Edge-critical subgraphs of Schrijver graphs II: The general case

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Abstract

We give a simple combinatorial description of an $(n-2k+2)$-chromatic edge-critical subgraph of the Schrijver graph $\mathrm{SG}(n,k)$, itself an induced vertex-critical subgraph of the Kneser graph $\mathrm{KG}(n,k)$. This extends the main result of [J. Combin. Theory Ser. B 144 (2020) 191–196] to all values of k, and sharpens the classical results of Lovász and Schrijver from the 1970s.

1 Introduction

Given integers $k \geq 1$ and $n \geq 2k$, the *Kneser graph* $\mathrm{KG}(n,k)$ is defined as follows: the vertices are all the k-element subsets of $[n] = \{1,\ldots,n\}$, and the edges are the pairs of disjoint subsets. A famous conjecture of Kneser [6], proved by Lovász [8], states that $\mathrm{KG}(n,k)$ is $(n-2k+2)$-chromatic. Schrijver [12] sharpened the result by identifying the elements of $[n]$ with the vertices of the n-cycle C_n, and showing that the *Schrijver graph* $\mathrm{SG}(n,k)$ — the subgraph of $\mathrm{KG}(n,k)$ induced by the vertices containing no pair of adjacent elements of C_n — is also $(n-2k+2)$-chromatic. Moreover, Schrijver proved that $\mathrm{SG}(n,k)$ is *vertex-critical*, i.e., the removal of any vertex decreases the chromatic number.

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There is a stronger (and arguably, more natural) notion of criticality: a graph is said to be *edge-critical*, or simply *critical*, if the removal of any edge decreases the chromatic number — in other words, if any proper subgraph (not necessarily induced) has a smaller chromatic number than the graph itself.

The Schrijver graph $\mathrm{SG}(n,k)$ is not edge-critical, unless $k = 1$ or $n = 2k+1$. This prompts the following natural question: can we give a simple combinatorial description of an $(n-2k+2)$-chromatic edge-critical subgraph of $\mathrm{SG}(n,k)$?

In a recent paper [5], such a construction was given for the case $k = 2$. Here we extend the construction to all values of k, thereby sharpening Schrijver's theorem.

An edge AB of $\mathrm{SG}(n,k)$ is said to be *interlacing* if the elements of A and B alternate as we go round C_n. Simonyi and Tardos [13] recently proved that any edge of $\mathrm{SG}(n,k)$ whose removal decreases the chromatic number is interlacing. Thus, a tempting candidate for an $(n-2k+2)$-chromatic edge-critical subgraph of $\mathrm{SG}(n,k)$ might be the spanning subgraph formed by the interlacing edges. However, Litjens et al. [7] have shown that this graph has chromatic number $\lceil n/k \rceil$, so interlacing edges are much too restrictive.

We introduce instead the notion of *almost-interlacing* edges (we postpone the definition to Section 3), and define $\mathrm{XG}(n,k)$ to be the spanning subgraph of $\mathrm{SG}(n,k)$ formed by the almost-interlacing edges. The main result of this paper is the following theorem:

Theorem 1.1. *For every $k \geq 1$ and every $n \geq 2k$, $\chi(\mathrm{XG}(n,k)) = n-2k+2$. Moreover, $\mathrm{XG}(n,k)$ is edge-critical.*

We remark that the definition of almost-interlacing edges is particularly simple for the case $k = 2$. Indeed, almost-interlacing edges of $\mathrm{SG}(n,2)$ correspond to crossing and transverse pairs defined in [5], so the graph $\mathrm{XG}(n,2)$ is precisely the graph G_n studied in [5].

In a forthcoming paper, we will relate the graph $\mathrm{XG}(n,k)$ to the graphs studied in [4], and show that $\mathrm{XG}(n,k)$ is a quadrangulation of $\mathbb{RP}^{n-2k}$ (see [3] for a definition). In conjunction with the results from [3], this gives a new proof of the first part of Theorem 1.1.

For terminology not defined here, we refer the reader to Bondy and Murty [1].

The paper is structured as follows. Preliminary definitions and observations are collected in Section 2. Section 3 gives the definition of the graph $\mathrm{XG}(n,k)$. The chromatic number of this graph is determined in Section 4, and the graph is shown to be edge-critical in Section 5.

2 Preliminaries

Let C_n be the n-cycle with vertex set $[n] = \{1, \ldots, n\}$ and edges between consecutive integers as well as between 1 and n. The vertices of the Schrijver graph $\mathrm{SG}(n,k)$ mentioned in Section 1 are independent sets in C_n of size k; two such sets are adjacent in $\mathrm{SG}(n,k)$ if they are disjoint.

We usually visualise C_n in such a way that the vertices $1, \ldots, n$ appear clockwise in the given order. The vertices of C_n will be referred to as *elements* to distinguish them from the vertices of $\mathrm{SG}(n,k)$ or of the graph $\mathrm{XG}(n,k)$ we will shortly define. Any arithmetic operations with the elements are performed modulo the equality $n + 1 = 1$.

Our arguments frequently use intervals in C_n. For $a, b \in [n]$, the interval $[a, b]$ is the set $\{a, a + 1, \ldots, b\}$. Thus, $[a, b]$ consists of a and the elements following a clockwise up to b. In case $b = a - 1$, the interval $[a, b]$ contains all elements of $[n]$. By a slight abuse of this notation, we will also write $[0, n]$ for the set $\{0, \ldots, n\}$.

Open or half-open versions of intervals, namely (a, b), $[a, b)$ or $(a, b]$, are defined as expected: for instance, $[a, b) = [a, b - 1]$. All of the following definitions are modified for these other versions of intervals in a straightforward way.

Let X be a subset of $[n]$. It will be convenient to let $[a, b]_X = [a, b] \cap X$. The set carries a natural ordering given by the interval; thus, for instance, the *first* element of $[a, b]_X$ is the element of this set encountered first when moving clockwise from a to b.

The following notions will be used often in our arguments. An *X-gap* is an interval (s, t) such that $s, t \in X$ and $(s, t)_X = \emptyset$. Furthermore, if $Y \subseteq [n]$, then we say that an X-gap (s, t) is an *X-gap in Y* (or that Y *has* this X-gap) if it is a Y-gap at the same time. See Figure 1 for an illustration. We stress that saying that Y has an X-gap (s, t) does *not* mean that $(s, t) \subseteq Y$.

To distinguish ordered pairs from open intervals, we use the notation $\langle a, b \rangle$ for an ordered pair consisting of elements a and b.

If I is an interval in $[n]$, we say that disjoint subsets A, B of $[n]$ *alternate on I* if the elements of A alternate with those of B as we follow C_n from the start to the end of I. Sets A, B whose elements alternate on the whole of $[n]$ are said to form an *interlacing pair*.

A crucial notion for our construction is that of an admissible interval. For disjoint subsets A, B of $[n]$, an interval $[d, c]$ is *weakly AB-admissible* if

$$|[d, c]_A| = |[d, c]_B| = c.$$

Furthermore, a weakly AB-admissible interval $[d, c]$ is *AB-admissible* if

$$c, d \notin A \cup B.$$

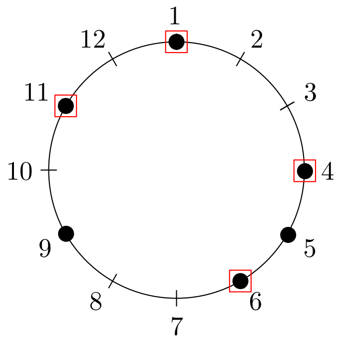


Figure 1: Subsets X (black dots) and $Y \subseteq X$ (red squares) of the vertex set of C_{12}. The X-gaps in Y are $(1,4)$ and $(11,1)$. The remaining four X-gaps, namely $(4,5)$, $(5,6)$, $(6,9)$ and $(9,11)$, are not X-gaps in Y.

We extend these notions to open or half-open intervals such as (d,c) or $(d,c]$ in precisely the same way, just replacing $[d,c]$ with the interval in question.

Let us examine some basic properties of weakly AB-admissible intervals, where A, B are disjoint subsets of $[n]$, each of size k. It is not yet required at this point that A and B be independent in C_n, so we may view AB as an edge of the Kneser graph $\mathrm{KG}(n,k)$.

Observation 2.1. *If AB is an edge of $\mathrm{KG}(n,k)$ and $[d,c]$ is a weakly AB-admissible interval, then $c \le k < d$.*

Proof. Note first that $c \le k$ follows directly from the definition of weakly AB-admissible interval. Since A and B are disjoint, we have $|[d,c]_{A\cup B}| = 2c$. It follows that $d > c$, for otherwise $2c \le |[d,c]| \le c$, leading to a contradiction as $c \ge 1$. Now

$$2k = |A \cup B| = |[d,c]_{A\cup B}| + |(c,d)_{A\cup B}| \le 2c + (d-c-1) = c+d-1,$$

and since $c \le k$, we must have $d > k$. □

Another basic property of weakly AB-admissible intervals is that they are nested, as shown by the first part of the following lemma:

Lemma 2.2. *Let AB be an edge of $\mathrm{KG}(n,k)$ and let $[d,c]$, $[d',c']$ be weakly AB-admissible intervals. Then the following hold:*

(i) $[d',c'] \subseteq [d,c]$ *or vice versa,*

(ii) *if* $[d',c'] \subseteq [d,c]$ *and* $c' < c$, *then the set* $[d,d')_{A\cup B}$ *is nonempty; if, moreover,* $[d,c]$ *is AB-admissible, then* $|(d,d')_{A\cup B}| \ge 2$.

Proof. (i) Suppose that the claim does not hold. By Observation 2.1 and by symmetry, we may assume that $c < c' < d < d'$. Then

$$2c' = |[d', c']_{A\cup B}| = |[d', c]_{A\cup B}| + |(c, c']_{A\cup B}| \leq 2c + (c' - c)$$

implying $c' \leq c$, a contradiction.

(ii) Our assumptions imply $c' < c < d \leq d'$. We have

$$\begin{aligned} 2c = |[d, c]_{A\cup B}| &= |[d, d')_{A\cup B}| + |[d', c']_{A\cup B}| + |(c', c]_{A\cup B}| \\ &\leq |[d, d')_{A\cup B}| + 2c' + (c - c'), \end{aligned} \tag{1}$$

so $|[d, d')_{A\cup B}| \geq c - c' \geq 1$.

If $c, d \notin A \cup B$, then the $(c - c')$ term in (1) improves to $(c - c' - 1)$ and furthermore, we can write $(d, d')_{A\cup B}$ in place of $[d, d')_{A\cup B}$. The second assertion follows. □

We conclude this section by the definition of switching, used in Section 3 to introduce the graph $\mathrm{XG}(n, k)$. Suppose that $c, d \in [n]$. *Switching* at $[d, c]$ is the operation transforming any pair AB of subsets of $[n]$ to another such pair $A'B'$ defined as follows:

$$\begin{aligned} A' &= A \triangle [d, c]_{A\cup B}, \\ B' &= B \triangle [d, c]_{A\cup B}, \end{aligned}$$

where $\triangle$ denotes symmetric difference. The pair $A'B'$ is the *result* of the switching.

