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# Investigation of critical rotor revolution in rotating space

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## 1. Introduction

The rotating space can be advantageously used for modelling rotating structures as e.g. rotors, turbine blades etc. This approach can be very useful especially in case of rotors having isotropic bearings and non-symmetrical cross sections. In such case result of modelling is time independent mathematical model with constant matrices of stiffness, mass, membrane and gyroscopic forces. It suggests itself a question what is the meaning of complex eigenvalues and mode shapes obtained from model corresponding to rotating space and how can be these quantities used for determining of critical angular rotor speed.

## 2. Mathematical model

As a starting point of modelling we can use expression of kinetic and potential energy to create conservative mathematical model of the rotor with circular cross section. According to

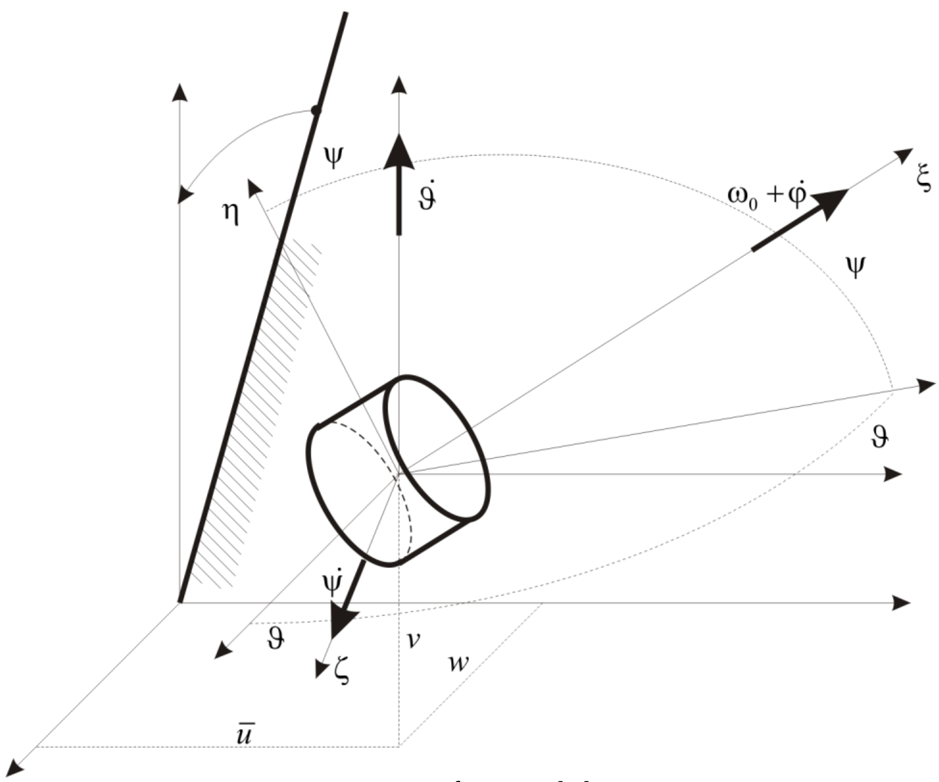


Fig. 1. Rotor infinitesimal element

Fig. 1, we can express displacement and velocity vectors of the gravity centre of an infinitesimal element in form

$$\mathbf{u}_S = \begin{bmatrix} \bar{u} \\ v \\ w \end{bmatrix}, \quad \mathbf{v}_S = \dot{\mathbf{u}}_S = \left.\frac{d\mathbf{u}_S}{dt}\right|_{\xi\eta\zeta} + \boldsymbol{\omega} \times \mathbf{u}_S \doteq \begin{bmatrix} \dot{\bar{u}} \\ \dot{v} - \omega w \\ \dot{w} + \omega v \end{bmatrix}, \tag{1}$$

where

$$\boldsymbol{\omega} = \begin{bmatrix} \dot{\varphi} + \omega + \dot{\vartheta}\psi \\ \dot{\vartheta} \\ \dot{\psi} \end{bmatrix}, \quad f^{\backslash} = \frac{\partial f}{\partial \xi}, \quad \dot{f} = \frac{\partial f}{\partial t}. \tag{2}$$

