

Forbidden induced pairs for perfectness and ω -colourability of graphs

Maria Chudnovsky* Adam Kabela† Binlong Li‡
 Petr Vrána§

Abstract

We characterise the pairs of graphs $\{X, Y\}$ such that all $\{X, Y\}$ -free graphs (distinct from C_5) are perfect. Similarly, we characterise pairs $\{X, Y\}$ such that all $\{X, Y\}$ -free graphs (distinct from C_5) are ω -colourable (that is, their chromatic number is equal to their clique number). More generally, we show characterizations of pairs $\{X, Y\}$ for perfectness and ω -colourability of all connected $\{X, Y\}$ -free graphs which are of independence at least 3, distinct from an odd cycle, and of order at least n_0 , and similar characterisations subject to each subset of these additional constraints. (The classes are non-hereditary and the characterisations for perfectness and ω -colourability are different.) We build on recent results of Brause et al. on $\{K_{1,3}, Y\}$ -free graphs, and we use Ramsey's Theorem and the Strong Perfect Graph Theorem as main tools. We relate the present characterisations to known results on forbidden pairs for χ -boundedness and deciding k -colourability in polynomial time.

1 Introduction

The study of forbidden induced subgraphs in relation to perfectness of graphs goes back to Berge [1] who conjectured that perfect graphs can be characterised by forbidding all induced odd cycles of length at least five and their complements; this was later proven by the first author and Robertson, Seymour and

*Department of Mathematics, Princeton University, U.S.A.

E-mail: mchudnov@math.princeton.edu.

†Faculty of Applied Sciences, University of West Bohemia, Czech Republic. Previous affiliation: Faculty of Informatics, Masaryk University, Czech Republic.

E-mail: kabela@kma.zcu.cz.

‡School of Mathematics and Statistics, Northwestern Polytechnical University, China.

E-mail: libinlong@mail.nwpu.edu.cn.

§Faculty of Applied Sciences, University of West Bohemia, Czech Republic.

E-mail: vranap@kma.zcu.cz.

Thomas [6] (and the result is referred to as the Strong Perfect Graph Theorem). For the sake of brevity, we ought to say that graph perfectness is one of the classical topics in graph theory and refer the reader to a monograph by Ramírez-Alfonsín and Reed [17] and to surveys by Hougardy [12] and by Roussel, Rusu and Thuillier [20].

In the spirit of graph perfectness, Gyárfás [9] initiated the study of hereditary classes of graphs whose chromatic number can be bounded by a function of their clique number. This topic (referred as χ -boundedness) is also widely investigated, in particular for classes of graphs defined by forbidden induced subgraphs. One of the major open conjectures in this area, stated by Gyárfás [9] and by Sumner [24], asserts that for every forest F , the class of all F -free graphs is χ -bounded. We refer the reader to surveys by Schiermeyer and Randerath [21] and by Scott and Seymour [22] (see also the recent result of Bonamy and Pilipczuk [2]).

Similarly, conditions on forbidden induced subgraphs are investigated in relation to the computational complexity of colourability. In particular, Král', Kratochvíl, Tuza and Woeginger [14] characterised graphs H such that, given an H -free graph and an integer k , the question whether the graph is k -colourable can be decided in polynomial time (namely, for H chosen as an induced subgraph of $K_1 \cup P_3$ or P_4 , and for every other graph H the problem is NP-complete), and they initiated the investigation of forbidden pairs. More results on the topic can be found surveyed by Golovach, Johnson, Paulusma and Song [10] who also asked for the complete classification of forbidden pairs in relation to the complexity of colourability. This question is still open. For perfect graphs the fact that k -colourability is decidable in polynomial time follows from the study of Grötschel, Lovász and Schrijver [11].

In relation to the present paper, we recall that every P_4 -free graph is perfect by Seinsche [23] (it also follows from the Strong Perfect Graph Theorem). On the other hand, if the chromatic number is bounded by a linear function of the clique number for the class of graphs defined by a forbidden induced subgraph H , then H is an induced subgraph of P_4 (for instance, this follows by combining the result of Erdős [8] who showed that there are graphs of arbitrarily large girth and chromatic number and the result Brause, Randerath, Schiermeyer and Vumar [5] who showed that there is no linear binding function for the class of $\{3K_1, 2K_2\}$ -free graphs). On the other hand, particular forbidden induced subgraphs (and sets of subgraphs) are known to give polynomial binding functions, and there is a number of pairs of forbidden induced subgraphs giving linear binding functions. The results and open questions of this type can be found surveyed in [21]; in particular, the question of characterising the pairs of connected forbidden subgraphs giving that the chromatic number is at most the clique number plus one. It seems natural to also ask about the improvement of a binding function when restricting a class by additional constraints. For instance, the first author and Seymour [7] showed that the chromatic number of a connected $K_{1,3}$ -free graph of independence at least 3 is at most twice its clique number; whereas $K_{1,3}$ -free

graphs in general admit a polynomial binding function (for instance, see [21]). Considering forbidden pairs $\{K_{1,3}, Y\}$, Brause et al. [4] showed that, depending on the choice of Y , the class of all connected $\{K_{1,3}, Y\}$ -free graphs distinct from an odd cycle (and of independence at least 3) either consists of perfect graphs or contains infinitely many graphs which are not ω -colourable (see Theorem 4 in Section 4). The dichotomic nature of this result is non-trivial (since the classes are non-hereditary, perfectness and ω -colourability are different concepts), and it motivates the general question of forbidden pairs $\{X, Y\}$, which is answered by the present study.

As the main results, we characterise the pairs of forbidden induced subgraphs in relation to perfectness and ω -colourability for classes of graphs restricted by additional constraints. The basic characterisations concern pairs $\{X, Y\}$ such that every $\{X, Y\}$ -free graph (distinct from C_5) is perfect, and pairs such that every $\{X, Y\}$ -free graph (distinct from C_5) is ω -colourable (see Theorem 1 in Section 3). More generally, we present characterisations of pairs $\{X, Y\}$ giving perfectness (ω -colourability) of all connected $\{X, Y\}$ -free graphs of independence at least 3 distinct from an odd cycle, and analogous characterisations subject to each subset of the additional constraints (stated formally in Theorem 2). Furthermore, we characterise pairs $\{X, Y\}$ in relation to perfectness (ω -colourability) of all but finitely many $\{X, Y\}$ -free graphs satisfying the constraints (see Theorem 3). In other words, there is n_0 (depending on $\{X, Y\}$) such that all considered $\{X, Y\}$ -free graphs on at least n_0 vertices are perfect (ω -colourable). The present characterisations related to perfectness are, in fact, different from those related to ω -colourability due to the consideration of the additional constraints. (The constraints are discussed in Section 4.)

We note that the present conditions on forbidden pairs (without the additional constraints on a class) were studied in relation to χ -boundedness and to computational complexity of colourability. In fact, under each of the conditions (or its generalisation), the class is known to be χ -bounded and the k -colourability is decidable in polynomial time (see the known results surveyed in [21] and [10, Theorem 21 (ii)], respectively). We remark that for some pairs, the χ -boundedness and the fact that k -colourability is decidable in polynomial time can also be deduced using the present results (for details, see Section 4). In this sense, Theorems 2 and 3 show that these forbidden pairs yield a stronger property.

The paper is structured as follows. In Section 2, we recall basic notation. In Section 3, we state the main results on the forbidden pairs (see Theorems 1, 2 and 3). In addition, we use the fact that a collection of pairs of graphs can be represented by a graph and we provide figures depicting the collections. In Section 4, we discuss the additional constraints and the pairings given by the characterisations, and we comment on the connections to known results. The key steps for proving the main results are broken down in Sections 5, 6, 7 and 8. In Section 5, we show several structural lemmas for classes defined by particular forbidden induced subgraphs. We use the classical result of Ramsey [18]

(recalled in Theorem 5) and the characterisation of connected Z_1 -free graphs by Olariu [16] (recalled in Lemma 6). In Section 6, we show a detailed statement on how conditions on forbidden pairs imply perfectness. Using the structural lemmas shown in the previous section, the perfectness follows either directly (for most pairs) or with the aid of the Strong Perfect Graph Theorem (recalled in Theorem 17). Similarly in Section 7, we show how conditions on forbidden pairs imply ω -colourability. We use the Strong Perfect Graph Theorem and one of the structural lemmas (shown in Section 5) and also Ramsey's Theorem. In Section 8, we exclude all possible remaining pairs. In particular, we recall that the pairs containing $K_{1,3}$ were resolved in [4], and we eliminate the remaining cases with the help of several constructions of families of graphs which are not perfect (not ω -colourable) and structural properties observed in Section 5. Finally in Section 9, we combine the ingredients (assembled in Sections 4 and 6, 7 and 8) and prove the main results.

2 Notation

In this short section, we recall basic definitions and concepts. We refer the reader to [3] for additional notation and for a detailed introduction to the concepts of subgraphs and graph colourings.

We recall that a graph G is k -colourable if the vertices of G can be coloured with k colours so that adjacent vertices obtain distinct colours; and the smallest integer k with this property is called the *chromatic number* of G , and denoted $\chi(G)$. We let $\omega(G)$ denote the *clique number* of a graph G , that is, the number of vertices of a maximum complete subgraph of G . Clearly, $\chi(G) \geq \omega(G)$ for every graph G . For the sake of simplicity, we say that a graph G is ω -colourable if $\chi(G) = \omega(G)$. We recall that a graph G is *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G .