It is easy to see that if AB is an edge of $\mathrm{KG}(n, k)$, then the result of switching AB at a weakly AB-admissible interval is again an edge of $\mathrm{KG}(n, k)$. A similar statement holds for $\mathrm{SG}(n, k)$ and switching at an AB-admissible interval.

Switching along a sequence $([d_i, c_i])_{i\in[m]}$ of intervals means switching at $[d_1, c_1], \dots, [d_m, c_m]$ in this order. (Switching along an empty sequence is the identity operation on pairs.)

Under an admissibility assumption, switching along a sequence of intervals maps any edge of the Schrijver graph $\mathrm{SG}(n, k)$ to an edge:

Observation 2.3. *Let AB be an edge of $\mathrm{SG}(n, k)$ and let $A'B'$ be the pair obtained by switching AB along a sequence S of AB-admissible intervals. The following holds:*

(i) $A'B'$ is again an edge of $\mathrm{SG}(n, k)$,

(ii) any weakly AB-admissible interval is weakly $A'B'$-admissible and vice versa.

3 Definition of $\mathrm{XG}(n,k)$

In this section, we define the graph $\mathrm{XG}(n,k)$. Let $k \geq 1$ and $n \geq 2k$. The vertex set of $\mathrm{XG}(n,k)$ coincides with that of $\mathrm{SG}(n,k)$, so the vertices of $\mathrm{XG}(n,k)$ are all k-element independent sets of C_n. The edges of $\mathrm{XG}(n,k)$ are all the almost-interlacing pairs, defined as follows.

A pair AB of vertices, where $A \cap B = \emptyset$, is *almost-interlacing* if there exists a set $X = C \cup D \subseteq [n]$ such that $C = \{c_1, \dots, c_m\}$ and $D = \{d_1, \dots, d_m\}$, with the following properties:

(1) $1 \leq c_1 < c_2 < \cdots < c_m \leq k-1$,

(2) $k+1 \leq d_m < d_{m-1} < \cdots < d_1 \leq n$,

(3) each interval $[d_i, c_i]$ is AB-admissible,

(4) switching along the sequence $([d_i, c_i])_{i \in [m]}$ changes AB to an interlacing pair.

Any set X satisfying this definition is called an *AB-alternator*. We often write it as $C \cup D$, with C and D as in the definition. The elements in C are the *control elements* of the AB-alternator, the elements c_i and d_i ($i \in [m]$) *correspond* to each other, and pairs $\langle c_i, d_i \rangle$ ($i \in [m]$) are the *control pairs* of the AB-alternator.

When referring to an $(A \cup B)$-alternator $C' \cup D'$ with $C' = \{c'_1, \dots, c'_m\}$ and $D' = \{d'_1, \dots, d'_m\}$, an ordering as in properties (1) and (2) is implicitly assumed.

Observe that $\mathrm{XG}(n,k)$ is a spanning subgraph of $\mathrm{SG}(n,k)$. Any pair of vertices AB that is an interlacing pair is an edge of $\mathrm{XG}(n,k)$, since in this case the empty set is trivially an AB-alternator.

Another example is shown in Figure 2, depicting an edge AB of $\mathrm{SG}(16,4)$. The set $\{2,3,7,11\}$ is an AB-alternator, so A and B are adjacent in $\mathrm{XG}(16,4)$. There is only one other AB-alternator, namely $\{2,3,7,10\}$.

Let us consider the special case of the definition for $k=2$. (See Figure 3 for an illustration.) Let AB be an edge of $\mathrm{SG}(n,2)$. We may assume that $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, where $a_1 < a_2$, $b_1 < b_2$ and $a_1 < b_1$. Possible AB-alternators are $\emptyset$ (in which case AB is an interlacing pair), or a set $\{1, d\}$, disjoint from $A \cup B$, such that $[d,1]_{A \cup B} = \{a_2, b_2\}$ (which is easily seen to be equivalent to $1 < a_1 < b_1 < b_2 < a_2$). In the paper [5], pairs of these two types are referred to as crossing and transverse pairs, respectively, and they coincide with the edges of the graph studied in that paper (denoted by G_n). Thus, as noted in Section 1, the present definition specialises to the one of [5] for $k=2$.

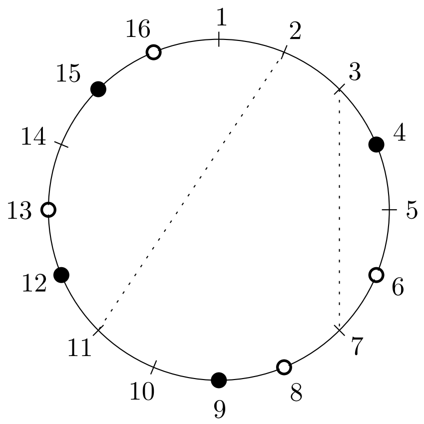


Figure 2: Vertices $A = \{4, 9, 12, 15\}$ (black dots) and $\{6, 8, 13, 16\}$ (white dots) of $\mathrm{XG}(16, 4)$ forming an almost-interlacing pair. The elements not in $A \cup B$ are shown as tick marks. Dotted lines mark the control pairs of the AB-alternator $\{2, 3, 7, 11\}$. Similar conventions are used in the other figures.

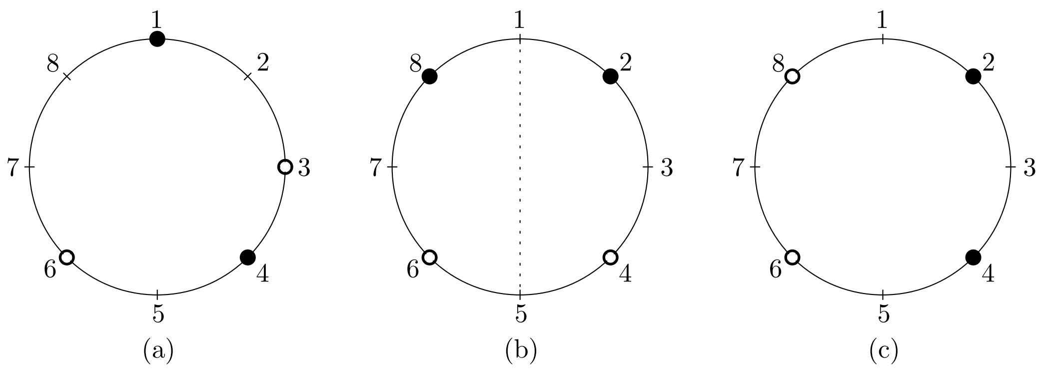


Figure 3: (a) and (b) Examples of edges in $\mathrm{XG}(8, 2)$. (c) A non-edge in $\mathrm{XG}(8, 2)$. The dotted line in picture (b) shows the only control pair of the AB-alternator $\{1, 5\}$.

Let us add some comments on the definition of edges of $\mathrm{XG}(n,k)$. Note that in condition (1), the bound $c_m \leq k$ is trivial (and stated in Observation 2.1), so (1) just strengthens this bound by one. Furthermore, the bound $k+1 \leq d_m$ in condition (2) is actually superfluous (though we include it for clarity) as it also follows from Observation 2.1. Using Lemma 3.1(i) below, the bounds in condition (2) can be strengthened to $d_m \geq k+2$ and $d_1 \leq n-2$.

We will now describe an algorithm that finds an AB-alternator $C \cup D$ if it exists, where AB is an edge of $\mathrm{SG}(n,k)$. It may be helpful to consult Figure 2 for an illustration. First we need another lemma.

Lemma 3.1. *Let $C \cup D$ be an AB-alternator for an edge AB of $\mathrm{SG}(n,k)$. The following hold:*

(i) if (x,y) is a $(D \cup \{k,n\})$-gap other than (n,k) or (k,n), then the size of $[x,y]_{A\cup B}$ is at least 2,

(ii) if (x,y) is an $(A\cup B)$-gap, then $|(x,y)_{C\cup D}|$ is odd if and only if $x,y \in A$ or $x,y \in B$.

Proof. We may assume that AB is not interlacing, for otherwise (i) is vacuous and (ii) is clearly true. Thus, let $D = \{d_1,\dots,d_m\}$ with $d_1 > \cdots > d_m$ and $m \geq 1$. For $i \in [m]$, let c_i be the control element corresponding to d_i.

(i) If $x,y \in D$, then the assertion follows from Lemma 2.2(ii) and the fact that each of the intervals $[d_i,c_i]$ is AB-admissible.

For the pair $\langle k, d_m\rangle$, we can write

$$\begin{aligned}2k = |A\cup B| &= |[d_m,c_m]_{A\cup B}| + |(c_m,k)_{A\cup B}| + |[k,d_m)_{A\cup B}| \\ &\leq 2c_m + (k-c_m-1) + |[k,d_m]_{A\cup B}|,\end{aligned}$$

so $|[k,d_m]_{A\cup B}| \geq k - c_m + 1 \geq 2$ since $c_m \leq k-1$.

Similarly, for the pair $\langle d_1, n\rangle$, we have

$$2c_1 = |[d_1,n]_{A\cup B}| + |[1,c_1]_{A\cup B}| \leq |[d_1,n]_{A\cup B}| + (c_1 - 1)$$

(using the fact that $c_1 \notin A\cup B$), and we find that $|[d_1,n]_{A\cup B}| \geq c_1 + 1 \geq 2$.

(ii) Let us say that a subset of $[n]$ is *separating* if it contains exactly one of x and y. Let s be the number of intervals $[d_i,c_i]$ ($i \in [m]$) which are separating. Observe that s has the same parity as $|(x,y)_{C\cup D}|$.

For $0 \leq j \leq m$, let A_jB_j be the pair obtained from AB by switching along $([d_i,c_i])_{i\in[j]}$; in particular, $A_0B_0 = AB$. For $j > 0$, it is not hard to see that A_j is separating if and only if exactly one of A_{j-1} and $[d_j,c_j]$ is separating. Now since A_mB_m is an interlacing pair, A_m is not separating. It follows that either A is separating and s is even, or A is not separating and s is odd. Since A is not separating if and only if $x,y\in A$ or $x,y \in B$, and by the above observation on the parity of s, this implies part (ii). $\square$

Let us return to the task of finding an AB-alternator for a given edge AB of $\mathrm{SG}(n,k)$. Consider any $(A \cup B)$-gap (x,y) with $x, y \in [k,n]$. If $x, y \in A$ or $x, y \in B$, then by Lemma 3.1(ii), our set D needs to contain an element in (x,y). The latter interval is nonempty since each of A and B is independent in C_n. Furthermore, by Lemma 3.1(i), D must contain exactly one element from this interval. The choice of the element from (x,y) is arbitrary; in fact, we will see that this is the only choice we have in the process. In the example of Figure 2, the set D must include the element 7 and one element from $\{10, 11\}$.

Similarly to the above, Lemma 3.1(ii) and (i) implies that if exactly one of x, y is in A, then $(x,y)_D$ must be empty, because its size is even and at most one. Finally, by Lemma 3.1(i), D contains no element between k and the first element of $[k,n]_{A\cup B}$, nor between the last element of the latter set and n.

Summing up, D is obtained by choosing exactly one element in each $(A \cup B)$-gap (x,y) with $x, y \in [k,n]$ and either $x, y \in A$ or $x, y \in B$. Let $D = \{d_1, \dots, d_m\}$ for some such choice. (Thus, for the pair in Figure 2, D equals $\{7, 10\}$ or $\{7, 11\}$.)