Vector $\boldsymbol{\omega}$ expresses resulting angular velocity of the infinitesimal element. Using FEM for modelling we can take these common used functions for displacement approximations

$$v(\xi) = \boldsymbol{\Phi}(\xi)\mathbf{S}_1^{-1}\mathbf{q}_1, \;\; \psi(\xi) = \frac{\partial v}{\partial \xi} = \boldsymbol{\Phi}^{\backslash}(\xi)\mathbf{S}_1^{-1}\mathbf{q}_1, \; \bar{u}(\xi) = \boldsymbol{\Psi}(\xi)\,\mathbf{S}_3^{-1}\mathbf{q}_3, \quad \varphi(\xi) = \boldsymbol{\Psi}(\xi)\mathbf{S}_3^{-1}\mathbf{q}_4,$$

$$w(\xi) = \boldsymbol{\Phi}(\xi)\mathbf{S}_1^{-1}\mathbf{P}\mathbf{q}_2, \; \vartheta(\xi) = -\frac{\partial w}{\partial \xi} = -\boldsymbol{\Phi}^{\backslash}(\xi)\mathbf{S}_1^{-1}\mathbf{P}\mathbf{q}_2, \tag{3}$$

where

$$\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix}, \quad \mathbf{P} = diag\{1, \quad -1, \quad 1, \quad -1\} \Rightarrow \mathbf{P}^{-1} = \mathbf{P},$$

$$\boldsymbol{\Phi}(\xi) = \begin{bmatrix} 1, & \xi, & \xi^2, & \xi^3 \end{bmatrix}, \; \boldsymbol{\Psi}(\xi) = \begin{bmatrix} 1, & \xi \end{bmatrix},$$

$$\mathbf{q}_1 = \begin{bmatrix} v_1, & \psi_1, & v_2, & \psi_2 \end{bmatrix}^T, \; \mathbf{q}_2 = \begin{bmatrix} w_1, & \vartheta_1, & w_2, & \vartheta_2 \end{bmatrix}^T, \; \mathbf{q}_3 = \begin{bmatrix} \bar{u}_1, & \bar{u}_2 \end{bmatrix}^T, \; \mathbf{q}_4 = \begin{bmatrix} \varphi_1, & \varphi_2 \end{bmatrix}^T.$$

Respecting foregoing relations and Fig. 1 we can express kinetic and potential energy of one finite rotor element by means of rotatory system coordinates in form

$$E_K = \frac{1}{2}\int_0^l \rho A \mathbf{v}_S^T \mathbf{v}_S d\xi + \frac{1}{2}\int_0^l \rho\, \boldsymbol{\omega}^T \mathbf{J} \boldsymbol{\omega}\; d\xi, \;\; \mathbf{J} = diag\left\{J_o, \frac{1}{2}J_o, \frac{1}{2}J_o\right\}. \tag{4}$$

Respecting (1) and (2) we can come to

$$\begin{aligned} E_K = {} & \frac{1}{2}\int_0^l \rho A\left(\dot{\bar{u}}^2 + \dot{v}^2 - 2\omega\dot{v}w + \omega^2 w^2 + \dot{w}^2 + 2\omega\dot{w}v + \omega^2 v^2\right) d\xi + \\ & + \frac{1}{2}\int_0^l \rho J_o\left(\dot{\varphi}^2 + \omega^2 + \underline{\dot{\vartheta}^2\psi^2 + 2\dot{\varphi}\omega + 2\dot{\varphi}\dot{\psi}\psi} + 2\omega\dot{\vartheta}\psi + \frac{1}{2}\dot{\vartheta}^2 + \frac{1}{2}\dot{\psi}^2\right) d\xi. \end{aligned} \tag{5}$$

