = \frac{1}{\eta_T}$$

then by Girsanov's theorem, the process $\overline{B}_t$, which is given by the following formula:

$$\overline{B}_t = B_t + \int_0^t \gamma(u, X_u)du$$

follows a standard 2-dimensions Brownian motion under the new measure $\overline{\mathrm{P}}^*$. Under this new measure the evolution of the processes X_t and Y_t can be written in terms of the Brownian motion $\overline{B}_t$ as the following:

$$\begin{aligned} dX_t &= \left(\mu(t, X_t) - \Sigma(t, X_t)\gamma(t, X_t)\right)dt + \\ &+ \Sigma(t, X_t)d\overline{B}_t \end{aligned} \tag{16}$$

$$d\eta_t = \eta_t\gamma(t, X_t)\cdot d\overline{B}_t \tag{17}$$

According to the abstract version of Bayes's formula, the price of the collar option at time can be written with respect to the new measure $\overline{\mathrm{P}}^*$ as:

$$V(t,x) = \overline{\mathrm{E}}^*\left\{e^{-r(T-t)}\phi(S_T)\eta_T \mid X_t = x\right\} \tag{18}$$

Then the Monte Carlo simulation for the approximation of (18) can be calculated in the following manner:

$$\begin{aligned} V(t,x) &\approx e^{-r(T-t)}\frac{1}{N}\sum_{k=1}^{N}\phi\left(S_T^{(k)}\right)\eta_T^{(k)} \approx \\ &\approx e^{-r(T-t)}\frac{1}{N}\sum_{k=1}^{N}\min\left(\max\left(S_T^{(k)},\ K_1\right),\ K_2\right)\eta_T^{(k)} \end{aligned} \tag{19}$$

where $S_T^{(k)}$ and $\eta_T^{(k)}$ are the k-th independent realization of process S_t and η_t at time T respectively. In order to simplify the notation, we denote $\mu(t,x)$ as μ, $\Sigma(t,x)$ as Σ, and $V(t,x)$ as V; we further denote the gradient of the state variables of $V(t,x)$ and the Hessian of the state variable of $V(t,x)$ as the following:

$$\nabla V = \begin{pmatrix} \dfrac{\partial V}{\partial s} \\ \dfrac{\partial V}{\partial y} \end{pmatrix}, \ \nabla^2 V = \begin{pmatrix} \dfrac{\partial^2 V}{\partial s^2} & \dfrac{\partial^2 V}{\partial s \partial y} \\ \dfrac{\partial^2 V}{\partial y \partial s} & \dfrac{\partial^2 V}{\partial y^2} \end{pmatrix}$$

Lemma 1: Assuming that the quantity $V(t,x)$ was known and the function $\gamma(t,x)$ is expressed as follow:

$$\gamma(t,x) = -\frac{1}{V(t,x)} \Sigma^T (\nabla V) \tag{20}$$

then the variance of $\phi(S_T)\eta_T$ is zero, where $\sum^T$ is transpose of $\sum$.

Proof: According to Ito's formula, we can compute:

$$\begin{aligned} dV(t,X_t) &= \left\{ \frac{\partial V}{\partial t} + \left(\mu(t,X_t) - \Sigma(t,X_t)\gamma(t,X_t) \right) \cdot \nabla V + \frac{1}{2} \left(\Sigma\Sigma^T \right) \cdot \left(\nabla^2 V \right) \right\} dt + \nabla V \cdot \left(\Sigma(t,X_t) d\overline{B}_t \right) \\ &= \left\{ rV(t,X_t) - \Sigma(t,X_t)\gamma(t,X_t) \cdot \nabla V \right\} dt + \nabla V \cdot \left(\Sigma(t,X_t) d\overline{B}_t \right) \end{aligned}$$

The second equality is the result of applying Feyman-Kac formula on (14) to obtain:

$$\frac{\partial V}{\partial t} + \mu(t,X_t) \cdot \nabla V + \frac{1}{2} \left(\Sigma\Sigma^T \right) \cdot \left(\nabla^2 V \right) = rV(t,X_t)$$

From the result of (17) we know that:

$$\begin{aligned} d\eta_t dV(t,X_t) &= \left[\eta_t \gamma(t,X_t) \cdot d\overline{B}_t \right] \left[\left\{ rV - \Sigma(t,X_t)\gamma(t,X_t) \cdot \nabla V \right\} dt + \nabla V \cdot \left(\Sigma(t,X_t) d\overline{B}_t \right) \right] \\ &= \eta_t \left[\gamma(t,X_t) \cdot d\overline{B}_t \right] \left[\nabla V \cdot \left(\Sigma(t,X_t) d\overline{B}_t \right) \right] \\ &= \eta_t \left(\Sigma(t,X_t)\gamma(t,X_t) \cdot \nabla V \right) dt \end{aligned}$$