Given a family \mathcal{H} of graphs, we say that a graph G is \mathcal{H} -free if G contains no member of \mathcal{H} as an induced subgraph (and we write $\{H_1, H_2, \dots, H_k\}$ -free for a family $\{H_1, H_2, \dots, H_k\}$, and H -free as a short for $\{H\}$ -free). We let K_n, P_n, C_n denote the complete graph, the path, the cycle on n vertices, respectively. We let K_{n_1, \dots, n_k} denote the complete k -partite graph whose parties are of sizes n_1, \dots, n_k . We let $K_{1,3}^+$ denote the graph obtained from $K_{1,3}$ by subdividing an edge, and D the graph obtained from K_4 by removing an edge, and Z_1 the graph obtained from K_3 by adding a pendant edge, and Z_2 the graph obtained from Z_1 by subdividing the pendant edge (see graphs $K_{1,3}^+, D, Z_1$ and Z_2 depicted in the bottom part of Figure 2; the graphs are commonly referred to as chair, diamond, paw and hammer, respectively).

We let $G \cup H$ denote the disjoint union of graphs G and H , and we write kG for the disjoint union of k copies of G . We say that G is of independence at least k if G contains induced kK_1 ; and the largest such integer k is called

the *independence number* of G . We let \overline{G} denote the complement of a graph G that is, the graph on the same vertex set as G whose vertices are adjacent if and only if the corresponding vertices of G are not. (For instance see graph $\overline{K_3 \cup P_4}$ depicted in Figure 2.)

3 Main results

We present characterisations of the pairs of forbidden induced subgraphs in relation to perfectness and ω -colourability of graphs, see Theorems 1, 2 and 3. The pairs of interest are collected in Definitions 1, 3 and 4, and the class notation is given in Definition 2. We invite the reader to consult Figures 1, 2, \dots , 6.

Definition 1. Consider graphs $3K_1, K_1 \cup P_3, 2K_1 \cup K_2, K_3, Z_1$ and D (defined in Section 2). Let \mathcal{P}_1 be the collection of all pairs \mathcal{X} of graphs such that at least one of the following conditions is satisfied.

- At least one of the graphs of \mathcal{X} is an induced subgraph of P_4 .
- \mathcal{X} is either $\{3K_1, K_3\}$ or $\{3K_1, Z_1\}$ or $\{3K_1, D\}$.
- \mathcal{X} is either $\{K_1 \cup P_3, K_3\}$ or $\{K_1 \cup P_3, Z_1\}$ or $\{K_1 \cup P_3, D\}$.
- \mathcal{X} is either $\{2K_1 \cup K_2, K_3\}$ or $\{2K_1 \cup K_2, Z_1\}$.

Furthermore, let Ω_1 be the collection consisting of $\{2K_1 \cup K_2, D\}$ and all pairs of \mathcal{P}_1 .

The following are the basic characterisations (see also Figure 1).

Theorem 1. *Let \mathcal{X} be a pair of graphs and \mathcal{P}_1 and Ω_1 be the collections described in Definition 1. Then every \mathcal{X} -free graph (distinct from C_5) is perfect if and only if \mathcal{X} belongs to \mathcal{P}_1 . Similarly, every \mathcal{X} -free graph (distinct from C_5) is ω -colourable if and only if \mathcal{X} belongs to Ω_1 .*

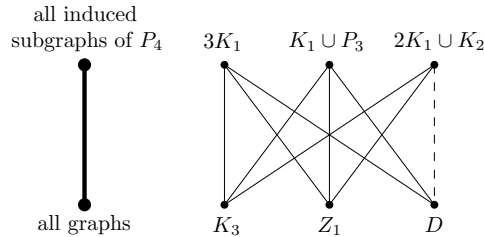


Figure 1: Depicting collections \mathcal{P}_1 and Ω_1 described in Definition 1. The graph in the picture represents collection Ω_1 of pairs of graphs as follows. Every vertex represents a graph and two vertices are adjacent if and only if the corresponding graphs form a pair. (Each of the bold vertices represents a family of graphs and the bold edge represents the collection of all pairs given by the families.) The dashed edge represents the pair belonging to Ω_1 but not to \mathcal{P}_1 . (We note that $3K_1, K_1 \cup P_3, 2K_1 \cup K_2$ is the complement of K_3, Z_1, D , respectively.)

As the main results, we extend the characterisations of Theorem 1 in relation to classes of graphs given by particular sets of additional constraints. The extended characterisations are presented in Theorems 2 and 3. In particular, we note that Theorem 1 is implied by the combination of items (1) and (6) of Theorem 2.

Definition 2. For the sake of brevity, we use the following notations.

- \mathcal{G} is the class of all graphs,
- \mathcal{G}_5 is the class of all graphs distinct from C_5 ,
- \mathcal{G}_o is the class of all graphs distinct from an odd cycle,
- \mathcal{G}_c is the class of all connected graphs,
- $\mathcal{G}_{c,5}$ is the class of all connected graphs distinct from C_5 ,
- \mathcal{G}_α is the class of all graphs of independence at least 3,
- $\mathcal{G}_{o,\alpha}$ is the class of all graphs of independence at least 3 distinct from an odd cycle,
- $\mathcal{G}_{c,\alpha}$ is the class of all connected graphs of independence at least 3,
- $\mathcal{G}_{c,o}$ is the class of all connected graphs distinct from an odd cycle,
- $\mathcal{G}_{c,o,\alpha}$ is the class of all connected graphs of independence at least 3 distinct from an odd cycle.

Definition 3. Consider graphs $3K_1$, $2K_1 \cup K_2$, $K_1 \cup P_3$, $2K_2$, $K_{1,3}$, $K_{1,3}^+$, P_5 , K_3 , $K_1 \cup K_3$, D , Z_1 , Z_2 , $\overline{K_1 \cup P_4}$ and $\overline{K_3 \cup P_4}$ defined in Section 2 (some of the graphs are also depicted in Figure 2), and collections \mathcal{P}_1 and Ω_1 described in Definition 1. Let \mathcal{I} be the collection of all pairs of graphs such that at least one member of the pair is $3K_1$. Let $\mathcal{P}_2, \mathcal{P}_{2c}, \mathcal{P}_3$ and \mathcal{P}_4 be the collections of pairs of graphs defined as follows.

- \mathcal{P}_2 consists of all pairs \mathcal{X} such that \mathcal{X} belongs to \mathcal{P}_1 or to \mathcal{I} or such that one member of \mathcal{X} is $K_1 \cup P_3$ and the other is an induced subgraph of $\overline{K_3 \cup P_4}$.
- \mathcal{P}_{2c} consists of all pairs of \mathcal{P}_2 and $\{K_{1,3}, 2K_2\}$ and $\{K_{1,3}, P_5\}$.
- \mathcal{P}_3 consists of all pairs of \mathcal{P}_1 and $\{K_{1,3}, K_3\}$ and $\{K_{1,3}, Z_1\}$ and $\{K_{1,3}^+, K_3\}$ and $\{K_{1,3}^+, Z_1\}$.
- \mathcal{P}_4 consists of all pairs of \mathcal{P}_{2c} and \mathcal{P}_3 and $\{K_{1,3}, K_1 \cup K_3\}$ and $\{K_{1,3}, Z_2\}$.

Similarly, let $\Omega_2, \Omega_{2c}, \Omega_3, \Omega_4$ be the collections of pairs of graphs defined as follows.

- Ω_2 consists of all pairs of \mathcal{P}_2 and $\{2K_1 \cup K_2, D\}$ and $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$.
- Ω_{2c} consists of all pairs of \mathcal{P}_{2c} and $\{2K_1 \cup K_2, D\}$ and $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$.
- Ω_3 consists of all pairs of \mathcal{P}_3 and $\{2K_1 \cup K_2, D\}$.
- Ω_4 consists of all pairs of \mathcal{P}_4 and $\{2K_1 \cup K_2, D\}$ and $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$.

The first of the main results is the following (see also Figures 2, 3 and 6 and the commentary in Section 4).

Theorem 2. *Let \mathcal{X} be a pair of graphs, and consider the class notation of Definition 2 and the collections described in Definitions 1 and 3. For each of the following choices of a class, every \mathcal{X} -free graph of the class is perfect if and only if \mathcal{X} belongs to the corresponding collection.*

- (1) For each of $\mathcal{G}_5, \mathcal{G}_o, \mathcal{G}_{c,5}$, the collection is \mathcal{P}_1 .
- (2) For each of $\mathcal{G}_\alpha, \mathcal{G}_{o,\alpha}$, it is \mathcal{P}_2 .
- (3) For $\mathcal{G}_{c,\alpha}$, it is \mathcal{P}_{2c} .
- (4) For $\mathcal{G}_{c,o}$, it is \mathcal{P}_3 .
- (5) For $\mathcal{G}_{c,o,\alpha}$, it is \mathcal{P}_4 .

Similarly, every \mathcal{X} -free graph of the class is ω -colourable if and only if \mathcal{X} belongs to the collection as follows.

- (6) For each of $\mathcal{G}_5, \mathcal{G}_o, \mathcal{G}_{c,5}$, it is Ω_1 .
- (7) For each of $\mathcal{G}_\alpha, \mathcal{G}_{o,\alpha}$, it is Ω_2 .
- (8) For $\mathcal{G}_{c,\alpha}$, it is Ω_{2c} .
- (9) For $\mathcal{G}_{c,o}$, it is Ω_3 .
- (10) For $\mathcal{G}_{c,o,\alpha}$, it is Ω_4 .