The highlighted terms can be left out. $J_o, \; \rho, \; A$ is cross section polar moment of inertia, density and cross section area, respectively. To express finite element potential energy of deformation we take into account displacement and strains of arbitrary point

$$u(\xi,\eta,\zeta) = \bar{u}(\xi) - \eta\psi(\xi) + \zeta\vartheta(\xi) = \bar{u}(\xi) - \eta v^{\backslash}(\xi) - \zeta w^{\backslash}(\xi), \tag{6}$$

$$\varepsilon(\xi,\eta,\zeta) = \frac{\partial u(\xi,\eta,\zeta)}{\partial \xi} = \bar{u}^{\backslash}(\xi) - \eta v^{\backslash\backslash}(\xi) - \zeta w^{\backslash\backslash}(\xi), \; \gamma(\xi,r) = r\varphi^{\backslash}(\xi) = \sqrt{\eta^2 + \zeta^2}\varphi^{\backslash}(\xi), \tag{7}$$

$$E_{Pd} = \frac{1}{2}\int_0^l \iint_A \boldsymbol{\varepsilon}^T \mathbf{E}\,\boldsymbol{\varepsilon}\, dA\, d\xi,\ \ \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon \\ \gamma \end{bmatrix},\ \ \mathbf{E} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}. \tag{8}$$

Substituting approximation relations (3) and applying Lagrange's equation we can come to coefficient matrices of one finite element [1-2]. Assembling individual finite element matrices into global ones we obtain equation of motion of the whole rotor in form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + (\mathbf{B} + \omega\mathbf{G})\dot{\mathbf{q}}(t) + \left(\mathbf{K} + \omega^2\mathbf{K}_D\right)\mathbf{q}(t) = \mathbf{0}, \tag{9}$$

where $\mathbf{M}, \mathbf{B}, \omega\mathbf{G}, \mathbf{K}, \omega^2\mathbf{K}_D$ mark matrix of mass, damping, gyroscopic forces, stiffness and membrane forces, respectively.

## 3. Critical angular speed investigation

Let us remind meaning of Campbell diagram. It is dependence of imaginary parts of eigenvalues on angular rotor speed (revolution). Let us say what is it critical rotor speed (in stationary coordinate space). It is such revolution, which are equal to imaginary part of some eigenvalue. It means that critical speed corresponds to intersection of quadrant axis with imaginary part of some eigenvalue dependence on revolution. It can be written down $\mathrm{Im}\{\lambda_i(\omega)\} = \omega$. Corresponding mode shape is complex vector which can be animated as bending curve rotating about rotor axis by angular speed $\omega$ (forward whirl-FW-increasing branch of dependence) or $-\omega$ (backward whirl-BW-decreasing branch of dependence). What does it mean in rotating space by $\omega$? This mean that bending curve is standing ($\omega = 0$) for FW and rotating by -2$\omega$ for BW. For the reason of symmetricity of Campbell diagram about the axis $\mathrm{Im}\{\lambda_i(\omega)\} = 0$ we will seek intersection of decreasing branch of $\mathrm{Im}\{\lambda_i(\omega)\}$ with straight line $\mathrm{Im}\{\lambda_i(\omega)\} = 0$ (FW) and increasing branch with straight line $\mathrm{Im}\{\lambda_i(\omega)\} = 2\omega$. It can be shown by means of simple example. Let us find critical speed of very simple rotor modelled by means of FEM in stationary and rotatory spaces and depicted in Fig. 2.

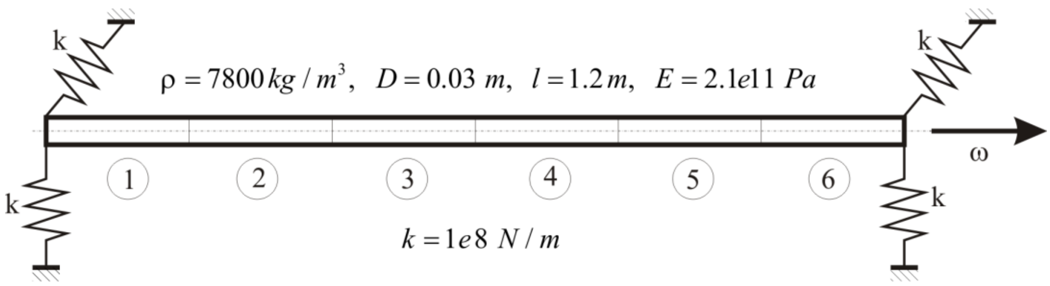


Fig. 2. Subdivision of rotor to finite elements

Campbell diagram of the rotor modelled in stationary coordinate system with zoom is depicted in Fig. 3. The following Fig. 4 presents Campbell diagram of the rotor modelled in rotatory coordinate system.