By the Ito product rule we can compute:

$$\begin{aligned} d\left(V(t,X_t)\eta_t \right) &= \eta_t dV(t,X_t) + V(t,X_t) d\eta_t + d\eta_t dV(t,X_t) \\ &= \eta_t \left\{ rV(t,X_t) - \Sigma(t,X_t)\gamma(t,X_t) \cdot \nabla V \right\} dt + \eta_t \nabla V \cdot \left(\Sigma(t,X_t) d\overline{B}_t \right) \\ &\quad + V(t,X_t)\eta_t \gamma(t,X_t) \cdot d\overline{B}_t + \eta_t \left(\Sigma(t,X_t)\gamma(t,X_t) \cdot \nabla V \right) dt \\ &= r\eta_t V(t,X_t) dt + \eta_t \nabla V \cdot \left(\Sigma(t,X_t) d\overline{B}_t \right) + \eta_t V(t,X_t)\gamma(t,X_t) \cdot d\overline{B}_t \\ &= r\eta_t V(t,X_t) dt + \eta_t \Sigma^T(t,X_t) \nabla V \cdot d\overline{B}_t + \eta_t V(t,X_t)\gamma(t,X_t) \cdot d\overline{B}_t \end{aligned}$$

and the following result

$$\begin{aligned} d\left(e^{-rt} V(t,X_t)\eta_t \right) &= V(t,X_t)\eta_t d\left(e^{-rt} \right) + e^{-rt} d\left(V(t,X_t)\eta_t \right) \\ &= e^{-rt} \eta_t \left\{ \Sigma^T(t,X_t) \nabla V \cdot d\overline{B}_t + V(t,X_t)\gamma(t,X_t) \cdot d\overline{B}_t \right\} \end{aligned}$$

In order to obtain $\phi(S_T)\eta_T$, we can integrate above equation from 0 to time T, we have:

$$e^{-rT} V(T,X_T)\eta_T = V(0,X_0) + \int_0^T e^{-ru} \eta_u \left(\Sigma^T(u,X_u) \nabla V + V(u,X_u)\gamma(u,X_u) \right) \cdot d\overline{B}_u$$

or:

$$\phi(S_T)\eta_T = V(T, X_T)\eta_T$$
$$= e^{rT} V(0, X_0) + e^{rT} \int_0^T e^{-ru} \eta_u \left(\Sigma^T(u, X_u) \nabla V + V(u, X_u) \gamma(u, X_u) \right) \cdot d\bar{B}_u$$

According to Ito isometry, the variance of $\phi(S_T)\eta_T$ can be easily obtain:

$$\mathrm{Var}_{\bar{\mathrm{P}}^*} \left(\phi(S_T)\eta_T \right) = e^{2rT} \bar{\mathrm{E}}^* \left\{ \int_0^T e^{-2ru} \eta_u^2 \left\| \Sigma^T(u, X_u) \nabla V + V(u, X_u) \gamma(u, X_u) \right\|^2 du \right\}$$

Hence if $V(t,x)$ was known, then (20) is the optimal choice for $\gamma(t, X_t)$ which gives the zero variance of $\phi(S_T)\eta_T$.

By using the approximated price $\tilde{V}(t,x)$ in (20), we can implement the Monte Carlo simulation according to (19).

3. Numerical Computation

From (10) in section 1, we know that the delta of the approximated price of the collar option under stochastic volatility is:

$$\frac{\partial \tilde{V}}{\partial s} = \frac{\partial V_0}{\partial s} - (T-t) \left(C_1 2s \frac{\partial^2 V_0}{\partial s^2} + \right.$$
$$\left. + C_1 s^2 \frac{\partial^3 V_0}{\partial s^3} + C_2 3s^2 \frac{\partial^3 V_0}{\partial s^3} + C_2 s^3 \frac{\partial^4 V_0}{\partial s^4} \right) \quad (21)$$

By the result of we can compute the following, see Appendix A.

$$2s \frac{\partial^2 V_0}{\partial s^2}, \; s^2 \frac{\partial^3 V_0}{\partial s^3}, \; 3s^2 \frac{\partial^3 V_0}{\partial s^3}, \; s^3 \frac{\partial^4 V_0}{\partial s^4}$$