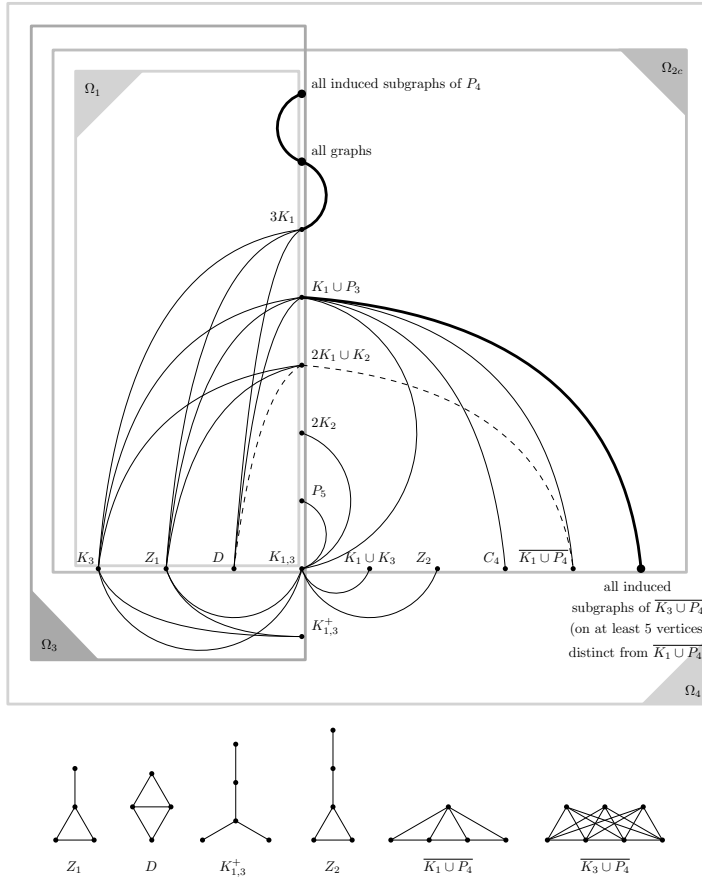


Figure 2: Collections $\Omega_1, \Omega_{2c}, \Omega_3, \Omega_4$ and $\mathcal{P}_1, \mathcal{P}_{2c}, \mathcal{P}_3, \mathcal{P}_4$ of pairs of graphs described in Definition 3 (top part of the figure), and graphs $Z_1, D, K_{1,3}^+, Z_2, \overline{K_1 \cup P_4}$ and $\overline{K_3 \cup P_4}$ (bottom part). In the top part, the vertices represent graphs and the edges represent pairs of graphs (similarly as in Figure 1). The grey rectangles represent the belonging of the pairs to collections $\Omega_1, \Omega_{2c}, \Omega_3, \Omega_4$, and also to collections $\mathcal{P}_1, \mathcal{P}_{2c}, \mathcal{P}_3, \mathcal{P}_4$ when not considering the pairs depicted by dashed edges.

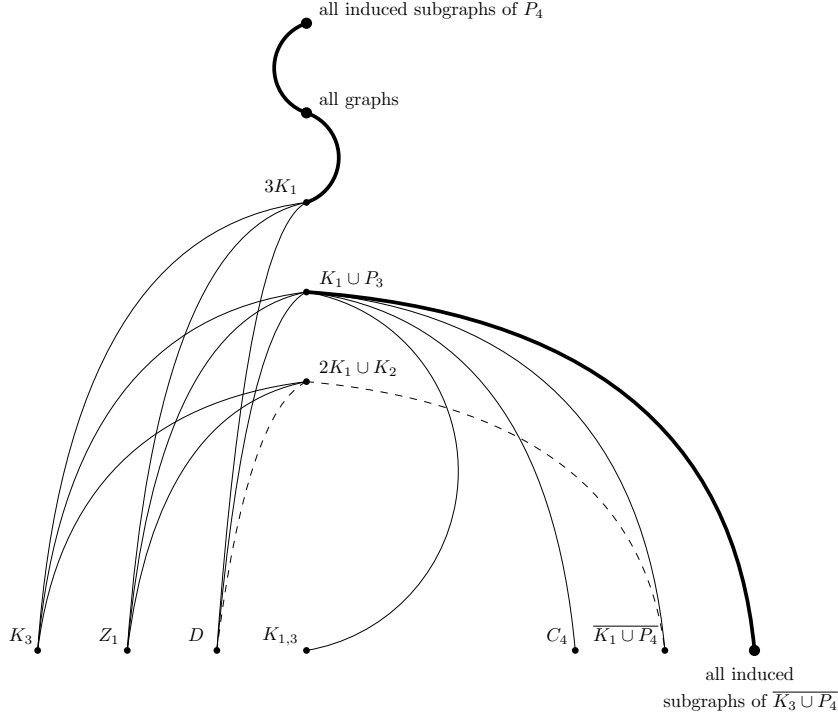


Figure 3: Collections Ω_2 and \mathcal{P}_2 described in Definition 3. The vertices represent graphs and the edges represent pairs of graphs and the dashed edges represent the pairs belonging to Ω_2 but not to \mathcal{P}_2 (similarly as in Figure 1).

In the final definition, we extend the collections as follows.

Definition 4. Consider collections $\mathcal{P}_1, \dots, \mathcal{P}_4$ and \mathcal{P}_{2c} described in Definitions 1 and 3. Let \mathcal{R} be the collection of all pairs of the form $\{kK_1, K_\ell\}$ where $k \geq 4$ and $\ell \geq 3$. Let $\mathcal{A}_{\mathcal{P}}, \mathcal{A}_1, \mathcal{A}_c, \mathcal{A}_3$ and $\mathcal{P}_1^+, \dots, \mathcal{P}_4^+$ and $\mathcal{P}_{1c}^+, \mathcal{P}_{2c}^+$ be the collections of pairs of graphs defined as follows.

- $\mathcal{A}_{\mathcal{P}}$ consists of all pairs of \mathcal{R} and $\{2K_1 \cup K_2, D\}$ and all pairs of the form $\{kK_1 \cup K_2, K_3\}$ where $k \geq 3$.
- \mathcal{A}_1 consists of all pairs $\{3K_1, K_{k+1}\}$ and $\{3K_1, \overline{kK_1 \cup K_2}\}$ where $k \geq 3$ (note that these pairs belong to \mathcal{I} and thus to $\mathcal{P}_2, \mathcal{P}_{2c}, \mathcal{P}_4$ and $\Omega_2, \Omega_{2c}, \Omega_4$).
- \mathcal{A}_c consists of all pairs $\{(k+1)K_1, Z_1\}$ and $\{kK_1 \cup K_2, Z_1\}$ where $k \geq 3$.
- \mathcal{A}_3 consists of $\{K_1 \cup K_{1,3}, K_3\}$ and $\{K_1 \cup K_{1,3}, Z_1\}$.
- \mathcal{P}_1^+ consists of all pairs of \mathcal{P}_1 and $\mathcal{A}_{\mathcal{P}}$ and \mathcal{A}_1 .
- \mathcal{P}_{1c}^+ consists of all pairs of \mathcal{P}_1 and $\mathcal{A}_{\mathcal{P}}$ and \mathcal{A}_1 and \mathcal{A}_c .
- \mathcal{P}_2^+ consists of all pairs of \mathcal{P}_2 and $\mathcal{A}_{\mathcal{P}}$.
- \mathcal{P}_{2c}^+ consists of all pairs of \mathcal{P}_{2c} and $\mathcal{A}_{\mathcal{P}}$ and \mathcal{A}_c .
- \mathcal{P}_3^+ consists of all pairs of \mathcal{P}_3 and $\mathcal{A}_{\mathcal{P}}$ and \mathcal{A}_1 and \mathcal{A}_c and \mathcal{A}_3 .
- \mathcal{P}_4^+ consists of all pairs of \mathcal{P}_4 and $\mathcal{A}_{\mathcal{P}}$ and \mathcal{A}_c and \mathcal{A}_3 .

Similarly, let \mathcal{A}_{Ω} and $\Omega_1^+, \dots, \Omega_4^+$ and $\Omega_{1c}^+, \Omega_{2c}^+$ be the collections of pairs of graphs defined as follows.

- \mathcal{A}_Ω consists of all pairs of $\mathcal{A}_\mathcal{P}$ and \mathcal{A}_c (except for the pair $\{2K_1 \cup K_2, D\}$ which is not necessary since it belongs to Ω_1 already) and all pairs of the form $\{(k+1)K_1, D\}$ and $\{kK_1 \cup K_2, D\}$ where $k \geq 3$ and $\{kK_1, \overline{\ell K_1 \cup K_2}\}$ where $k \geq 4$ and $\ell \geq 3$.
- Ω_1^+ consists of all pairs of Ω_1 and \mathcal{A}_Ω and \mathcal{A}_1 . (In other words, it consists of all pairs of \mathcal{P}_{1c}^+ and all pairs of the form $\{(k+1)K_1, D\}$ and $\{kK_1 \cup K_2, D\}$ and $\{(k+1)K_1, \overline{\ell K_1 \cup K_2}\}$ where $k \geq 3$ and $\ell \geq 3$.)
- Ω_{1c}^+ is the same as Ω_1^+ .
- Ω_2^+ consists of all pairs of Ω_2 and \mathcal{A}_Ω .
- Ω_{2c}^+ consists of all pairs of Ω_{2c} and \mathcal{A}_Ω .
- Ω_3^+ consists of all pairs of Ω_3 and \mathcal{A}_Ω and \mathcal{A}_1 and \mathcal{A}_3 .
- Ω_4^+ consists of all pairs of Ω_4 and \mathcal{A}_Ω and \mathcal{A}_3 .