We will show that this determines the set C whenever there exists an AB-alternator. The following lemma provides a tool.

Lemma 3.2. *Let $d \in [n]$ and let X be a vertex of $\mathrm{SG}(n,k)$. There is at most one element $c \in [k-1]$ such that $|[d,c]_X| = c$ and $c \notin X$.*

Proof. For $x \in [k-1]$, let

$$f(x) = |[d,x]_X| - x.$$

The function f is non-increasing. For each $x \in [k-2]$, we have

$$f(x+1) = \begin{cases} f(x) & \text{if } x+1 \in X, \\ f(x)-1 & \text{otherwise.} \end{cases}$$

Thus, if $f(x) = f(x+1)$ and $x \le k-3$, then $f(x+1) > f(x+2)$ by the independence of A. It follows that we have $f(x) = 0$ for at most two values of x. Supposing (for the sake of a contradiction) that the lemma does not hold, there are two such values, say c and $c+1$, where $c \in [k-2]$. Since $f(c+1) = f(c)$, we have $c+1 \in X$, so $c+1$ does not satisfy the conditions, a contradiction. □

For each $i \in [m]$, C has to contain an element c_i such that $[d_i, c_i]$ is AB-admissible. Since c_i has to satisfy the condition of Lemma 3.2 with

$X = A$, there is at most one such element. Furthermore, c_i is independent of the choice of d_i: more precisely, if (x, y) is the $(A \cup B)$-gap such that $d_i \in (x, y)$, and if $d'_i \in (x, y)$, then $[d'_i, c]_{A\cup B} = [d_i, c]_{A\cup B}$ for any $c \in [k-1]$. It follows that if an AB-alternator does exist, then each element of C is uniquely determined by Lemma 3.2. Our algorithm returns $C \cup D$ when this is the case, and reports that there is no AB-alternator otherwise. (In the example of Figure 2, we have $c_1 = 2$ and $c_2 = 3$, so one of the sets $\{2, 3, 7, 10\}$ or $\{2, 3, 7, 11\}$ is returned.)

To obtain a unique choice for the AB-alternator when it exists, we impose the extra condition that for each $i \in [m]$, $d_i + 1 \in A \cup B$. This amounts to choosing the largest possible element for each d_i. The resulting AB-alternator is called *standard*. Speaking of the control elements or control pairs for the edge AB, we mean the control elements or pairs of the standard AB-alternator.

4 Chromatic number

In this section, we prove the first part of Theorem 1.1 — namely, that $\chi(\mathrm{XG}(n, k)) = n - 2k + 2$ for every $k \geq 1$ and every $n \geq 2k$. It is enough to prove the inequality $\chi(\mathrm{XG}(n, k)) \geq n - 2k + 2$, the other inequality being a direct consequence of the fact that $\mathrm{XG}(n, k)$ is a subgraph of $\mathrm{KG}(n, k)$.

The case $k = 2$ of Theorem 1.1 was proved in [5] using the so-called Mycielski construction. Here we prove the general case using the same idea, but rely instead on the *generalised* Mycielski construction, introduced by Stiebitz [14] (see also [2, 11]).

Given a graph $G = (V, E)$ and an integer $r \geq 1$, the graph $M_r(G)$ has vertex set $(V \times [0, r-1]) \cup \{z\}$, and there is an edge $(u, 0)(v, 0)$ and $(u, i)(v, i+1)$ (for every $i \in [0, r-2]$) whenever $uv \in E$, and an edge $(u, r-1)z$ for all $u \in V$. The construction is illustrated in Figure 4.

For every integer $t \geq 2$, we denote by $\mathcal{M}_t$ the set of all 'generalised Mycielski graphs' obtained from K_2 by $t - 2$ iterations of $M_r(\cdot)$, where the value of r can vary from iteration to iteration. That is, $H \in \mathcal{M}_t$ if and only if there exist integers $r_1, r_2, \ldots, r_{t-2} \geq 1$ such that

$$H \cong M_{r_{t-2}}(M_{r_{t-3}}(\ldots M_{r_2}(M_{r_1}(K_2))\ldots)).$$

Using topological methods, Stiebitz [14] (see also [2, 9]) proved the following result. A 'discrete' proof, based on a combinatorial lemma of Fan, can be found in [10].

Theorem 4.1 (Stiebitz [14]). *If $G \in \mathcal{M}_t$, then $\chi(G) = t$.*

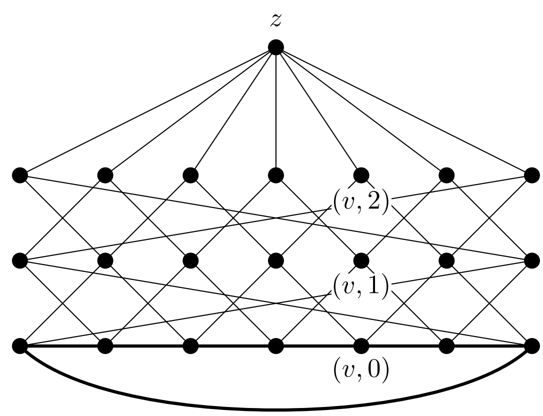


Figure 4: The generalised Mycielski construction applied to C_7 (bold) resulting in the graph $M_3(C_7)$.

We now come to the key lemma of this section.

Lemma 4.2. *For every $k \geq 1$ and every $n \geq 2k$, $M_k(\mathrm{XG}(n-1,k))$ admits a homomorphism to $\mathrm{XG}(n,k)$.*

Proof. We shall explicitly describe a homomorphism f from $M_k(\mathrm{XG}(n-1,k))$ to $\mathrm{XG}(n,k)$. Let A be a vertex of $\mathrm{XG}(n-1,k)$ and let $(A,0),\ldots,(A,k-1)$ be its copies in $M_k(\mathrm{XG}(n-1,k))$. In order to keep all vertex names capitalised, we choose to denote the vertex z in the generalised Mycielski construction by Z.

Suppose that $A = \{a_1,\ldots,a_k\}$, where $a_1 < \cdots < a_k$. Let $0 \leq i \leq k$. We define the set $\Lambda_{n,i} \subseteq [n]$ as follows:

$$\Lambda_{n,i} = \begin{cases} \{n-i+1, n-i+3, \ldots, n\} \cup \{2,4,\ldots,i-1\} & \text{if } i \text{ is odd,} \\ \{n-i+1, n-i+3, \ldots, n-1\} \cup \{1,3,\ldots,i-1\} & \text{if } i \text{ is even.} \end{cases}$$

Thus, for instance, $\Lambda_{n,0} = \emptyset$, $\Lambda_{n,1} = \{n\}$ and $\Lambda_{n,2} = \{1, n-1\}$.

We will now define a map $f : V(M_k(\mathrm{XG}(n-1,k))) \to V(\mathrm{XG}(n,k))$. Given a vertex A of $\mathrm{XG}(n-1,k)$ and an integer $j \in [k]$, let $A^j = [d,j]_A$, where d is the maximum integer such that $|[d,j]_A| = j$. Furthermore, let $A^0 = \emptyset$. We set

$$f : (A,j) \mapsto (A \setminus A^j) \cup \Lambda_{n,j}, \text{ where } 0 \leq j \leq k-1,$$
$$Z \mapsto \Lambda_{n,k}.$$

Note that the image of f is contained in the vertex set of $\mathrm{XG}(n,k)$. Informally, $f(A,j)$ can be seen as the result of the following process: viewing

A as a subset of $V(C_n)$, A^j consists of the j elements of A that are closest to j counterclockwise; push them clockwise in such a way that the first one stops at j and the remaining ones are tightly packed (still forming an independent set), and rotate them back by one element. The other $k-j$ elements of A are not affected.

To verify that f is a homomorphism, it is enough to check that f maps edges of $M_k(\mathrm{XG}(n-1,k))$ to edges of $\mathrm{XG}(n,k)$. Fix an arbitrary edge AB of $\mathrm{XG}(n-1,k)$, and let $C \cup D \subseteq [n-1]$ be the standard AB-alternator. Let $\{\langle c_i, d_i \rangle : i \in [m]\}$ be its set of control pairs.

We will show that f maps the edges $(A,0)(B,0)$, $(A,j)(B,j+1)$ (for any $j \in [0,k-1]$), as well as $(A,k-1)Z$, to edges of $\mathrm{XG}(n,k)$, by finding an appropriate alternator $C' \cup D'$.

First, consider the edge $(A,0)(B,0)$ of $M_k(\mathrm{XG}(n-1,k))$. Since $f((A,0)) = A$ and $f((B,0)) = B$, the required alternator is obtained by taking $C' = C$ and $D' = D$. (Note that the definition is still satisfied if A and B are viewed as vertices of $\mathrm{XG}(n,k)$ rather than $\mathrm{XG}(n-1,k)$.)

Edges of type $(A,k-1)Z$ are another easy case: we have $f(Z) = \Lambda_{n,k}$ and $f((A,k-1))$ contains $\Lambda_{n,k-1}$ as a subset, which means that $f((A,k-1))f(Z)$ must actually be an interlacing pair, and hence an edge of $\mathrm{XG}(n,k)$ (with empty alternator).

It remains to consider the edge $(A,j)(B,j+1)$, where $j \in [0,k-1]$. Let $A' = f((A,j))$ and $B' = f((B,j+1))$. The sets A' and B' are disjoint since $A \cap B = \emptyset$ and $\Lambda_{n,j} \cap \Lambda_{n,j+1} = \emptyset$.

Given $r \in [0,m]$, let $A_r B_r$ be the pair obtained from AB by switching along $([d_i,c_i])_{i\in[r]}$. Since $A_m B_m$ is interlacing, there is $d \in [k+1,n]$ such that $[d,j+1]$ is weakly $A_m B_m$-admissible. Choose d to be maximal with this property. By Observation 2.3(ii), $[d,j+1]$ is weakly AB-admissible.

For any $i \in [m]$, we have $[d_i,c_i] \subseteq [d,j+1]$ or vice versa by Lemma 2.2(i). If there is $t \in [m]$ such that $c_t < j+1$, then let t be maximal with this property; otherwise, let $t = 0$.

We now aim to show that the pair $A'B'$ is, in a sense, not too different from $A_t B_t$.

Let X, Y be disjoint vertices of the graph H (that is, vertices such that $X \cap Y = \emptyset$), where H is either $\mathrm{XG}(n-1,k)$ or $\mathrm{XG}(n,k)$. Let I be the interval $[d,j+1] \subseteq V(C_n)$, where d is as above. (Thus, $n \in I$ even if H is $\mathrm{XG}(n-1,k)$.)

Let us say that the pair XY is *nice* if the following hold:

(N1) $X \setminus I = A \setminus I$ and $Y \setminus I = B \setminus I$,

(N2) the sets X and Y alternate on I and the first element of $I \cap (X \cup Y)$

belongs to X if and only if the first element of $I \cap (A \cup B)$ belongs to A.

Claim 1. *The pair $A_t B_t$ is nice.*

Condition (N1) in the definition follows from the fact that for each of the intervals $[d_i, c_i]$ with $i \leq t$, we have $c_i < j+1$ and therefore $[d_i, c_i] \subseteq I$ by Lemma 2.2(i). Thus, switching at such intervals does not affect the elements outside I.