Note that V_0 and V_1 are independent of y hence:

$$\nabla \tilde{V} = \left(\frac{\partial \tilde{V}}{\partial s}, \frac{\partial \tilde{V}}{\partial y} \right)^T = \left(\frac{\partial \tilde{V}}{\partial s}, 0 \right)^T$$

By substituting $\tilde{V}$ in lemma 1, we have:

$$\gamma(t,s,y) = \frac{-1}{\tilde{V}} \begin{bmatrix} f(y)s & v\rho\sqrt{2\alpha} \\ 0 & v\sqrt{2\alpha(1-\rho^2)} \end{bmatrix}$$
$$\begin{bmatrix} \frac{\partial \tilde{V}}{\partial s} \\ 0 \end{bmatrix} = \frac{-1}{\tilde{V}} \begin{bmatrix} f(y)s \frac{\partial \tilde{V}}{\partial s} \\ 0 \end{bmatrix} \quad (22)$$

In the following numerical experiment, we will compare the variance for the basic Monte Carlo simulation and the variance for the importance sampling Monte Carlo simulation. The basic Monte Carlo simulation refers to calculate the price of the collar option under measure P^*, is based on calculating the sample mean by (15). The importance sampling Monte Carlo simulation refers to calculate the option price under measure $\bar{\mathrm{P}}^*$, is based on calculating the sample by (19), when the optimal choice of γ is obtained through (22). Euler scheme is employed to simulate the discretization of the diffusion process of X_t which will be used in the basic and importance sampling Monte Carlo simulation. The numerical experiment is based on the following relevance parameters (Fouque & Tullie, 2002):

$$m = -2.6, \;\; v = 1, \;\; \rho = -0.3, \;\; \bar{\sigma} = 0.2,$$
$$T = 1, \;\; t = 0, \;\; K_1 = 50, \;\; K_2 = 150$$

The rate of mean-reverting α is assumed to range from $\alpha = 50$ to $\alpha = 400$. The volatility risk price $\Lambda(y)$ is chosen to be zero; the volatility function is assumed to be $f(y) = \max\left[\min\left[e^y, 0.5 \right], 0.0001 \right]$ which ensures that the volatility is bounded. The starting values of the diffusion process are chosen to be $S_0 = 110$ and $Y_0 = -2.32$. Total number of realizations is $N = 500$ in each simulation with time step $\Delta t = 10^{-3}$. The algorithm implementation steps are provided in Appendix B. The result of the numerical experiment is presented in the following tables.

It can be easily observed from the results of Tab. 1 that importance sampling Monte Carlo successfully reduce the variance of the option price. Furthermore, the variance reduction is more significant in the regime where the rate of mean-reversion is large. Tab. 2 and Tab. 3

Tab. 1: Comparison of Monte Carlo simulation of variance and option price with different α

α	Basic Monte Carlo	Importance sampling Monte Carlo
400	1.211321 (106.59)	0.040211 (106.64)
200	1.042794 (107.87)	0.018769 (107.84)
100	0.535174 (108.45)	0.030871 (108.33)
50	0.786324 (108.60)	0.030489 (108.53)

Source: own

Tab. 2: Comparison of Monte Carlo simulation of variance and option price with different $\bar{\sigma}$ when $\alpha = 100$

$\bar{\sigma}$	Basic Monte Carlo	Importance sampling Monte Carlo
0.25	0.481509 (108.589)	0.008291 (108.299)
0.20	0.594106 (108.2456)	0.036746 (108.2492)
0.15	0.55949 (108.1709)	0.040022 (108.3748)
0.10	0.465523 (107.7254)	0.011623 (108.3127)

Source: own

Tab. 3: Comparison of Monte Carlo simulation of variance and option price with different $\bar{\sigma}$ when $\alpha = 200$

$\bar{\sigma}$	Basic Monte Carlo	Importance sampling Monte Carlo
0.25	0.460534 (107.733)	0.0192175 (107.7128)
0.20	0.711191 (107.807)	0.0163262 (107.7302)
0.15	1.32531 (107.7402)	0.014895 (107.8398)
0.10	0.718793 (107.924)	0.0281919 (107.9423)