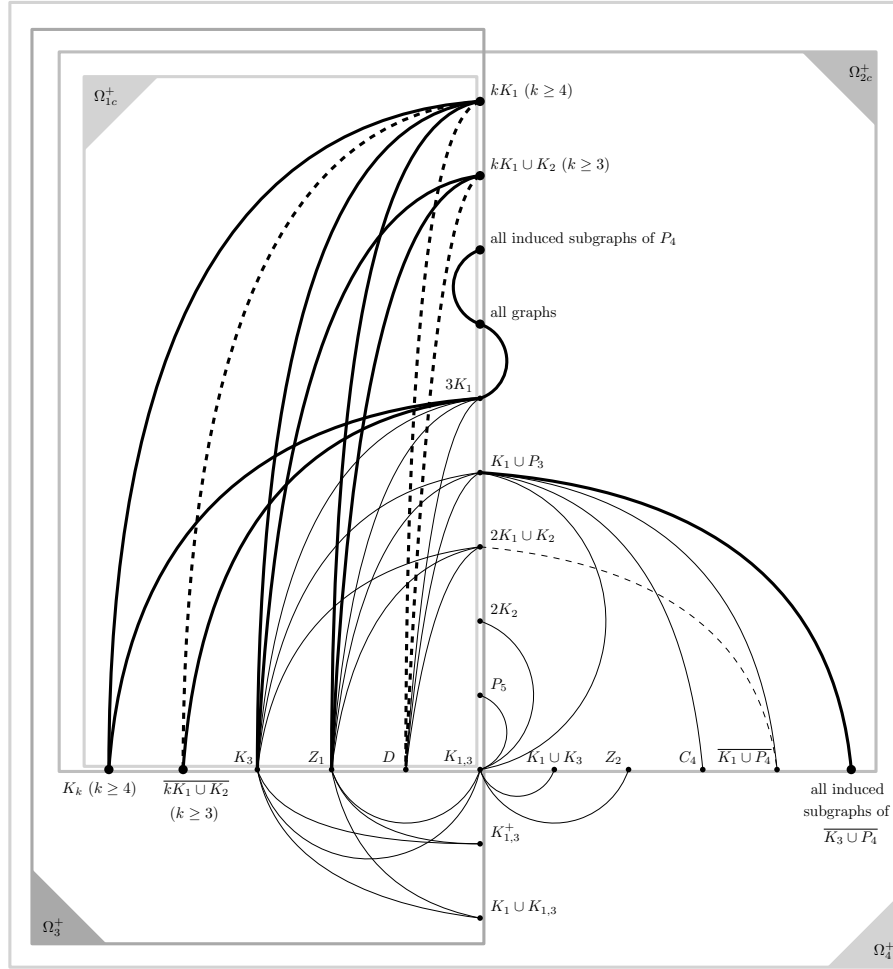


Figure 4: Collections $\Omega_{1c}^+, \Omega_{2c}^+, \Omega_3^+, \Omega_4^+$ and $\mathcal{P}_{1c}^+, \mathcal{P}_{2c}^+, \mathcal{P}_3^+, \mathcal{P}_4^+$ described in Definition 4. The edges represent pairs of graphs and the grey rectangles represent the belonging to collections (similarly as in Figure 2).

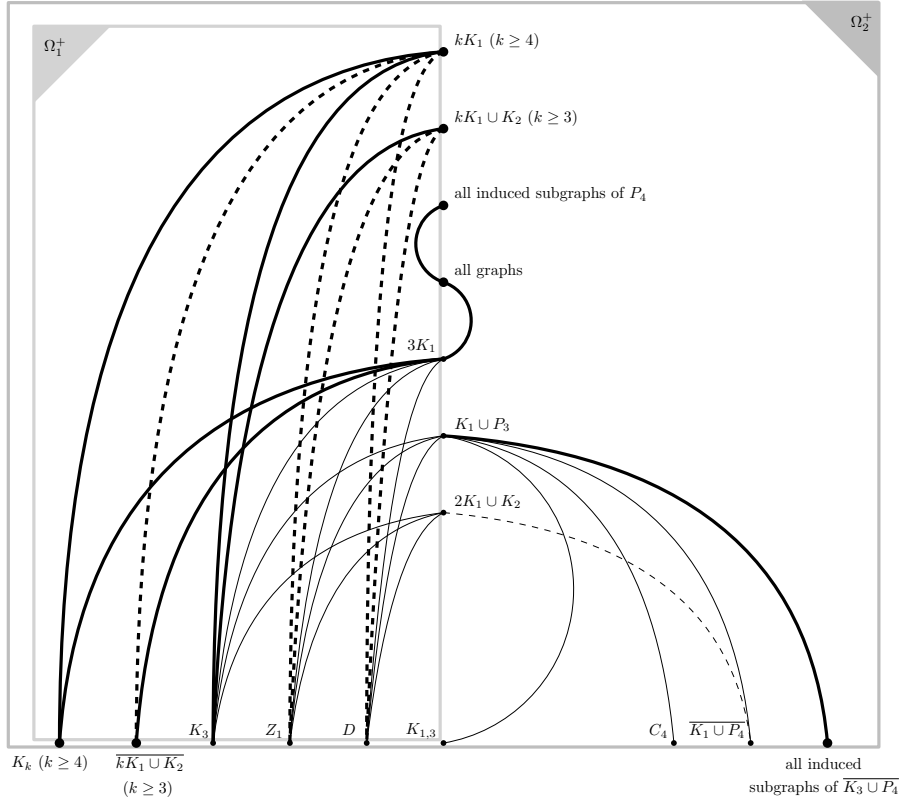


Figure 5: Collections Ω_1^+, Ω_2^+ and $\mathcal{P}_1^+, \mathcal{P}_2^+$ described in Definition 4.

The second of the main results is the following (see also Figures 4, 5 and 6).

Theorem 3. *Let \mathcal{X} be a pair of graphs, and consider the class notation of Definition 2 and the collections described in Definition 4. For each of the following choices of a class, the set of all non-perfect \mathcal{X} -free graphs in the class is finite if and only if \mathcal{X} belongs to the corresponding collection.*

- (1) For each of $\mathcal{G}, \mathcal{G}_o$, the collection is \mathcal{P}_1^+ .
- (2) For \mathcal{G}_c , it is \mathcal{P}_{1c}^+ .
- (3) For each of $\mathcal{G}_\alpha, \mathcal{G}_{o,\alpha}$, it is \mathcal{P}_2^+ .
- (4) For $\mathcal{G}_{c,\alpha}$, it is \mathcal{P}_{2c}^+ .
- (5) For $\mathcal{G}_{c,o}$, it is \mathcal{P}_3^+ .
- (6) For $\mathcal{G}_{c,o,\alpha}$, it is \mathcal{P}_4^+ .

Similarly, the class contains only finitely many \mathcal{X} -free graphs which are not ω -colourable if and only if \mathcal{X} belongs to the collection.

- (7) For each of $\mathcal{G}, \mathcal{G}_o, \mathcal{G}_c$, it is Ω_1^+ .
- (8) For each of $\mathcal{G}_\alpha, \mathcal{G}_{o,\alpha}$, it is Ω_2^+ .
- (9) For $\mathcal{G}_{c,\alpha}$, it is Ω_{2c}^+ .
- (10) For $\mathcal{G}_{c,o}$, it is Ω_3^+ .
- (11) For $\mathcal{G}_{c,o,\alpha}$, it is Ω_4^+ .

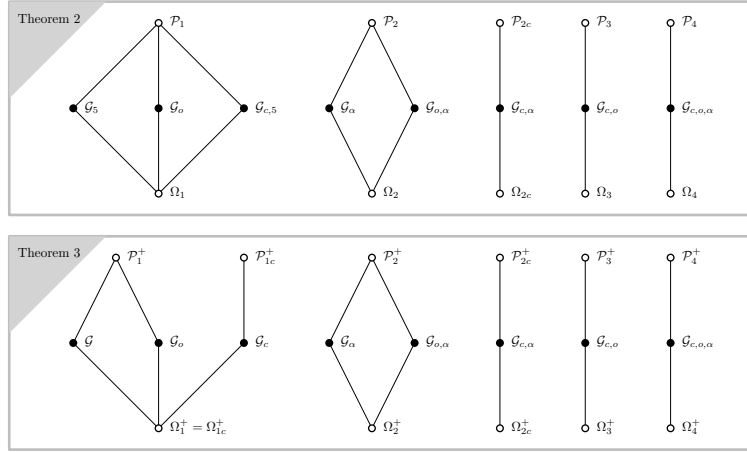


Figure 6: The bipartite graphs depicting the characterisations of Theorems 2 and 3. Each black vertex represents a class of graphs, each white vertex represents a collection of pairs, and each edge represents a characterisation.

4 Additional commentary on the main results

In this short section, we comment on the pairings and the additional constraints, and on the present characterisations in the context of known results.

The basic intuition on the forbidden pairs in Theorem 1 is as follows. As recalled in Section 1, if X or Y is an induced subgraph of P_4 , then all $\{X, Y\}$ -free graphs are perfect. Assuming that $\{X, Y\}$ contains no such graph, we note that if there are only finitely many $\{X, Y\}$ -free graphs which are not perfect (not ω -colourable), then there are only finitely many X -free odd cycles and only finitely many Y -free graphs which are complements of an odd cycle (possibly for X and Y swapped), and furthermore C_5 is $\{X, Y\}$ -free. In particular, we view the exclusion of C_5 as a natural additional constraint for the present study and we consider classes \mathcal{G}_5 and $\mathcal{G}_{c,5}$ in Theorem 2 (rather than \mathcal{G} and \mathcal{G}_c).