The critical speeds (intersections in Fig. 4) can be found by means of iteration rules

$$\omega_i^{(r)} = \frac{1}{2}\left|\mathrm{Im}\left\{\lambda_i^{(r-1)}\right\}\right|, \quad \text{(backward whirl)} \tag{10}$$

$$\omega_i^{(r)} = \omega_i^{(r-1)} + \mathrm{Im}\left\{\lambda_i^{(r-1)}\right\}, \quad \text{e.g.} \quad \omega_i^{(0)} = 10\ rad/s. \quad \text{(forward whirl)} \tag{11}$$

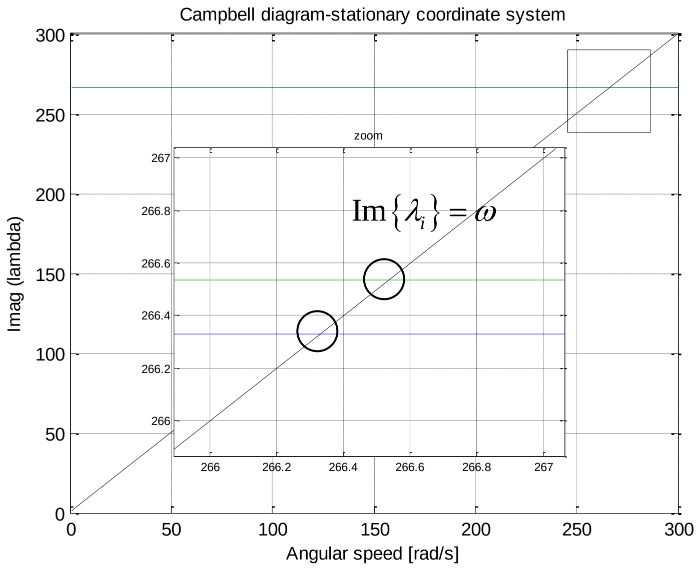


Fig. 3. Campbell diagram for rotor model in stationary coordinate system

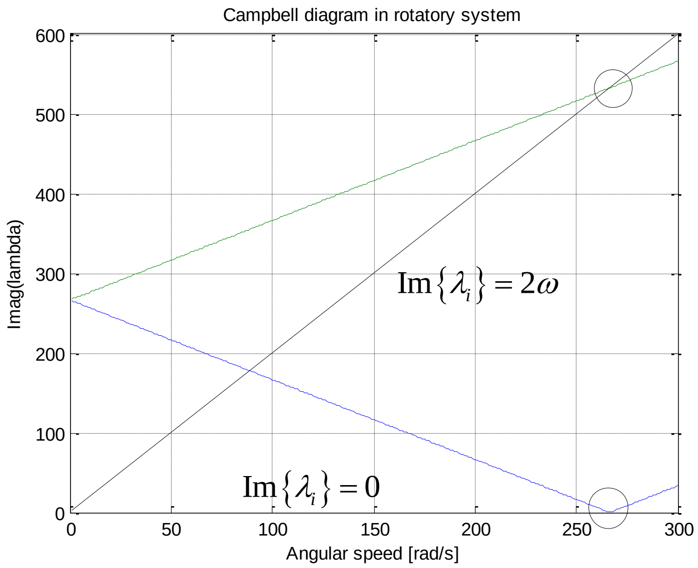


Fig. 4. Campbell diagram for rotor model in rotatory coordinate system

## 4. Conclusion

The relation (10) and (11) need not be convergent in some cases because imaginary parts of some eigenvalues can change its order in the iteration process. For this reason, the recommendation of the paper author is use Campbell diagram and zoom of the intersections.

## References


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