Source: own

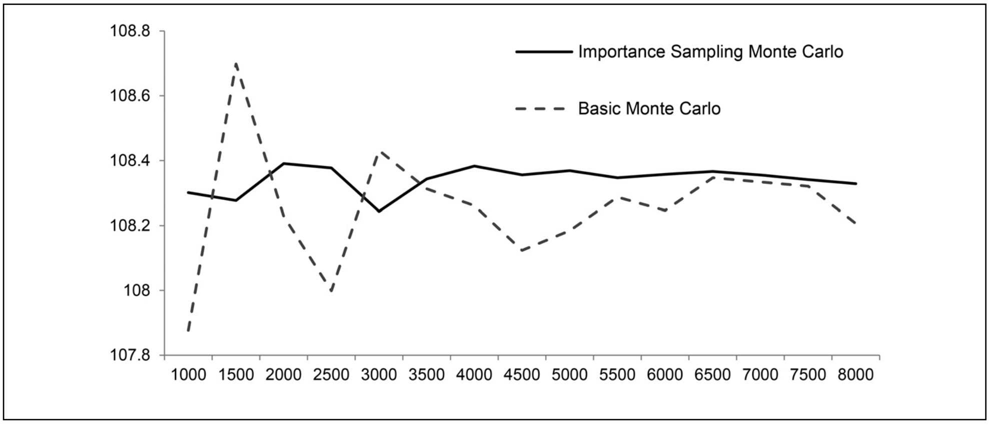
demonstrate the comparison of Monte Carlo simulation of variance and option price between basic Monte Carlo and importance sampling Monte Carlo. The mean-reversion rate is fixed at 100 and 200 while the effective volatility $\bar{\sigma}$ ranges from 0.1 to 0.25. The variance reduction is also significant, but unlike the case of different mean-reversion rate, the effective volatility seems independent of the performance of variance reduction.

Fig. 1 shows the numerical result of two Monte Carlo simulations as a function of the number of realizations with the mean-reversion rate equal to 100. It can be clearly shown by the figure that the basic Monte Carlo simulation performs poorly when compared to the importance sampling Monte Carlo simulation.

Conclusions

In this paper, we study the importance sampling variance reduction technique in the pricing problem of collar option under the context of fast mean-reverting stochastic volatility. The importance sampling technique in this paper is based on the approximation of the option price which was derived from the pricing partial differential equation by the singular perturbation. The numerical experiment for

Fig. 1: Monte Carlo simulation with mean-reversion rate $\alpha = 100$



Source: own

the collar option demonstrates the significant reduction of the option price variance from the basic Monte Carlo simulation to the importance sampling Monte Carlo simulation. This method can be easily carried out on other derivatives.

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Appendix A

As shown by Fouque et al. (2003), under the assumption that α is large and $0 < \varepsilon \le 1$, the approximated price of the collar option can be given explicitly by:

$$\tilde{V} = V_0 + \sqrt{\varepsilon}V_1 = V_0 - (T-t)\left(C_1 s^2 \frac{\partial^2 V_0}{\partial x^2} + C_1 s^3 \frac{\partial^3 V_0}{\partial x^3} \right)$$

where V_0 is the Black-Scholes price with constant volatility $\bar{\sigma}$. The payoff of the collar option at expiration is:

$$V_0(T,s) = \min\left(\max\left(s, K_1\right), K_2\right) = K_1 + (S_T - K_1)^+ - (S_T - K_2)^+$$

It can be easily seen that the arbitrage price of the collar option at $t < T$, can be explicitly represented by:

$$V_0(t,s) = K_1 e^{-r(T-t)} + C(t,s;K_1) - C(t,s;K_2)$$

where $C(t,s;K_i)$ is the Black-Scholes call option price with strike price K_i, and constant volatility $\bar{\sigma}$. Hence the Black Scholes price with constant volatility $\bar{\sigma}$ is given as (10). From the result of (10), the first derivative of V_0 respect to s can be obtained as follows:

$$\frac{\partial V_0}{\partial s} = N(d_1^{K_1}) - N(d_1^{K_2})$$

By direct computation, we can obtain the following results:

$$2s \frac{\partial^2 V_0}{\partial s^2} = \frac{2}{\bar{\sigma}\sqrt{T-t}}\left(n(d_1^{K_1}) - n(d_1^{K_2}) \right)$$

$$s^2 \frac{\partial^3 V_0}{\partial s^3} = \frac{-n(d_1^{K_1})}{\bar{\sigma}\sqrt{T-t}}\left(\frac{d_1^{K_1}}{\bar{\sigma}\sqrt{T-t}} + 1 \right) - \frac{-n(d_1^{K_2})}{\bar{\sigma}\sqrt{T-t}}\left(\frac{d_1^{K_2}}{\bar{\sigma}\sqrt{T-t}} + 1 \right)$$