Strengthening this additional constraint, we also consider graphs of independence at least 3 and graphs distinct from an odd cycle (the motivation comes from [7, 4]). In this regard, the present study is a direct continuation of the paper of Brause et al. [4] who showed the following result on forbidden pairs (and a similar result considering induced subgraphs of Z_1 and P_4 and no constraint on the independence).

Theorem 4. *Let Y be a graph and \mathcal{G} be the class of all connected $\{K_{1,3}, Y\}$ -free graphs of independence at least 3 distinct from an odd cycle. If Y is an induced subgraph of P_5 or of Z_2 , then all graphs of \mathcal{G} are perfect. Otherwise, there are infinitely many graphs of \mathcal{G} which are not ω -colourable.*

In a sense, the dichotomic nature of Theorem 4 says that ‘the perfect-and-only partners’ for $K_{1,3}$ are P_5 and Z_2 and their induced subgraphs. With Theorems 2

and 3 on hand, we can see that this dichotomy also works the other way around, that is, P_5 and Z_2 (and also $2K_2$ and $K_1 \cup K_3$) are partnered with $K_{1,3}$ (and P_4). Similarly, $K_1 \cup P_3$ is partnered with $\overline{K_3 \cup P_4}$ (and its induced subgraphs), and $\overline{K_3 \cup P_4}$ (and each of its induced subgraphs on at least 5 vertices, and C_4) is partnered with $K_1 \cup P_3$ (some details on induced subgraphs of $\overline{K_3 \cup P_4}$ can be found in Observations 14 and 15).

In addition, we recall that a graph is perfect if and only if its complement is perfect by the result of Lovász [15] (it also follows from the Strong Perfect Graph Theorem). In particular, we note that there is certain symmetry about collection \mathcal{P}_1 . Specifically, a pair $\{X, Y\}$ belongs to \mathcal{P}_1 if and only if $\{\overline{X}, \overline{Y}\}$ does (see Figure 1). Also, there is a similar symmetry about \mathcal{P}_1^+ . Furthermore, we can readily obtain the characterisation of the forbidden pairs in relation to perfectness of graphs of clique number at least 3 by taking the collection of all pairs which are complementary to the pairs of \mathcal{P}_2 (and similarly for \mathcal{P}_2^+). For completeness, we note that the collection \mathcal{I} arises as a formal consequence of the additional constraint, and similarly for collection \mathcal{R} .

As remarked in Section 1, the present conditions on forbidden pairs were studied in relation to χ -boundedness and deciding k -colourability in polynomial time, and for some pairs these properties can also be deduced using the present results. In particular, we note that applying the ‘if part’ of item (10) of Theorem 3 to connected components of a graph yields the following corollary. For every pair \mathcal{X} of Ω_3^+ , there exists a constant c such that every \mathcal{X} -free graph is $(\omega + c)$ -colourable. Similarly, a combination of item (6) of Theorem 3 and the results of Král’ et al. [14] and Grötschel et al. [11] gives the following. For every pair \mathcal{X} of \mathcal{P}_4^+ , the k -colourability of an \mathcal{X} -free graph can be decided in polynomial time. We note that, for each of these corollaries, a more general result is known (see [21] and [10, Theorem 21 (ii)]). We only mention the corollaries to put the present study in context and to note that some forbidden pairs (known from the investigation of χ -boundedness and computational complexity of k -colourability), in fact, imply a stronger property given by the hypotheses and conclusions of Theorems 2 and 3. We conclude the section with a comment on conditions giving that the chromatic number is at most the clique number plus one (results of this type can be found in [21]). Considering connected graphs (of independence at least 3), we note that such conditions follow from the ‘if part’ of items (9), (10) of Theorem 2 and items (10), (11) of Theorem 3.

5 Structure given by forbidden subgraphs

In this section, we investigate structural properties of graphs given by particular forbidden subgraphs, and we assemble several auxiliary statements. In particular, we outline the structure of $\{X, K_3\}$ -free and $\{X, Z_1\}$ -free graphs for several choices of X , and we characterise $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free graphs containing in-

duced C_5 . We start the exposition by recalling the classical result of Ramsey [18].

Theorem 5 (Ramsey's Theorem). *Let k and ℓ be positive integers. There is an integer r such that every graph (on at least r vertices) contains induced kK_1 or K_ℓ .*

We let $R(k, \ell)$ denote the *Ramsey number* corresponding to the pair $\{k, \ell\}$, that is, the smallest integer r satisfying Theorem 5. We also recall the following result of Olariu [16].

Lemma 6 (Olariu's Characterisation). *Every connected Z_1 -free graph is K_3 -free or complete multipartite.*

With the structural similarity of K_3 -free and Z_1 -free graphs on hand (given by Lemma 6), we show several statements.

Observation 7. *Every connected $\{K_{1,3}^+, K_3\}$ -free graph (distinct from an odd cycle) is bipartite.*

Proof. For the sake of a contradiction, we suppose that there is a set of vertices C inducing an odd cycle in G . Clearly, G contains a vertex, say x , not belonging to C but adjacent to a vertex of C (since G is connected and distinct from an odd cycle). Furthermore, two adjacent vertices of C cannot both belong to $N(x)$ (since G is K_3 -free), and therefore at most $\frac{|C|-1}{2}$ vertices belong to $C \cap N(x)$ (since $|C|$ is odd). Consequently, we note that G contains $K_{1,3}^+$ as an induced subgraph, a contradiction. \square

Lemma 8. *Let G be a connected $\{K_1 \cup K_{1,3}, K_3\}$ -free graph. If G is of independence at least 5, then G is either a path or a cycle or a complete bipartite graph. In particular, there is an integer n so that if G has at least n vertices, then it satisfies the condition. For instance, we can choose $n = R(5, 3)$.*

Proof. We consider a maximum independent set, say I , of vertices of G (assuming that $|I| \geq 5$). In addition, we can assume that G is neither a path nor a cycle. In particular, G contains a vertex of degree at least 3 (since G is connected). Clearly, every such vertex is the centre of an induced copy of $K_{1,3}$ (since G is K_3 -free). Furthermore, we observe that G contains such vertex, say x , not belonging to I (since G is $K_1 \cup K_{1,3}$ -free and $|I| \geq 5$). In the remainder of the proof, we shall use the fact that every vertex adjacent to at least three vertices of I is, in fact, adjacent to all vertices of I (since G is $K_1 \cup K_{1,3}$ -free).

We show that x is adjacent to at least three vertices of I (and thus to all). To the contrary, we suppose that x is adjacent to at most two vertices of I . Clearly, x is adjacent to at least one vertex of I (by the choice of x and I). Since G is $K_1 \cup K_{1,3}$ -free, there is a vertex x' adjacent to x and to at least three vertices of I . Thus, x' is adjacent to all vertices of I , which contradicts the fact that G is K_3 -free.

We consider a vertex y not belonging to $I \cup \{x\}$. Clearly, y is adjacent to a vertex of I , and thus y is not adjacent to x (since G is K_3 -free). Consequently, we get that y is adjacent to at least $|I| - 2$ vertices of I (since G is $K_1 \cup K_{1,3}$ -free), and thus y is adjacent to all vertices of I . It follows that G is a complete bipartite graph. \square

Lemma 9. *For every positive integer k , there exists n such that every $\{kK_1 \cup K_2, K_3\}$ -free graph (on at least n vertices) is bipartite. For instance, $n = R(k - 1, 3) + \frac{8k(k^2 - 1)}{2k + 1} + 2k + 2$ will do.*

Proof. We consider n chosen as suggested and a $\{kK_1 \cup K_2, K_3\}$ -free graph G which is not bipartite, and we show that G has at most $n - 1$ vertices. Since G is not bipartite, there is an induced cycle C_ℓ where ℓ is odd. Clearly, this cycle contains $\frac{\ell - 3}{2}K_1 \cup K_2$ as an induced subgraph. Hence, we have $5 \leq \ell \leq 2k + 2$.