Let us verify condition (N2). Since $A_m B_m$ is an interlacing pair and $I \subseteq [d_i, c_i]$ for any $i > t$, A_t and B_t must alternate on I. For the rest of condition (ii), we may assume that $t > 0$. Let x be the first element of $I \cap (A \cup B)$; since $A \cup B = A_t \cup B_t$, this is also the first element of $I \cap (A_t \cup B_t)$. By Lemma 2.2(ii), x is not contained in $[d_t, c_t]$ (nor in any $[d_i, c_i]$ with $i < t$), and therefore $x \in A_t$ if and only if $x \in A$. This concludes the proof of the claim.

Claim 2. *Any nice pair XY of disjoint vertices of* $\mathrm{XG}(n, k)$ *forms an edge of* $\mathrm{XG}(n, k)$*.*

It is clear from the definition of nice pair that XY can be obtained from (the nice pair) $A_t B_t$ by first extending the underlying cycle C_{n-1} to C_n (just inserting the element n) and then moving the elements of $X \cup Y$ within I without changing their order on C_n.

It follows that switching along $([d_{t+1}, c_{t+1}], \ldots, [d_m, c_m])$ changes XY to an interlacing pair, just as in the case of $A_t B_t$. (Recall that I is a subset of each of these intervals by the choice of t.) Summing up, $\{c_{t+1}, \ldots, c_m\} \cup \{d_{t+1}, \ldots, d_m\} \subseteq [n]$ is an XY-alternator.

The following claim relates the above observations to $A'B'$.

Claim 3. *One of the following conditions holds:*

(i) $A'B'$ is a nice pair,

(ii) the interval I is $A'B'$-admissible and the pair $A''B''$ obtained by switching $A'B'$ at I is nice.

First of all, observe that since I is weakly AB-admissible, both A^j and B^{j+1} are contained in I. Furthermore, both $\Lambda_{n,j+1}$ and $\Lambda_{n,j}$ are contained in I: indeed, the weakly $A'B'$-admissible interval $I = [d, j+1]$ must satisfy $d \leq n - j$, while at the same time $\Lambda_{n,j} \cup \Lambda_{n,j+1} = [n-j, j]$. This proves condition (N1) for both of the pairs involved in (i) and (ii).

We have in fact $B^{j+1} = I \cap B$ and $A^j = (I \cap A) \setminus \{a\}$ for some $a \in I \cap A$. There are essentially three possibilities for a, illustrated in Figure 5:

if $j+1 \notin A$, then a is the first element of $[d, j+1]_A$ and it may or may not equal d, while if $j+1 \in A$, then $a = j+1$.

All the elements of B^{j+1} are replaced in B' by $\Lambda_{n,j+1}$; similarly, all the elements of A^j are replaced in A' by $\Lambda_{n,j}$. Hence, A' and B' alternate on $[n-j, j]$, and therefore they alternate on I regardless of the position of the remaining element a of $[d, j+1]_{A' \cup B'}$.

If condition (N2) holds for $A'B'$, then we are done. Assume thus that this is not the case. We have $a \neq d$, for otherwise a would be the first element of both $I \cap (A \cup B)$ and $I \cap (A' \cup B')$ while $a \in A \cap A'$, implying (N2). For a similar reason (using the fact that $A'B'$ alternates in I), we find $a \neq j+1$. Consequently, neither d nor $j+1$ belong to A'. They do not belong to B' either: this is clear in the case of $j+1$, and $d \in B'$ would only be possible if $d = n-j$, but then $|[d, j+1]_A| = j+1$ would force $j+1 \in A$ and hence $a = j+1$, a contradiction. We have proved that $[d, j+1]$ is $A'B'$-admissible.

Let x be the first element of $[d, j+1]_{A' \cup B'} = [d, j+1]_{A'' \cup B''}$ and note that x belongs to A' if and only if it belongs to B''. Thus, condition (N2) is satisfied for exactly one of the pairs $A'B'$ and $A''B''$. This proves the claim.

Let us finish the proof of the lemma. If condition (i) of Claim 3 holds, then $A'B'$ is an edge of $\mathrm{XG}(n,k)$ by Claim 2. If condition (ii) holds, then we obtain an $A'B'$-alternator by setting $C' = C \cup \{j+1\}$ and $D' = D \cup \{d\}$, completing the discussion for edges of type $(A, j)(B, j+1)$ as well as the whole proof. $\square$

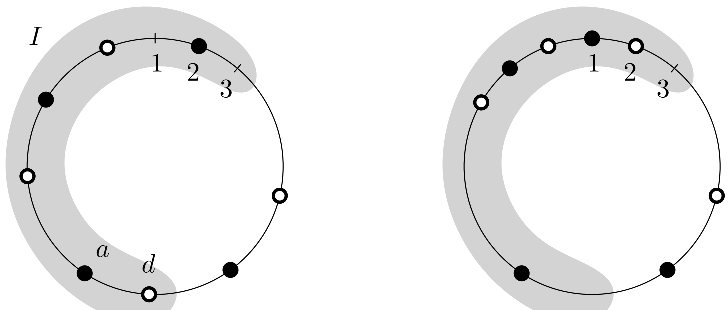
We are now ready to prove that $\chi(\mathrm{XG}(n,k)) \geq n - 2k + 2$. First, observe that if G, H are graphs such that G is homomorphic to H (i.e., there exists a homomorphism from G to H), then $M_k(G)$ is homomorphic to $M_k(H)$. Hence, by repeated applications of Lemma 4.2, the graph

$$H = M_k(M_k(\ldots M_k(\mathrm{XG}(2k,k))\ldots)),$$

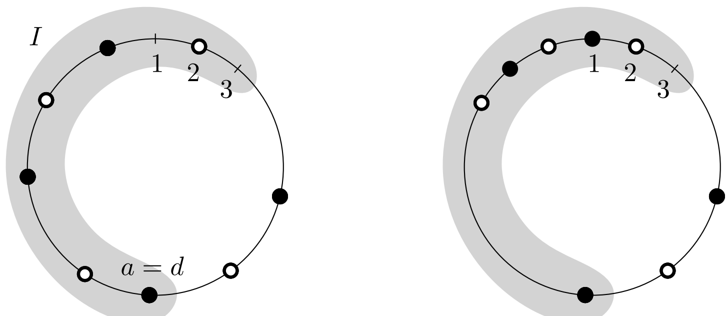
where $M_k(\cdot)$ is applied $n-2k$ times, is homomorphic to $\mathrm{XG}(n,k)$. Since $\mathrm{XG}(2k,k)$ is isomorphic to K_2, $H \in \mathcal{M}_{n-2k}$, so using Theorem 4.1, we conclude that $\chi(\mathrm{XG}(n,k)) \geq n - 2k + 2$.

5 Criticality

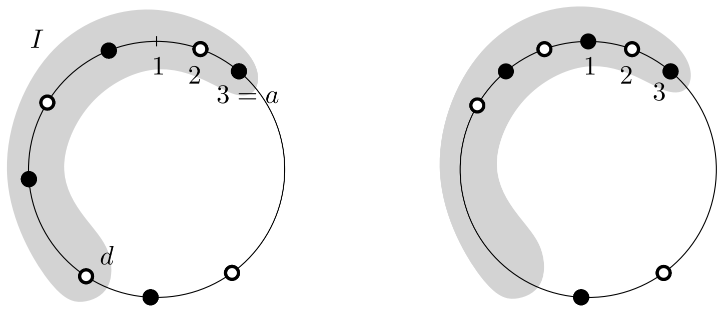
In this section, we prove the second part of Theorem 1.1, namely that $\mathrm{XG}(n,k)$ is edge-critical. Let AB be an edge of $\mathrm{XG}(n,k)$ and let $G = \mathrm{XG}(n,k) - AB$. We show that G is $(n-2k+1)$-colourable.



(a) a differs from both d and $j+1$.



(b) $a = d$.



(c) $a = j + 1$.

Figure 5: Possible cases in the proof of Claim 3, shown for $k = 4$, $j = 2$ and AB interlacing. The figures on the left show the pair AB, those on the right show $A'B'$. Black dots represent A or A', white dots represent B or B', the interval $I = [d, j + 1]$ is shown gray. The cases are distinguished by the position of the element a of $(I \cap A) \setminus A^j$. The pair $A'B'$ is interlacing except in (a), in which case a switch at $[d, j + 1]$ is needed to make it interlacing.

Let $C \cup D$ be the standard AB-alternator, where $C = \{c_1, \dots, c_m\}$, $D = \{d_1, \dots, d_m\}$ and

$$c_1 < c_2 < \cdots < c_m \leq k - 1 < d_m < d_{m-1} < \cdots < d_1.$$

We recall that the sets A, B, C, D are pairwise disjoint.

Let us give an informal overview of the colouring procedure. The precise description will be given later as Rules (R1)–(R7) on page 27. This overview will also serve to motivate and introduce (precise) definitions of the necessary concepts.

Let us call a vertex of G *essential* if it is a subset of $W := A \cup B \cup C \cup D$. It will turn out that we may restrict our attention to colouring the essential vertices of G, since there is an easy and safe rule to colour the other ones.

The main idea for colouring an essential vertex X is to determine the colour based on the weight of X on certain intervals. Here, the *weight* of X on an interval I is the cardinality of $X \cap I$. The intervals taken into consideration are determined by the AB-alternator $C \cup D$: namely, these are the intervals

$$[d_i, c_i) \quad \text{and} \quad (d_i, c_i], \quad \text{where } 1 \leq i \leq m.$$

We call them *standard* intervals. Recall that the weight of A (and B) on the interval $[d_i, c_i)$ is exactly c_i, by the definition of AB-alternator. The same applies to $(d_i, c_i]$. We compare the weight of X on such an interval to c_i, which is viewed as the 'standard' weight.

Thus, if I is one of the intervals $[d_i, c_i)$ and $(d_i, c_i]$ (where $1 \leq i \leq m$), then we say that X is *heavy* on I if its weight on I is greater than c_i. Furthermore, X is *light* or *balanced* on I if its weight on I is smaller than c_i or equal to c_i, respectively. See Figure 6 for an illustration of these notions.

We will see in Lemma 5.1 below that it is safe to assign the same colour to two vertices X and Y such that X is heavy on some $[d_i, c_i)$ while Y is light on this interval, because such a pair of vertices cannot be adjacent.

We might therefore think of colouring all essential vertices that are heavy or light on $[d_i, c_i)$ with one colour. This would not work, however, since nothing prevents G from containing an edge between vertices X and Y both of which are heavy on $[d_i, c_i)$. Luckily, Proposition 5.3 shows that it helps to require i to be *minimal* such that either of X and Y is heavy on $[d_i, c_i)$. This leads us to define an essential vertex X to be *min-heavy* on the interval $[d_i, c_i)$ ($1 \leq i \leq m$) if it is heavy on $[d_i, c_i)$ and not heavy on any interval $[d_j, c_j)$ nor $(d_j, c_j]$ with $j < i$. For instance, the vertex X from Figure 6 is min-heavy on $[d_1, c_1)$ but not on $[d_2, c_2)$.