$$\begin{aligned} s^3 \frac{\partial^4 V_0}{\partial s^4} = & \frac{n(d_1^{K_1})}{\bar{\sigma}\sqrt{T-t}}\left(\frac{d_1^{K_1}}{\bar{\sigma}\sqrt{T-t}} + 2 \right)\left(\frac{d_1^{K_1}}{\bar{\sigma}\sqrt{T-t}} + 1 \right) \\ & - \frac{n(d_1^{K_2})}{\bar{\sigma}\sqrt{T-t}}\left(\frac{d_1^{K_2}}{\bar{\sigma}\sqrt{T-t}} + 2 \right)\left(\frac{d_1^{K_2}}{\bar{\sigma}\sqrt{T-t}} + 1 \right) \\ & + \frac{1}{\bar{\sigma}^3 (T-t)^{3/2}}\left[n(d_1^{K_2}) - n(d_1^{K_1}) \right] \end{aligned}$$

by direct calculation, we can easily obtain the approximated price (12).

Appendix B

When $\tilde{V}$ is derived, Monte Carlo simulation can be implemented according to (19). From the results of the approximated price of collar option which is given by (12), we can obtain the expression of $\partial\tilde{V}/\partial s$ which is shown as (21). In the numerical experiment, we assume *Λ(y)* is negligible and $f(y) \quad \max\left[\min\left[e \;, 0.5\right], 0.0001\right]$. From the results of (16), (17), (22) and the preceding assumptions, the processes of η, S and Y can be written as:

$$d\eta_t = -\eta_t e^{Y_t} S_t((\partial\tilde{V}/\partial s)\tilde{V}^{-1})\, d\overline{W}_t^*$$

$$dS_t = \left(rS_t + e^{2Y_t} S_t((\partial\tilde{V}/\partial s)\tilde{V}^{-1})\, dt\right) + S_t e^{Y_t}\, d\overline{W}_t^*$$

$$dY_t = \left(\alpha(m - Y_t) + \rho\nu\sqrt{2\alpha} e^{Y_t} S_t((\partial\tilde{V}/\partial s)\tilde{V}^{-1})\right)dt + \nu\sqrt{2\alpha}(\rho d\overline{W}_t^* + \sqrt{1-\rho^2}\, d\overline{Z}_t^*)$$

Next, we use Euler scheme to simulate the discretization of the above diffusion process and carry out the importance sampling Monte Carlo simulation according to the following steps.

Step 1: Set the initial value of η, S, Y. Set the following for time step Δt and other parameters.

Step 2: Generate two independent standard normal variables z_1 and z_2, then construct another random variable as:

$$z_\rho = \rho z_1 + \sqrt{1-\rho^2}\, z_2$$

Step 3: Simulate a potential price path under the stochastic volatility price processes by the following Euler scheme:

$$\eta_{t_{i+1}} = \eta_{t_i} - \eta_{t_i} f(Y_{t_i}) S_{t_i}\,((\partial\tilde{V}/\partial s)\,\tilde{V}^{-1})_i)\sqrt{\Delta t}\, z_1$$

$$S_{t_{i+1}} = S_{t_i} + \left(rS_{t_i} + f^2(Y_{t_i}) S_{t_i}^2((\partial\tilde{V}/\partial s)\,\tilde{V}^{-1})_i\right)\Delta t + f(Y_{t_i}) S_{t_i}\sqrt{\Delta t}\, z_1$$

$$Y_{t_{i+1}} = Y_{t_i} + \left(\alpha(m - Y_{t_i}) + \rho\nu\sqrt{2\alpha} f(Y_{t_i}) S_{t_i}\,((\partial\tilde{V}/\partial s)\,\tilde{V}^{-1})_i\right)\Delta t + \nu\sqrt{2\alpha}\sqrt{\Delta t}\, z_\rho$$

Step 4: Loops step 2 and step 3 to produce simulated paths and obtain $S_T^{(k)}$ and $\eta_T^{(k)}$ at the end of each path, where $S_T^{(k)}$ and $\eta_T^{(k)}$ are the k-th independent simulated values at expiration time T.

Step 5: Average the discounted prices to obtain the final result by:

$$e^{-r(T-t)} \frac{1}{N} \sum_{k=1}^{N} \min\left(\max\left(S_T^{(k)},\, K_1\right),\, K_2\right)\eta_T^{(k)}$$