We fix a set, say C , of vertices inducing C_ℓ , and let O be the set of all vertices of G which are adjacent to no vertex of C . We note that the graph induced by O is $\{(k - \frac{\ell - 3}{2})K_1, K_3\}$ -free. By Theorem 5, we get

$$|O| \leq R\left(k - \frac{\ell - 3}{2}, 3\right) - 1 \leq R(k - 1, 3) - 1.$$

In addition, we let $N = V(G) \setminus (C \cup O)$, and we show that $|N| \leq \frac{8k(k^2 - 1)}{2k + 1}$. We consider a pair $\{v, e\}$ such that v is a vertex of C and e is an edge of the cycle induced by C and v is not incident with e . For each such pair, we let $N_{v, \bar{e}}$ denote the set of all vertices of N adjacent to v and adjacent to none of the vertices incident with e . We note that $N_{v, \bar{e}}$ is an independent set (since G is K_3 -free), and $|N_{v, \bar{e}}| \leq k - 1$ (since G is $kK_1 \cup K_2$ -free). We observe that there are $\ell(\ell - 2)$ distinct such pairs $\{v, e\}$, and thus $\ell(\ell - 2)$ sets $N_{v, \bar{e}}$. Furthermore, we observe that every vertex of N belongs to at least $\frac{\ell - 1}{2}$ of these sets. Consequently, we get

$$|N| \leq \frac{2\ell(\ell - 2)(k - 1)}{\ell - 1} \leq \frac{8k(k^2 - 1)}{2k + 1}.$$

We conclude that G has $|O| + |N| + \ell$ vertices, and $|O| + |N| + \ell \leq n - 1$. \square

Observation 10. *Let k and ℓ be positive integers and G be a kK_1 -free graph (on at least $(k - 1)(\ell - 1) + 1$ vertices) whose every component is a complete multipartite graph. Then G contains K_ℓ as a subgraph.*

Proof. For the sake of a contradiction, we suppose that G is K_ℓ -free. We let G_1, \dots, G_p denote the graphs given by components of G and $\alpha_1, \dots, \alpha_p$ denote their independence numbers. We note that $\alpha_1 + \dots + \alpha_p \leq k - 1$, and G_i has at most $\alpha_i(\ell - 1)$ vertices for every $i = 1, \dots, p$. Thus, G has at most $(k - 1)(\ell - 1)$ vertices, a contradiction. \square

Corollary 11. *Let k be an integer greater than 2 and let n be an integer given by Lemma 9 (with respect to k). Let G be a $\{kK_1 \cup K_2, Z_1\}$ -free graph, and M be the subgraph of G (possibly empty) given by all components of G which are complete multipartite (possibly trivial), and N be the subgraph given by the remaining components. Then at least one of the following statements is satisfied.*

- (1) N is bipartite (possibly empty).
- (2) N has at most $n - 1$ vertices; and furthermore if M has at least $n(k - 2) - 2k + 5$ vertices, then M contains K_{n-1} as a subgraph.

Proof. We recall that every component of G is K_3 -free or complete multipartite (by Lemma 6), and hence N is K_3 -free. We can assume that N is not bipartite. In particular, N has at most $n - 1$ vertices (by Lemma 9). Furthermore, N contains induced $K_1 \cup K_2$, and so M is $(k - 1)K_1$ -free. We conclude that if M has at least $(k - 2)(n - 2) + 1$ vertices, then it contains K_{n-1} (by Observation 10). \square

In addition, we state the following characterisation (a similar result was shown by Rao [19]). The proof is given at the end of the present section.

Lemma 12. *Let G be a graph containing C_5 as an induced subgraph. Then G is $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free if and only if it can be obtained from some of the graphs G_1, \dots, G_{14} (depicted in Figure 7) by blowing-up vertex i to an independent set and blowing-up vertices c_1, \dots, c_5 to complete graphs. More precisely, the blow-up process is as follows. For vertex i (if present in the graph), add any number of new vertices (possibly none) adjacent precisely to the neighbours of i ; and then similarly for each c_j in sequence (where $1 \leq j \leq 5$), add in sequence any number of new vertices adjacent precisely to c_j and to the neighbours of c_j .*

We also show the following technical statement on particular forbidden subgraphs of a forest (it will be used for forests and for complements of forests).

Lemma 13. *Let F be a forest (on at least 4 vertices) distinct from P_4 and $K_{1,3}$. The following statements are satisfied.*

- (1) *If F is $\{4K_1, 2K_1 \cup K_2, 2K_2\}$ -free, then F is precisely the graph $K_1 \cup P_3$.*
- (2) *If F is $\{2K_1 \cup P_3, 2K_2, K_{1,4}\}$ -free, then F is either $K_1 \cup P_3$ or $K_1 \cup P_4$ or $K_1 \cup K_{1,3}$ or $K_{1,3}^+$ or $(k + 2)K_1$ or $kK_1 \cup K_2$ (for some $k \geq 2$).*

Proof. In order to show statement (1), we observe the following facts.

- The maximum degree of F is at most 2 (since F is a $\{4K_1, 2K_1 \cup K_2\}$ -free forest distinct from $K_{1,3}$).
- F has at most two components, (since it has at least 4 vertices, and it is $\{4K_1, 2K_1 \cup K_2\}$ -free).
- If F has two components, then one is trivial (since it is $2K_2$ -free).

Consequently, we get that all edges of F belong to a common path and that F has at most two components. Thus, F contains induced $K_1 \cup P_3$ (since F has at

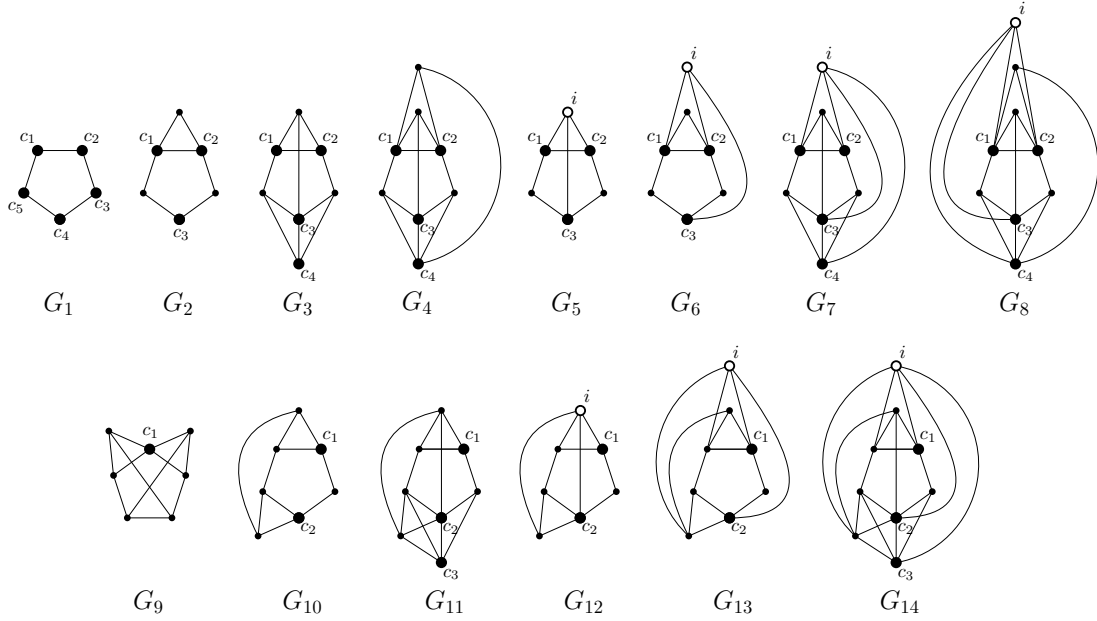


Figure 7: Graphs G_1, \dots, G_{14} whose particular vertices are labelled i and c_1, \dots, c_5 . We distinguish vertex i by depicting it as a large empty circle, and vertices c_1, \dots, c_5 by depicting them as large full circles. The graphs encode possible structure of a non-perfect $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free graph where each empty, full circle represents blowing-up a vertex to an independent set, to a complete graph, respectively.

least 4 vertices, and it is distinct from P_4). Furthermore, F cannot contain more vertices (since it is $2K_1 \cup K_2$ -free).

We show (2). We can assume that F has at least two edges (otherwise F is $(k+2)K_1$ or $kK_1 \cup K_2$ for some $k \geq 2$, and the statement is satisfied). We note that F has precisely one non-trivial component (since it is $2K_2$ -free); and we let T denote the tree given by this component.

In addition, we can assume that F is distinct from $K_1 \cup P_3$ and $K_1 \cup P_4$. We observe that T is not a path (since F has at least 4 vertices and it is distinct from P_4 and it is $\{2K_1 \cup P_3, 2K_2\}$ -free). Hence, T contains $K_{1,3}$, and so F contains induced $K_1 \cup K_{1,3}$ or $K_{1,3}^+$ (since F is distinct from $K_{1,3}$ and it is $K_{1,4}$ -free). We conclude that F is, in fact, $K_1 \cup K_{1,3}$ or $K_{1,3}^+$ (since it is $\{2K_1 \cup P_3, 2K_2, K_{1,4}\}$ -free). \square

We also state the following two facts on induced subgraphs of $\overline{K_3 \cup P_4}$, and we give short proofs. (This could also be shown simply by checking all graphs on at most 8 vertices with the help of a computer.)

Observation 14. *There are precisely five distinct induced subgraphs of $\overline{K_3 \cup P_4}$ on 4 vertices, and precisely five such subgraphs on 5 vertices. Namely, $P_4, K_{1,3}, Z_1, D, C_4$ and $\overline{K_1 \cup P_4}, K_{1,2,2}, \overline{K_2 \cup P_3}, K_{1,1,3}, K_{2,3}$.*

Proof. We discuss graphs obtained from $\overline{K_3 \cup P_4}$ by removing 3 vertices. We consider the maximum independent set of $\overline{K_3 \cup P_4}$ and let i be the number of vertices of this set which are being removed. For the case that $i = 3$, we note that the resulting graph is P_4 . For $i = 2$, we get Z_1 or D , and for $i = 1$, we get D or C_4 . Finally $i = 0$, gives $K_{1,3}$. Similarly for removing 2 vertices, we discuss the cases and get the subgraphs on 5 vertices. \square

Observation 15. *A graph is an induced subgraph of $\overline{K_3 \cup P_4}$ if and only if it is $\{4K_1, 2K_1 \cup K_2, K_1 \cup P_3, 2K_2, K_1 \cup K_3, K_4, C_5, \overline{P_5}, K_{3,3}, K_{2,2,2}\}$ -free.*

Proof. We note that $\overline{K_3 \cup P_4}$ satisfies the property, and hence it is satisfied by each of its induced subgraphs.