As one can expect, there is a symmetric notion for light vertices: X is *max-light* on $[d_i, c_i)$ if it is light on this interval and not light on any $[d_j, c_j)$

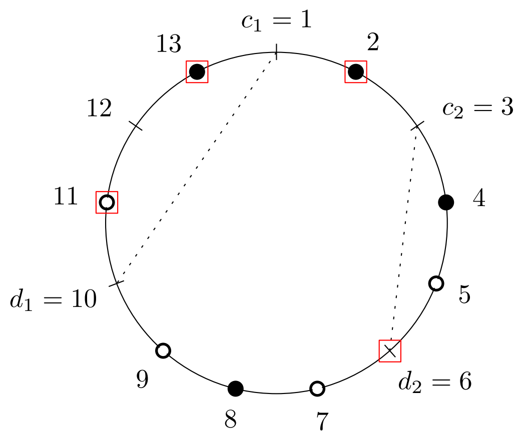


Figure 6: Vertices A (black circles) and B (white circles) of $\mathrm{XG}(13, 4)$. Dotted lines show the control pairs of the standard AB-alternator. Note that the only element not in W is 12. An essential vertex X of $\mathrm{XG}(13, 4)$ is shown with red boxes. The vertex X is heavy on $[d_1, c_1)$, $(d_1, c_1]$ and $[d_2, c_2)$, while it is balanced on $(d_2, c_2]$. Observe that A and B are balanced on every $[d_i, c_i)$ and $(d_i, c_i]$, where $i \in \{1, 2\}$.

nor $(d_j, c_j]$ with $j > i$. Being min-heavy or max-light on the interval $(d_i, c_i]$ is defined in an analogous manner.

Now the idea is to assign one colour to all essential vertices which are either min-heavy or max-light on $[d_i, c_i)$, and another colour to those which are min-heavy or max-light on $(d_i, c_i]$. This will indeed be done, but by itself, this rule might still assign the same colour to adjacent vertices, as illustrated in Figure 7.

However, it turns out that — as in Figure 7 — this only happens if one of the vertices, say X, contains a *balanced pair*, namely a pair $\langle c_i, d_i \rangle$, where $1 \le i \le m$, c_i and d_i belong to X, and X is balanced on $[d_i, c_i)$ (and therefore also on $(d_i, c_i]$). To prevent the above problem, we will colour such vertices first. We do not need any extra colour for that, since it will be shown that no essential vertex containing a balanced pair $\langle c_i, d_i \rangle$ is adjacent to a vertex that is min-heavy or max-light on $(d_i, c_i]$. Hence, it is safe to assign it the same colour used for the latter group of vertices.

It is not hard to see that by now, any essential vertex X that is heavy or light on some standard interval will have obtained a colour. This leaves us with essential vertices which are balanced on every standard interval — we call them simply *balanced*.

Among these vertices, those containing at least one vertex from $C \cup D$ are also already coloured, because Proposition 5.2 below shows that each of

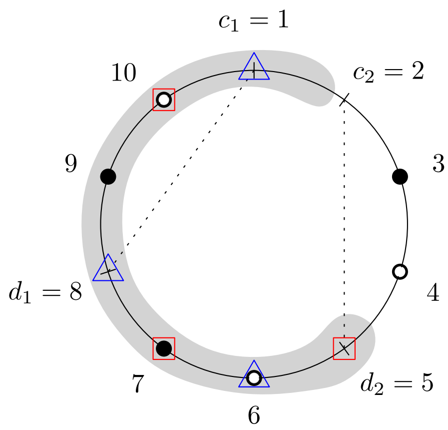


Figure 7: Two *adjacent* vertices of XG$(10, 3)$ (shown with red boxes and blue triangles), both of which are min-heavy on the standard interval $[d_2, c_2)$ (shaded grey). Vertices A and B and the control pairs of an AB-alternator are shown using the same conventions as in Figure 6.

them contains a balanced pair.

Vertices having no colour at this point are exactly the balanced vertices which are subsets of $A \cup B$. We call such vertices *regular*. Note that two examples of regular vertices are A and B, and that any two adjacent regular vertices partition $A \cup B$.

Let us pause to discuss the actual set of colours we are going to use to colour G. It will be convenient to associate colours with certain elements of $[n]$. For each such element j, there will be a colour $\boxed{j}$. If $j \notin W$, then $\boxed{j}$ will be used to colour inessential vertices of G containing j. If $j \in C \cup D$, then $\boxed{j}$ will be assigned to essential vertices which contain a balanced pair that includes j, or which are min-heavy or max-light on one of the standard intervals delimited by j. There will be no colours associated with $j \in A \cup B$, but instead there will be an additional colour $\boxed{0}$. Note that the total number of colours is $n - 2k + 1$, which is the right value.

To deal with regular vertices, we first observe that adjacent regular vertices X and Y partition the set $A \cup B$. This means that any regular vertex X has at most one regular neighbour in G, namely $X^* := (A \cup B) \setminus X$.

In most cases, one or both of X and X^* will contain a W-gap. Therefore, we first focus on colouring vertices of this type, for which we will use the colours $\boxed{j}$, where $j \in [k-1] \setminus (A \cup B)$, together with the colour $\boxed{0}$. The basic idea here is the concept of 'depth' of a vertex X containing a W-gap. This will be introduced later, but for the time being let us say that the depth of X is an integer in $[k]$ that encodes the position of the topmost W-gap in X (where 'topmost' refers to the ordering of $[n]$).

This notion has two important properties, which follow from Lemmas 5.5 and 5.8 below. If X and Y are adjacent vertices, both having W-gaps, then the following hold:

- X and Y have different depths,
- the depth of X is not contained in $A \cup B$ (unless it equals k).

The first property suggests that we might colour a regular vertex X having a W-gap with colour $\boxed{j}$, where j is the depth of X, and this is what we do (using colour $\boxed{0}$ for depth k). The only exception is if $j \in A \cup B$, in which case we do not have any corresponding colour to use. The second property, however, implies that X is not adjacent to X^*. Indeed, if they were, then the depth of X would not belong to $A \cup B$ by the second property above. Thus, X may be given the special colour $\boxed{0}$. The same can be done for vertices of depth k, and the first of the above properties ensures that this causes no problem.

Having coloured vertices with a W-gap, we have simplified the situation substantially. Indeed, we observe that if adjacent vertices X, X^* are still uncoloured, then they alternate on every interval delimited by the elements of $C \cup D$. In particular, for each $d_i \in D$, X contains exactly one of the two elements preceding d_i in W and exactly one of the two following d_i in W. In one of the four possible cases, we call X 'skew' at d_i (see page 27 for a precise definition). If X is skew at some d_i, we take the minimal such i and colour X with $\boxed{d_i}$, checking that this does not conflict with the previous use of this colour.

In the last step, we colour all the remaining vertices with $\boxed{0}$ and verify that this is an independent set by narrowing the list of possible edges on it down to the single edge AB, which is precisely the edge we removed from $\mathrm{XG}(n, k)$ to get G. Again, we check that there is no conflict with previous applications of colour $\boxed{0}$.

Having completed this high-level description of the colouring process, let us now state and prove the necessary tools. The first lemma represents the main idea of this section.

Lemma 5.1 (Disbalance Lemma)**.** *Suppose that X, Y are disjoint vertices of G, $c \in [k-1]$ and $d \in [n]$. The pair XY is* not *an edge of G if one of the following conditions holds:*

(i) $|[d, c]_X| > c$ *and* $|[d, c]_Y| < c$, *or*

(ii) $|[d, c)_X| > c$ *and* $|[d, c)_Y| < c$.

Proof. Assume condition (i). For the sake of a contradiction, assume that XY is an edge of G, and consider the standard XY-alternator $C' \cup D'$, where $C' = \{c'_1, \dots, c'_\ell\}$ and $D' = \{d'_1, \dots, d'_\ell\}$. (Recall our convention, introduced on page 6, that the elements of C' are listed in increasing order and those of D' in decreasing order.)

Suppose first that

$$\text{for each } j \in [\ell], [d'_j, c'_j] \subseteq [d, c] \text{ or vice versa.} \tag{2}$$

For $0 \leq j \leq \ell$, let $X_j Y_j$ be the result of switching XY along $([d'_i, c'_i])_{i \in [j]}$. Let

$$\beta(X_j Y_j) = \left| \left| [d, c]_{X_j} \right| - \left| [d, c]_{Y_j} \right| \right| .$$

Since $X_\ell Y_\ell$ is an interlacing pair, we have $\beta(X_\ell Y_\ell) \leq 1$. We claim that for $j > 0$, it holds that $\beta(X_j Y_j) = \beta(X_{j-1} Y_{j-1})$. This is clear if $[d, c] \subseteq [d'_j, c'_j]$, for then the effect of the switch at $[d'_j, c'_j]$ within $[d, c]$ is just to interchange membership in X_{j-1} and Y_{j-1}. On the other hand, if $[d'_j, c'_j] \subseteq [d, c]$, then $[d'_j, c'_j]$ is $X_{j-1}Y_{j-1}$-admissible by Observation 2.3(ii), and therefore $\left|[d, c]_{X_j}\right| = \left|[d, c]_{X_{j-1}}\right|$ and similarly $\left|[d, c]_{Y_j}\right| = \left|[d, c]_{Y_{j-1}}\right|$. The claim follows.

Since $XY = X_0 Y_0$, we have shown that $\beta(XY) \leq 1$. This contradiction with condition (i) implies that our assumption (2) does not hold.

Thus, let j be the least index such that $\left|[d, c] \cap \{c'_j, d'_j\}\right| = 1$.

Suppose that $d'_j \notin [d, c]$. Since $|[d, c]_X| > c$ and $c'_j \in [d, c]$, we have $\left|[d'_j, c]_X\right| > c$. On the other hand, $\left|[d'_j, c'_j]_X\right| = c'_j$, and thus

$$\left|(c'_j, c]_X\right| \geq c + 1 - c'_j. \tag{3}$$

Since X is an independent set in C_n, we have $\left|(c'_j, c]_X\right| \leq (c - c'_j + 1)/2$. Combining this with (3), we derive $c < c'_j$, a contradiction with the assumption that $c'_j \in I$.

The argument for the case $c'_j \notin [d, c]$ is similar. Analogously to (3), we find that $\left|(c, c'_j]_Y\right| \geq c'_j - c + 1$. On the other hand, $(c, c'_j]_Y$ is independent and thus its size is at most $(c'_j - c)/2$, an improvement by $1/2$ coming from the fact that $c'_j \notin Y$ as c'_j is a control element for XY. As a consequence, the resulting bound $c > c'_j + 1$ is even stronger than its analogue in the preceding case.

A similar computation works for condition (ii). □

The next two propositions describe properties of irregular vertices which are later used to assign colours to them or argue that the colouring is valid.

Proposition 5.2. *Let X be an essential vertex of G. If X is not regular, then there exists $i \in [m]$ satisfying one of the following:*

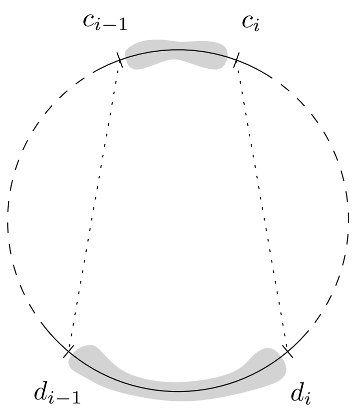


Figure 8: The set U_i, defined above Proposition 5.3, consists of all elements of $A \cup B$ contained in the shaded area. The dotted lines show control pairs of the standard AB-alternator, the dashed lines represent parts of the cycle C_n.