We consider a $\{4K_1, 2K_1 \cup K_2, K_1 \cup P_3, 2K_2, K_1 \cup K_3, K_4, C_5, \overline{P_5}, K_{3,3}, K_{2,2,2}\}$ -free graph G , and we discuss two cases and show that it is an induced subgraph of $\overline{K_3 \cup P_4}$.

First, we suppose that G contains an independent set, say I , of size 3. We note that every vertex of $V(G) \setminus I$ is adjacent to all vertices of I (since G is $\{4K_1, 2K_1 \cup K_2, K_1 \cup P_3\}$ -free), and that the graph $G - I$ is $\{K_3, C_4, C_5, 3K_1, 2K_2\}$ -free (since G is $\{K_4, K_{2,2,2}, C_5, K_{3,3}, 2K_2\}$ -free). In particular, $G - I$ is a $\{3K_1, 2K_2\}$ -free forest, and thus an induced subgraph of P_4 and the statement follows.

Next, we suppose that G is $3K_1$ -free, and we consider the complement of G . Similarly, we note that \overline{G} is a forest (since G is $\{3K_1, 2K_2, C_5, \overline{P_5}\}$ -free). Furthermore, \overline{G} is $\{4K_1, 3K_2, K_{1,3}, P_5\}$ -free (since G is $\{K_4, K_{2,2,2}, K_1 \cup K_3, \overline{P_5}\}$ -free). We observe that \overline{G} is an induced subgraph of $K_2 \cup P_4$, and the statement follows. \square

We conclude this section by proving Lemma 12.

Proof of Lemma 12. We consider the graphs obtained by the construction. The ‘if part’ of the statement follows by observing that they are $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free.

We show the ‘only if part’ of the statement. We consider a set of vertices, say C , inducing C_5 . For every vertex u of $V(G) \setminus C$, we get that $2 \leq |N(u) \cap C| \leq 3$ (since G is $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free); and we say that u is *blue* if $N(u) \cap C$ induces P_3 , and u is *red* if $N(u) \cap C$ induces K_2 or $K_1 \cup K_2$ (see Figure 8).

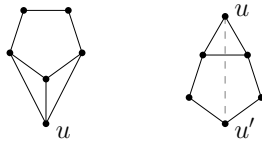


Figure 8: Possible graphs induced by $C \cup \{u\}$. Edge uu' (depicted as dashed grey) may or may not be present in the graph; and so the picture on the right-hand side, in fact, represents two graphs. Depending on $N(u) \cap C$, vertex u is blue (left-hand side) or red (right-hand side).

In addition, we consider a set, say A , consisting of two vertices of $V(G) \setminus C$, and we shall discuss the graph induced by $C \cup A$. We let $X = C \cup A$, and let r be the number of red vertices in A , and H_1, \dots, H_8 be the graphs depicted in Figure 9 (considered with the dashed grey edges). We show three claims.

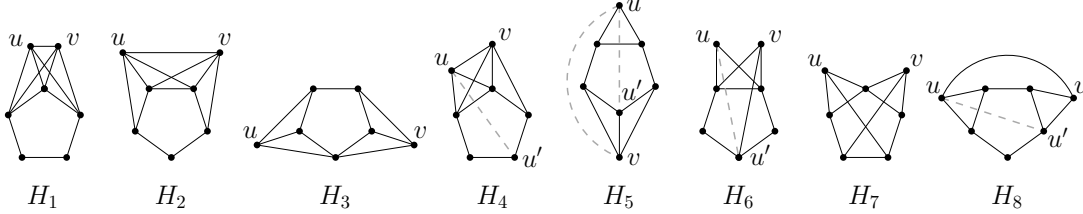


Figure 9: Graphs H_1, \dots, H_8 . The vertices of A are labelled u and v , and one particular vertex of C is labelled u' in graphs H_4, H_5, H_6 and H_8 . We distinguish edge uv of H_5 and all edges uu' by depicting them as dashed grey.

Claim 12.1. *If $r = 0$, then X induces either H_1 or H_2 or H_3 .*

Proof of Claim 12.1. For the sake of a contradiction, we suppose that X induces none of H_1, H_2, H_3 . Considering the number of vertices of C which are adjacent to both vertices of A , we discuss three cases, see Figure 10 (first row), and we observe that the graph induced by X is not $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free, a contradiction. ■

Claim 12.2. *If $r = 1$, then the graph induced by X can be obtained from either H_4 or H_5 by removing some of the dashed grey edges (possibly none).*

Proof of Claim 12.2. For the sake of a contradiction, we suppose that the graph induced by X violates the claim. We discuss all cases, see Figure 10 (second row), and we observe that the graph is not $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free, a contradiction. ■

We use the similarity of the adjacencies of blue vertices and vertices of C given by Claims 12.1 and 12.2, and we consider a set, say C' , of five non-red vertices inducing C_5 (so that C' and A are disjoint). We let $X' = C' \cup A$ and consider the graph induced by X' .

Claim 12.3. *If $r = 2$, then the graph induced by X' is either H_7 or it can be obtained from H_6 or H_8 by removing some of the dashed grey edges (possibly none). Furthermore, if X' induces H_7 , then G has precisely two red vertices.*

Proof of Claim 12.3. Similarly as above, we suppose that the graph induced by X' violates the first statement of the claim, and we observe that it is not $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free, see Figure 10 (third row).

The second statement of the claim is also shown by contradiction. We suppose that X' induces H_7 and that there is another red vertex, say w , and we discuss

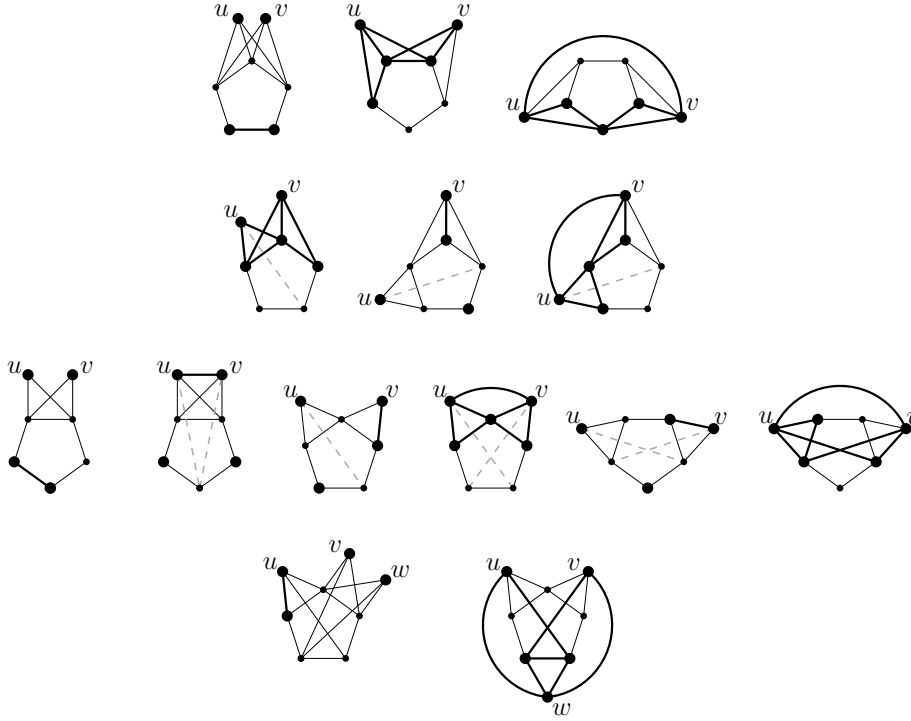


Figure 10: Particular adjacencies among the vertices of A (labelled u and v) and C , and among A and C' (third row), and among A and C' and vertex w (fourth row). The dashed grey edges indicate that the vertices may or may not be adjacent; for instance, the first picture in the second row represents two graphs (both containing induced $\overline{K_1 \cup P_4}$). In each picture, induced $2K_1 \cup K_2$ or $\overline{K_1 \cup P_4}$ is highlighted (the vertices and edges are depicted as bold).

the graph induced by $X' \cup \{w\}$. We apply the first statement of the claim to all 2-element subsets of $A \cup \{w\}$ and observe how this reduces the number of cases. We discuss the remaining cases, see Figure 10 (fourth row), and we conclude that the graph induced by $X' \cup \{w\}$ is not $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free, a contradiction. ■

In addition, we consider an edge e whose both ends belong to C and the set, say R_e , of all red vertices adjacent to both ends of e ; and we choose e so that $|R_e|$ is maximised. Using Claim 12.3, we observe that at most one red vertex does not belong to R_e . We let o be the vertex of C which is adjacent to no end of e , and we let O be the set consisting of vertex o and all vertices added for o by the blow-up process. We show the following.