(a) $\{c_i, d_i\}$ *is a balanced pair in* X*,*

(b) X *is min-heavy on* $[d_i, c_i)$ *or on* $(d_i, c_i]$*,*

(c) X *is max-light on* $[d_i, c_i)$ *or on* $(d_i, c_i]$*.*

Proof. Suppose that X is not regular. If there exists $j \in [m]$ such that X is heavy or light on $[d_j, c_j)$ or $(d_j, c_j]$, then an index i satisfying (b) or (c) can be obtained by making an appropriate extremal choice of j. We can thus assume that X is balanced.

Since X is not regular, it contains an element from $C \cup D$ — say, $d_\ell \in X$. (A symmetric argument works in the other case.) Being balanced, X contains c_ℓ elements of $[d_\ell, c_\ell)$, and therefore $|(d_\ell, c_\ell)_X| = c_\ell - 1$. Since $|(d_\ell, c_\ell]_X| = c_\ell$, we have $c_\ell \in X$. Thus, $\{c_\ell, d_\ell\}$ is a balanced pair in X. □

For convenience, we set $d_0 = c_1$, $c_0 = d_1$, $d_{m+1} = c_m$ and $c_{m+1} = d_m$ for the rest of this section. For $i \in [m+1]$, we define

$$U_i = (d_i, d_{i-1})_W \cup (c_{i-1}, c_i)_W.$$

(See the illustration in Figure 8.) Note that for each i, $U_i \subseteq A \cup B$ and $|U_i|$ is even, namely

$$|U_i| = \begin{cases} 2c_1 & \text{if } i = 1, \\ 2(k - c_m) & \text{if } i = m+1, \\ 2(c_i - c_{i-1}) & \text{otherwise.} \end{cases}$$

Proposition 5.3. *Let X be an essential vertex of G and $i \in [m]$.*

(i) If X is min-heavy on $[d_i, c_i)$ (respectively, $(d_i, c_i]$), then either $i > 1$ and $\{c_{i-1}, d_{i-1}\}$ is a balanced pair in X, or X contains more than half of the elements in the set $U_i \cup \{d_i\}$ (respectively, $U_i \cup \{c_i\}$).

(ii) If X is max-light on $[d_i, c_i)$ (respectively, $(d_i, c_i]$), then either $i < m$ and $\{c_{i+1}, d_{i+1}\}$ is a balanced pair in X, or X contains more than half of the elements in the set $U_{i+1} \cup \{c_i\}$ (respectively, $U_{i+1} \cup \{d_i\}$).

Proof. We prove (i) only for the case of X min-heavy on $[d_i, c_i)$ since the other case is completely analogous. For $i = 1$, the claim is trivially true since X is heavy on $[d_1, c_1) = U_1 \cup \{d_1\}$. Suppose then that $i > 1$.

Since X is heavy on $[d_i, c_i)$, $|[d_i, c_i)_X| \geq c_i + 1$. Let us assume that X contains less than half of the elements of the (odd-sized) set $U_i \cup \{d_i\}$ — that is, $|X \cap (U_i \cup \{d_i\})| \leq c_i - c_{i-1}$. Hence

$$|[d_{i-1}, c_{i-1}]_X| \geq c_{i-1} + 1. \tag{4}$$

On the other hand, X is heavy on neither $[d_{i-1}, c_{i-1})$ nor $(d_{i-1}, c_{i-1}]$, so $|[d_{i-1}, c_{i-1})_X| \leq c_{i-1}$ and $|(d_{i-1}, c_{i-1}]_X| \leq c_{i-1}$. Comparing with (4), we see that $c_{i-1}, d_{i-1} \in X$. Furthermore, $|[d_{i-1}, c_{i-1})_X| = c_{i-1}$, so $\{c_{i-1}, d_{i-1}\}$ is a balanced pair in X.

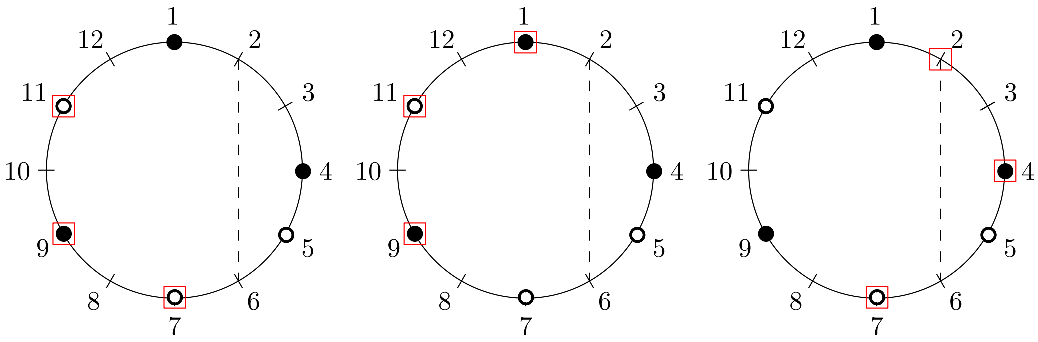
The proof of (ii) is similar and we only comment on the case of X max-light on $[d_i, c_i)$ and $i < m$. We have $|[d_i, c_i)_X| \leq c_i - 1$. If X contains less than half of the elements in $U_{i+1} \cup \{c_i\}$, then $|(d_{i+1}, c_{i+1})_X| \leq (c_i - 1) + (c_{i+1} - c_i) = c_{i+1} - 1$. However, X is not light on $[d_{i+1}, c_{i+1})$ nor on $(d_{i+1}, c_{i+1}]$, so $c_{i+1}, d_{i+1} \in X$ and $|[d_{i+1}, c_{i+1})_X| = c_{i+1}$. It follows that $\{c_{i+1}, d_{i+1}\}$ is a balanced pair in X. □

Let X be an essential vertex of G and let $d \in [n]$. By Lemma 3.2, there is at most one element $c \in [k-1]$ such that $|[d, c]_X| = c$ and $c \notin X$. If such an element exists, we call it the *depth of d in X* and define $\delta(X, d) = c$; otherwise, we let $\delta(X, d) = k$.

This notion will only be used for vertices X containing a W-gap. For such a vertex, a W-gap (s, t) in X is *topmost* if t is as large as possible. The *depth $\delta(X)$ of X* is defined as $\delta(X, t)$, where (s, t) is the topmost W-gap in X. (See the examples in Figure 9.)

In a series of lemmas, we infer now the basic properties of this newly-introduced notion.

Lemma 5.4. *Let X be an essential vertex of G. If (s, t) is the topmost W-gap in X and $\delta(X) < k$, then $k \leq s < t$.*



(a) (b) (c)

Figure 9: Essential vertices X of XG$(12,3)$ (shown with red boxes) with different depths and different topmost W-gaps (s,t). The vertices A and B are the same in all cases (black and white dots, respectively). (a) The depth of X is 1, $(s,t)=(9,11)$. (b) The depth of X is 2, $(s,t)=(9,11)$. (c) The depth of X is $k=3$, $(s,t)=(2,4)$.

Proof. Let $\delta(X)=\delta(X,t)$ be denoted by δ. We show that $\delta \notin [s,t)$. Suppose the contrary. Then

$$\delta = |[t,\delta]_X| = |[t,s]_X| = k$$

since (s,t) is a W-gap in X, a contradiction with $\delta < k$.

Additionally, it is straightforward to show that $\delta \neq t$: otherwise, the definition of depth would imply that $|[\delta,\delta]_X| = \delta$ and hence $\delta = 1$ and $\delta \in X$, contradicting the property $\delta \notin X$ which is immediate from the same definition.

Thus, $\delta \in (t,s)$. We have

$$k = |[t,s]_X| = |[t,\delta)_X| + |(\delta,s]_X| \leq \delta + (s-\delta),$$

which implies that $k \leq s$. Furthermore, it must be that $s < t$, for otherwise $t < \delta$ and we could not have $|[t,\delta)_X| = \delta$. $\square$

Part (ii) of the following lemma shows that adjacent essential vertices having W-gaps have different depths, so it makes sense to colour such a vertex according to its depth.

Lemma 5.5. *Let X, Y be adjacent essential vertices of G such that X has a W-gap and $\delta(X) < k$. Then the following hold:*

(i) $\delta(X) \notin X \cup Y$,

(ii) if Y has a W-gap, then $\delta(X) \neq \delta(Y)$.

Proof. Let (s,t) be the topmost W-gap in X. Consider the standard XY-alternator $C' \cup D'$. Lemma 3.1(ii) implies that (s,t) contains an element of $C' \cup D'$. By Lemma 5.4, $k \leq s < t$ and therefore there is an element $d' \in (s,t)_{D'}$. Let c' be the corresponding control element in C'. Since $|[t,c']_X| = |[d',c']_X| = c'$ and $c' \notin X$, c' is the depth of X.

Part (i) follows from the observation that by the definition of XY-alternator, $c' \notin X \cup Y$.

We prove (ii). Suppose that Y has a W-gap and the topmost W-gap in Y is (s',t'). Furthermore, suppose that the depth of Y is also c'. By Lemma 5.4 again, we have $k \leq s' < t'$. By symmetry, we may assume that $t' < t$, and therefore $t' < s$. According to the definition of depth,

$$|[t',c']_Y| = c' = |[t,c']_X| .$$

Since $t' < s$, we have $|[s,c']_Y| \leq c' - 1$ while $|[s,c']_X| = c' + 1$. Lemma 5.1 implies that X and Y are non-adjacent, a contradiction. □

In the three lemmas that follow, we will be concerned with the depth of regular vertices.

Lemma 5.6. *If X is a regular vertex of G that has a W-gap, then $\delta(X) \notin C$.*

Proof. Suppose for the sake of contradiction that X is regular and $\delta(X) \in C$; say, $\delta(X) = c_i$, where $i \in [m]$. Let (s,t) be the topmost W-gap in X. Thus, $|[t,c_i]_X| = c_i$ and $|[s,c_i]_X| = c_i + 1$. Since X is regular, $d_i \notin \{s,t\}$ and X is balanced on $(d_i,c_i]$, i.e., $|(d_i,c_i]_X| = c_i$. Hence $s < d_i < t$, contradicting the assumption that (s,t) is a W-gap. The lemma follows. □

Suppose that X is a regular vertex. As already mentioned, the only subset of $[n]$ which may be a regular vertex of G adjacent to X in G is the set of the elements of $A \cup B$ not contained in X — that is, $(A \cup B) \setminus X$. We let this set be denoted by X^*. Observe that if X^* is a vertex of G, it is indeed regular.

Lemma 5.7. *Let X be a regular vertex of G such that X and X^* are adjacent vertices and X has a W-gap. Let $C' \cup D'$ be the standard XX^*-alternator. The following hold:*

(i) every W-gap in X contains an element of C' or an element of D' but not both,

(ii) if some W-gap in X contains an element of D' then the topmost W-gap in X has this property too,

(iii) $\delta(X) < k$ if and only if some W-gap in X contains an element of D'.

Proof. (i) Let (s,t) be a W-gap in X. Furthermore, let $C' = \{c'_1, \ldots, c'_\ell\}$ and $D' = \{d'_1, \ldots, d'_\ell\}$. By Lemma 3.1(ii) (applied to the edge XX^*), $(s,t)_{C' \cup D'} \neq \emptyset$. Suppose (for the sake of a contradiction) that (s,t) contains elements from both C' and D'; then it must either contain both c'_1 and d'_1, or contain both c'_ℓ, d'_ℓ. Suppose the latter. By Lemma 3.1(i), $[k, d'_\ell]_{X \cup X^*} \neq \emptyset$. Since $c'_\ell < k$ and $X \cup X^* = A \cup B$, we find that $[c'_\ell, d'_\ell]_{A \cup B} \neq \emptyset$, contradicting the assumption that (s,t) is a W-gap. The argument for the case $c'_1, d'_1 \in (s,t)$ is analogous.

(ii) Suppose that a W-gap (s,t) in X contains an element of D' and that (s',t') is the topmost W-gap in X. By Lemma 5.4, $k \leq s < t$; since $t \leq t'$ by the choice of (s',t'), it follows that $s \leq s'$ because (s',t') is a W-gap. Hence $k \leq s' < t'$ and (s',t') must contain an element of D' by (i).

(iii) If $\delta(X) < k$ and (s',t') is the topmost W-gap in X, then Lemma 5.4 implies that $k \leq s' < t'$, so (s',t') contains no element of C'. By (i), it contains an element of D'.

Conversely, suppose that a W-gap (s,t) contains an element d'_i of D' ($i \in [\ell]$). By (ii), we may assume that it is the topmost W-gap in X. For the control element c'_i corresponding to d'_i, we have $c'_i \notin X$ and

$$|[t, c'_i]_X| = |[d'_i, c'_i]_X| = c'_i,$$

so the depth of X, which is defined as $\delta_X(t)$, is $c'_i < k$. □

The last in our series of lemmas complements Lemma 5.5(ii) in the sense that for X regular, the depths of X and X^* (when defined) are different even if they may equal k. However, the proof of this last lemma is quite a bit longer.

Lemma 5.8. *Let X be a regular vertex of G such that X^* is a vertex of G adjacent to X, X has a W-gap and $\delta(X) = k$. Then X^* has a W-gap and $\delta(X^*) < k$.*

Proof. Let $C' \cup D'$ be the standard XX^*-alternator, where $C' = \{c'_1, \ldots, c'_\ell\}$ and $D' = \{d'_1, \ldots, d'_\ell\}$.

Let (s,t) be the topmost W-gap in X. By parts (i) and (iii) of Lemma 5.7, (s,t) contains an element of C'. Let the least such element be c'_j. Consider the corresponding element $d'_j \in D'$ and let (s^*, t^*) be the $(X \cup X^*)$-gap containing d'_j. In a series of claims, we will prove that (s^*, t^*) is a W-gap in X^* and derive that $\delta(X^*) < k$.

Claim 4. *The elements s^* and t^* belong to X^*.*

We show first that $t^* \in X^*$. Note that t^* is the first and s is the last element of $[d'_j, c'_j]_{X\cup X^*}$, and that $s \in X$. The claim follows in case $j = 1$, for by the definition of $\mathrm{XG}(n,k)$, X and X^* alternate on $[d'_1, c'_1]$ and $|[d'_1, c'_1]_{X\cup X^*}|$ is even. If $j > 1$, then none of t^* and s belong to $[d'_{j-1}, c'_{j-1}]$ by the choice of j and by Lemma 3.1(i). Since switching along the sequence $([d'_p, c'_p])_{p\in[j-1]}$ changes XX^* to a pair alternating on $[d'_j, c'_j]$, and t^* and s are unaffected by the switching, we have $t^* \in X^*$ as claimed.

By Lemma 3.1(i), d'_j is the only element of $C' \cup D'$ in (s^*, t^*). Since $t^* \in X^*$, part (ii) of the same lemma implies that $s^* \in X^*$.

Claim 5. *(s^*, t^*) is a W-gap.*

Suppose by way of contradiction that the claim does not hold. Since (s^*, t^*) is an $(X \cup X^*)$-gap and $W = A \cup B \cup C \cup D = (X \cup X^*) \cup (C \cup D)$, the interval (s^*, t^*) must contain an element of $C \cup D$.

Suppose first that for some $q \in [m]$, $c_q \in (s^*, t^*)$. Since the interval (s^*, t^*) contains an element of C as well as an element of D', it contains either the element k or the element n. This leads to two symmetric cases.

Let us consider in detail the case $k \in (s^*, t^*)$. We have $c_q > c'_j$, for otherwise (s^*, t^*) would contain t and fail to be an $(X\cup X^*)$-gap. For a similar reason, we must have $d_q \notin (s^*, t^*)$, since by Lemma 3.1(ii), the interval $[k, d_q]$ contains at least two elements of $A \cup B = X \cup X^*$.

Putting the above observations together, we find that

$$c'_j < c_q < d'_j < d_q. \tag{5}$$

Informally, the control pairs $\langle c'_j, d'_j \rangle$ and $\langle c_q, d_q \rangle$ (of an XX^*-alternator and an AB-alternator, respectively) 'cross' each other. We will derive a contradiction by an argument similar to that used to establish Lemma 5.1.

The interval $[d_q, c_q]$ contains $2c_q$ elements of $A \cup B$. On the other hand,

$$\begin{aligned} |[d_q, c_q]_{A\cup B}| &= |[d_q, c_q]_{X\cup X^*}| \le \left|[d'_j, c_q]_{X\cup X^*}\right| \\ &= \left|[d'_j, c'_j]_{X\cup X^*}\right| + \left|[c'_j, c_q]_{X\cup X^*}\right| \le 2c'_j + (c_q - c'_j - 1), \end{aligned}$$

which implies that $c_q < c'_j$, a contradiction.

The other case, namely $n \in (s^*, t^*)$, can be obtained by a completely symmetric argument. This time, the inequalities analogous to (5) are $c_q < c'_j < d_q < d'_j$, and a computation similar to the above yields $c'_j < c_q$, which provides a contradiction.

We have thus ruled out the possibility that (s^*, t^*) contains an element of C. Thus, it must contain an element, say d_r, of D ($r \in [m]$). Consider the corresponding element c_r of C.

Since (s^*, t^*) contains no elements of $X \cup X^* = A \cup B$, we have $[d_r, c'_j]_A = [d'_j, c'_j]_A$ and similarly for B in place of A. Hence,

$$\big|[d_r, c'_j]_{A\cup B}\big| = \big|[d'_j, c'_j]_{A\cup B}\big| = \big|[d'_j, c'_j]_{X\cup X^*}\big| = 2c'_j.$$

By Lemma 5.1(i), we must have $\big|[d_r, c'_j]_A\big| = \big|[d_r, c'_j]_B\big| = c'_j$. Since the same equality is true with c'_j replaced by c_r (by the definition of AB-alternator), Lemma 3.2 implies that $c'_j = c_r$. Consequently, $c_r \in (s, t)$, so contrary to the assumption, (s, t) is not a W-gap. This proves the claim.

Since the W-gap (s^*, t^*) in X^* contains $d'_j \in D'$, Lemma 5.7(iii) implies that $\delta(X^*) < k$. Thus, the proof of the lemma is complete. $\square$

One last piece of terminology we will need in order to lay out the colouring rules is the following. For $i \in [m+1]$, an essential vertex X of G is *skew at d_i* if X contains the largest element of $(d_{i+1}, d_i)_W$ and the second smallest element of $(d_i, d_{i-1})_W$. (By Lemma 3.1(i), each of the latter two sets contains at least two elements.)

We are now ready to define a colouring of G using the following set of colours:

$$\left\{\boxed{i} : i \in [n] \setminus (A \cup B)\right\} \cup \left\{\boxed{0}\right\}.$$

Since $|A| = |B| = k$, the total number of colours is $n - 2k + 1$.

A vertex X of G is assigned a colour by the following rules, in the stated order of precedence:

(R1) If X is inessential and therefore contains an element of $[n] \setminus W$, then it gets colour $\boxed{j}$, where j is the least such element.

(R2) If X contains a balanced pair, then X gets colour $\boxed{c_i}$, where $i \in [m]$ is least such that $\{c_i, d_i\}$ is a balanced pair in X.

(R3) If X is min-heavy or max-light on $(d_i, c_i]$ for some $i \in [m]$, then X gets colour $\boxed{c_i}$.

(R4) If X is min-heavy or max-light on $[d_i, c_i)$ for some $i \in [m]$, then X gets colour $\boxed{d_i}$.

(R5) If X has a W-gap and $\delta(X) = j$, then X gets colour $\boxed{j}$ if $j \in [k-1] \setminus (A \cup B)$, and colour $\boxed{0}$ otherwise (that is, if $j = k$ or $j \in [k-1] \cap (A \cup B)$).

(R6) If X is skew at d_i for some $i \in [m]$, then X gets colour $\boxed{d_i}$, where i is least with this property.

(R7) If none of the above applies, X gets colour $\boxed{0}$.

We will now show that each colour class of this colouring is an independent set in G.

Proposition 5.9. *Rules (R1)–(R7) determine a valid colouring of G.*

Proof. We will discuss each colour class in turn. If $j \in [n] \setminus (W \cup [k-1])$, then colour $\boxed{j}$ is only assigned by Rule (R1), namely to those vertices that contain element j. The colour class is therefore independent.

Claim 1. *If $j \in [k-1] \setminus W$, then the colour class of $\boxed{j}$ is independent.*

Colour $\boxed{j}$ may be assigned to X by Rules (R1) and (R5) if X satisfies one of the following conditions:

- $j \in X$,
- X has a W-gap and $\delta(X) = j$.

Suppose that vertices X, Y both get colour $\boxed{j}$. We prove that they are non-adjacent in G.

Case 1.1: $j \in X$.

If $j \in Y$, then X and Y are not disjoint and XY is not an edge. If Y has a W-gap and $\delta(Y) = j$, then $\delta(Y) \in X \cup Y$, so XY is not an edge by Lemma 5.5(i).

Case 1.2: *X has a W-gap and $\delta(X) = j$.*

By symmetry and the preceding case, we may assume that Y contains a W-gap and $\delta(Y) = j$. In this case, X and Y are non-adjacent by Lemma 5.5(ii).

The proof of Claim 1 is complete.

At this point, it remains to consider all the colours $\boxed{j}$ with $j \in C \cup D$ and the colour $\boxed{0}$. This is done in the following three claims.

Claim 2. *For $i \in [m]$, the colour class of $\boxed{c_i}$ is independent.*

Colour $\boxed{c_i}$ is assigned by Rules (R2), (R3) and (R5) to essential vertices X satisfying one of the following:

- X contains a balanced pair $\{c_i, d_i\}$,
- X contains no balanced pair and X is min-heavy on $(d_i, c_i]$,
- X contains no balanced pair and X is max-light on $(d_i, c_i]$,
- X has a W-gap and $\delta(X) = c_i$.

By Proposition 5.2 and the position of Rule (R5), it may be assumed that if X has a W-gap, then X is regular. But then Lemma 5.6 shows that $\delta(X) \notin C$. We may therefore assume that all vertices assigned colour $\boxed{c_i}$ satisfy one of the first three conditions above.

Let X and Y be vertices of G assigned colour $\boxed{c_i}$. We prove that XY is not an edge of G.

Case 2.1: *X contains a balanced pair $\{c_i, d_i\}$.*

If Y contains $\{c_i, d_i\}$ as well, then X and Y are intersecting and therefore non-adjacent in G.

Suppose that Y is min-heavy on $(d_i, c_i]$. We may assume that $c_i \notin Y$ for otherwise X and Y intersect. Thus, $|(d_i, c_i)_Y| > c_i$ while $|(d_i, c_i)_X| = c_i - 1$. Lemma 5.1(ii) (with $c = c_i$ and $d = d_i+1$) implies that X, Y are non-adjacent.

Finally, if Y is max-light on $(d_i, c_i]$, then a symmetric argument applies. We may assume that $d_i \notin Y$, and therefore $|[d_i, c_i]_Y| < c_i$ while $|[d_i, c_i]_X| = c_i + 1$. Again, it follows from Lemma 5.1(ii) that XY is not an edge.

Case 2.2: *X is min-heavy on $(d_i, c_i]$.*

By symmetry and the preceding case, we may assume that $\{c_i, d_i\}$ is not a balanced pair in Y. In addition, it may be assumed (by the position of Rule (R2)) that Y does not contain any balanced pair.

If Y is min-heavy on $(d_i, c_i]$, then X and Y intersect by Proposition 5.3(i).

On the other hand, if Y is max-light on $(d_i, c_i]$, then $|(d_i, c_i]_Y| < c_i$. Since X is min-heavy on this interval, we have $|(d_i, c_i]_X| > c_i$, so Lemma 5.1(i) (with $c = c_i$ and $d = d_i + 1$) shows that X, Y are non-adjacent.

Case 2.3: *X is max-light on $(d_i, c_i]$.*

It may be assumed that Y is max-light on $(d_i, c_i]$ as well. Furthermore, as in the preceding case, Y may be assumed to contain no balanced pair. Proposition 5.3(ii) then implies that X and Y intersect.

This concludes the proof of Claim 2.

Claim 3. *For $i \in [m]$, the colour class of $\boxed{d_i}$ is independent.*

Note that since $d_i > k$ by the definition of AB-alternator, Rule (R5) does not assign colour $\boxed{d_i}$. Thus, colour $\boxed{d_i}$ is only assigned by Rules (R4) and (R6) to vertices X satisfying one of the following:

- X contains no balanced pair and is min-heavy on $[d_i, c_i)$,
- X contains no balanced pair and is max-light on $[d_i, c_i)$,
- X is regular and skew at d_i.

Suppose that X, Y are vertices of G assigned colour $\boxed{d_i}$. We may assume that X and Y are disjoint. We prove that X and Y are non-adjacent in G, beginning with the case of X satisfying the last condition above.

Case 3.1: *X is regular and skew at d_i.*

We will assume that $i > 1$, the $i = 1$ case being analogous. By the position of Rule (R5), X has no W-gap. Consider the set $S = U_i \cup \{d_i\}$ and recall that $|S| = 2(c_i - c_{i-1}) + 1$. Since X is regular, $|X \cap S| = c_i - c_{i-1}$.

We distinguish three subcases according to the type of Y.

Suppose that Y is min-heavy on $[d_i, c_i)$. By Proposition 5.3(i), $|Y \cap S| = c_i - c_{i-1} + 1$, so the sets $X \cap S$ and $Y \cap S$ partition S. Now $d_i \notin X$ (since X is regular), so $d_i \in Y$. Since $C \cup D$ is the standard AB-alternator, the smallest element of $(d_i, d_{i-1})_W$ is $d_i + 1$. We have $d_i + 1 \notin X$ as X is skew at d_i and has no W-gap. It follows that $\{d_i, d_i + 1\} \subseteq Y$, a contradiction with the independence of Y in C_n.

Suppose next that Y is max-light on $[d_i, c_i)$. Let d^- be the last element of $(d_{i+1}, d_i)_W$. (Recall our convention that $d_{m+1} = c_m$, see page 21.) Since $d^- \in X$, we have $d^- \notin Y$. Furthermore, $|[d^-, c_i)_X| = c_i + 1$ as X is regular on $[d_i, c_i)$, while $|[d^-, c_i)_Y| < c_i$. Lemma 5.1(ii) implies that X and Y are non-adjacent in G.

Finally, if Y is skew at d_i, then X and Y both contain the second smallest element of $(d_i, d_{i-1})_W$, a contradiction.

Case 3.2: *X is min-heavy on $[d_i, c_i)$.*

It may be assumed that Y is min-heavy or max-light on $[d_i, c_i)$.

The set Y cannot be min-heavy on $[d_i, c_i)$ for by Proposition 5.3(i), X and Y would intersect. If Y is max-light on $[d_i, c_i)$, then $|[d_i, c_i)_Y| < c_i$ and $|[d_i, c_i)_X| > c_i$, so XY is not an edge by Lemma 5.1(ii).

Case 3.3: *X is max-light on $[d_i, c_i)$.*

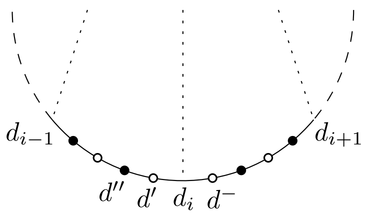
By the preceding cases, Y may be assumed to be max-light on $[d_i, c_i)$. Proposition 5.3(i) implies that X and Y intersect, a contradiction.

The proof of Claim 3 is complete.

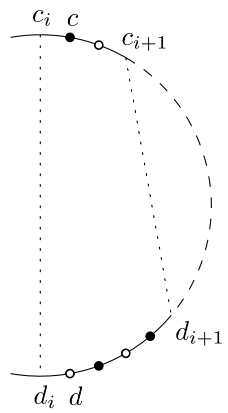
Claim 4. *The colour class of $\boxed{0}$ is independent.*

Colour $\boxed{0}$ is assigned by Rule (R5) to vertices X having a W-gap and having depth in the set $[1, k-1]_{A \cup B} \cup \{k\}$, and by Rule (R7) to vertices satisfying none of the conditions in Rules (R1)–(R6). Let us say that X is of type (R5) or (R7) accordingly. All of these vertices are regular (hence subsets of W) by Proposition 5.2; additionally, type (R7) vertices have no W-gap, and are not skew at any d_i ($i \in [m]$).

Recall that for a regular vertex X, we defined $X^* = (A \cup B) \setminus X$ and noted that this is the only candidate for a regular vertex adjacent to X in G.



(a) The vertices d', d'' and d^-.



(b) The vertices c and d.

Figure 10: Notation used in the proof of Claim 4. The black and white dots represent the elements of A and B, respectively. The elements of X are not shown. Dotted lines represent control pairs of the standard AB-alternator, dashed lines are irrelevant portions of the cycle C_n.

Our colouring procedure assigns a colour different from $\boxed{0}$ to every irregular vertex. Suppose then that X is regular and X^* is a vertex of G (and therefore a regular vertex), and both get colour $\boxed{0}$. We intend to show that X and X^* are non-adjacent.

Case 4.1: *X is a vertex of type (R5).*

If $\delta(X) = k$ then X is not adjacent to X^* by Lemma 5.8. On the other hand, if $\delta(X) \in [1, k-1]_{A\cup B}$, then $\delta(X) \in A \cup B = X \cup X^*$ and therefore XX^* is not an edge by Lemma 5.5.

By symmetry, this leaves us with the following case.

Case 4.2: *X and X^* are both of type (R7).*

We suppose that X and X^* are adjacent and intend to reach a contradiction by showing that X equals A or B (in which case the same holds for X^*). The set $(d_1, c_1)_W$ consists of $2c_1$ elements and each of X and X^* contains c_1 of them. Since none of X and X^* has a W-gap, they alternate on (d_1, c_1). Thus, we may assume by symmetry that $(d_1, c_1)_X = (d_1, c_1)_A$.

We prove by induction that for each i, $0 \leq i \leq m$, that $(d_{i+1}, d_i)_X = (d_{i+1}, d_i)_A$; recalling the convention that $c_1 = d_0$, the base case is established. Consider $i \geq 1$. Let d^- be the largest element of $(d_{i+1}, d_i)_W$, and let d', d'' be the smallest and second smallest element of $(d_i, d_{i-1})_W$, respectively. (Recall that each of these sets has size at least 2 by Lemma 3.1(i).) The notation is illustrated in Figure 10.

By the induction hypothesis, $(d_i, d_{i-1})_X = (d_i, d_{i-1})_A$. Since X is not skew at d_i, we have $d^- \in X$ if and only if $d'' \notin X$. The same holds for A, which is also not skew at d_i. Furthermore, d'' belongs to X if and only if it belongs to A, because X and A agree on (d_i, d_{i-1}). Summarising, $d^- \in X$ if and only if $d^- \in A$. Since each element of $(d_{i+1}, d_i)_W$ belongs to X or X^*, and since X and X^* have no W-gap, this implies that $(d_{i+1}, d_i)_X = (d_{i+1}, d_i)_A$ as required.

By the above, $[k, n]_X = [k, n]_A$. Since X^* is regular, $[k, n]_{X^*} = [k, n]_B$.

To prove that $X = A$, it remains to show that $(c_i, c_{i+1})_X = (c_i, c_{i+1})_A$ for each $i \in [m-1]$. We proceed by induction on i, starting with the case $i = 0$ which is already proved.

Let $i > 0$. We may assume that $(c_i, c_{i+1})_W \neq \emptyset$, for otherwise we are done. By Lemma 3.1(ii) and the assumption that X, X^* are adjacent, d_i is contained in a control pair for the edge XX^*. Since X and X^* are regular, it follows from Lemma 3.2 that the control pair must be $\langle c_i, d_i \rangle$. Let d be the largest element of $(d_{i+1}, d_i)_W$ and let c be the smallest element of $(c_i, c_{i+1})_W$.

Since switching along the sequence $([d_j, c_j])_{j \in [i]}$ transforms the pair AB to a pair alternating on $[d_{i+1}, c_{i+1}]$, and since $|[d_i, c_i]_{A \cup B}|$ is even, the element c belongs to A if and only if d belongs to B. We know from the induction hypothesis and the preceding part of the proof that $[d_i, c_i]_X = [d_i, c_i]_A$, so a similar argument shows that $c \in X$ if and only if $d \in X^*$. Knowing that $[k, n]_B = [k, n]_{X^*}$, we may combine these equivalences and infer that $c \in A$ if and only if $c \in X$.

Since X and X^* have no W-gap and cover $(c_i, c_{i+1})_W$, this determines $(c_i, c_{i+1})_X$ and shows that

$$(c_i, c_{i+1})_X = (c_i, c_{i+1})_A,$$

completing the proof of the induction step as well as the whole proof that $X = A$ and $X^* = B$. Since the edge AB does not exist in G, this finishes the proof of Claim 4.

The proof of Proposition 5.9 is complete. □

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