Claim 12.4. *At most two vertices of R_e have the property that they are not adjacent to all vertices of O . Furthermore, if there are two such vertices, then every red vertex belongs to R_e and every vertex of O is adjacent to precisely one of the two.*

Proof of Claim 12.4. We note that the first statement of the claim follows from

the second statement. Hence, it is sufficient to show the second statement, and proceed by contradiction. We consider two distinct vertices u, v of R_e , and two vertices u', v' of O such that u is not adjacent to u' and v is not adjacent to v' (and note that u' and v' are distinct by Claim 12.3). For the sake of a contradiction, we suppose that there is a vertex x such that either x is red and not belonging to R_e , or x belongs to O and it is not true that x is adjacent to precisely one of u, v (and thus it is adjacent to both by Claim 12.3). Using Claims 12.1, 12.2 and 12.3, we note that $\{u, v, u', v'\}$ induces P_4 and that x is adjacent to u, v, u' and v' , a contradiction (see Figure 11). ■

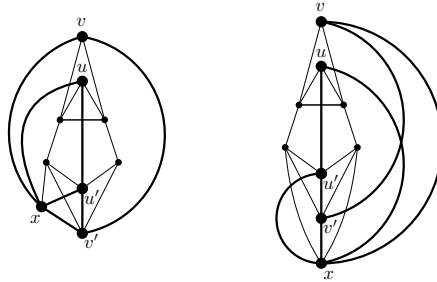


Figure 11: Graphs induced by $C \cup \{u, v, u', v', x\}$. In both pictures, induced $\overline{K_1 \cup P_4}$ is highlighted.

We apply Claims 12.1, 12.2 and 12.3 to all 2-element subsets of $V(G) \setminus C$, and we discuss what we know about the structure of G . We recall that Claim 12.1 gives the adjacencies between all non-red vertices. In particular, if G has no red vertex, then it can be obtained from G_1 by the blow-up process (described in Lemma 12); and so we can assume that G has a red vertex. We note that Claim 12.2 restricts the relative positions of red and blue vertices (viewed from C). Finally, Claim 12.3 gives the adjacencies between all red vertices, and also some of the adjacencies between red and non-red vertices. It remains to discuss the adjacencies corresponding to dashed grey edges of graphs H_6 and H_8 , see Figure 9.

Since at most one red vertex does not belong to R_e , most of the remaining adjacencies are given by Claim 12.4. We discuss two cases. For the case that every red vertex belongs to R_e , we observe that G can be obtained from some of G_2, \dots, G_8 by the blow-up process. Otherwise, we similarly conclude that G can be obtained from some of G_9, \dots, G_{14} . □

6 Showing perfectness

In the present section, we collect sufficient conditions for a graph to be perfect and we show the following.

Proposition 16. *Let G be a graph. If some of the following conditions is satisfied, then G is perfect.*

- (1) G is a $\{2K_1 \cup K_2, D\}$ -free graph distinct from C_5 and from the graphs E_1, \dots, E_4 depicted in Figure 12.
- (2) G is a $\{K_1 \cup P_3, \overline{K_3 \cup P_4}\}$ -free graph of independence at least 3.
- (3) G is a connected $\{K_{1,3}^+, Z_1\}$ -free graph distinct from an odd cycle.
- (4) G is a connected $\{K_1 \cup K_{1,3}, Z_1\}$ -free graph distinct from an odd cycle and G has at least n vertices where n is given by Lemma 8.
- (5) G is a connected $\{kK_1 \cup K_2, Z_1\}$ -free graph (where $k \geq 3$) on at least n vertices where n is given by Lemma 9.
- (6) G is a $\{kK_1 \cup K_2, K_3\}$ -free graph (where $k \geq 3$) on at least n vertices where n is given by Lemma 9.
- (7) G is a $\{3K_1, \overline{kK_1 \cup K_2}\}$ -free graph (where $k \geq 3$) on at least n vertices where n is given by Lemma 9.
- (8) G is a $\{2K_1 \cup K_2, Z_1\}$ -free graph distinct from C_5 .
- (9) G is a $\{K_1 \cup P_3, Z_1\}$ -free graph distinct from C_5 .
- (10) G is a $\{K_1 \cup P_3, D\}$ -free graph distinct from C_5 .

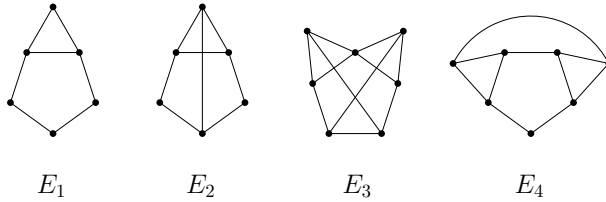


Figure 12: Graphs E_1, \dots, E_4 . We note that E_1 is, in fact, the complement of E_2 , and E_3 is the complement of E_4 . (The graphs are also included in Figure 7. Namely, E_1, E_2, E_3, E_4 is isomorphic to G_2, G_5, G_9, G_{10} , respectively.)

For conditions (3), (4) \dots , (10) we argue using the structural statements shown in Section 5, and for (1) and (2) using the following result shown in [6].

Theorem 17 (Strong Perfect Graph Theorem). *A graph is perfect if and only if neither the graph nor its complement contain an induced cycle whose length is odd and at least 5.*

Proof of Proposition 16. We let \mathcal{C} denote the family of all cycles whose length is odd and at least 7, and $\overline{\mathcal{C}}$ denote the family of all graphs whose complement belongs to \mathcal{C} . We show that each of the conditions (1), (2), \dots , (10) implies that G is perfect.

First, we suppose that condition (1) is satisfied. For the sake of a contradiction, we suppose that G is not perfect. By Theorem 17, G contains a graph from $\{C_5\} \cup \mathcal{C} \cup \overline{\mathcal{C}}$ as an induced subgraph. Furthermore, we note that G is $\mathcal{C} \cup \overline{\mathcal{C}}$ -free (since no graph of \mathcal{C} is $2K_1 \cup K_2$ -free and no graph of $\overline{\mathcal{C}}$ is D -free), and hence, G

contains C_5 as an induced subgraph. (Now, the statement can be deduced using Lemma 12. For the sake of clarity, we give a short proof not using the lemma.) We consider a set C of vertices inducing C_5 in G . We note that every vertex of G is adjacent to a vertex of C (since G is $2K_1 \cup K_2$ -free), and that G contains a vertex not belonging to C (since G is distinct from C_5). Considering such vertex x , we observe that the graph induced by $C \cup \{x\}$ is either E_1 or E_2 (since G is $\{2K_1 \cup K_2, D\}$ -free).

In addition, we consider two vertices, say x and x' , not belonging to C , and we discuss the graph induced by $C \cup \{x, x'\}$ (see Figure 13). We observe that $C \cup \{x, x'\}$ induces either E_3 or E_4 . Clearly, this reasoning applies to every pair

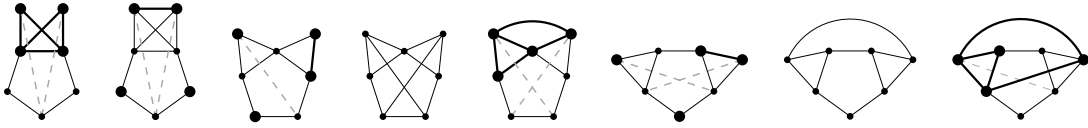


Figure 13: Possible ways of connecting x and x' to C (the edges depicted as dashed grey may or may not be present in the graph). Some of the resulting graphs contain induced $2K_1 \cup K_2$ or induced D (highlighted in the picture by depicting the vertices and edges as bold).

of vertices not belonging to C , and thus G has at most 7 vertices. We conclude that G is one of the graphs E_1, \dots, E_4 , a contradiction.

We suppose that (2) is satisfied. For the sake of a contradiction, we suppose that G is not perfect, that is, contains a graph from $\{C_5\} \cup \mathcal{C} \cup \bar{\mathcal{C}}$ as an induced subgraph (by Theorem 17). Clearly, G is \mathcal{C} -free (since it is $K_1 \cup P_3$ -free). Hence, G contains a graph from $\{C_5\} \cup \bar{\mathcal{C}}$ as an induced subgraph; and let A be a set of vertices inducing such subgraph. We note that every vertex of G is adjacent to a vertex of A (since G is $K_1 \cup P_3$ -free).

In addition, we consider a set I of three independent vertices of G , and a pair N of non-adjacent vertices of A . For every vertex x of $I \setminus N$, we observe that x is adjacent to at least one vertex of N (otherwise, there would be a pair N' of non-adjacent vertices of $A \setminus N(x)$ so that there is a vertex of $N(x)$ which is adjacent to precisely one vertex of N' , contradicting the assumption that G is $K_1 \cup P_3$ -free). Consequently, we note that no vertex of N belongs to I (since G is $K_1 \cup P_3$ -free). In particular, a vertex of N is adjacent to at least two vertices of I , and hence to all vertices of I (since G is $K_1 \cup P_3$ -free). It follows that the other vertex of N is also adjacent to all vertices of I . We recall that the choice of N was arbitrary, and thus every vertex of A is, in fact, adjacent to all vertices of I . We consider a subset, say P , of A inducing P_4 , and we conclude that $I \cup P$ induces $\bar{K}_3 \cup P_4$, a contradiction.

We suppose that (3) or (4) or (5) is satisfied. In either case, G is K_3 -free or complete multipartite by Lemma 6. Furthermore if G is K_3 -free, then we get that it is bipartite (we apply Observation 7 or Lemma 8 or Lemma 9, respectively). Clearly, every bipartite or complete multipartite graph is perfect.

Similarly, for the case that (6) is satisfied, we get that G is bipartite (by Lemma 8), and thus perfect.

We suppose that (7) is satisfied. We note that the complement of G is $\{kK_1 \cup K_2, K_3\}$ -free, and so it satisfies condition (6). We recall that a graph is perfect if and only if its complement is perfect (by [15] or by Theorem 17), and the perfectness of G follows.

We suppose that (8) is satisfied. If G is connected, then we note that condition (3) is satisfied and the statement follows. Otherwise, we get that every component of G is $K_1 \cup K_2$ -free. In particular, G is P_4 -free, and we recall that this implies perfectness (for instance, by [23] or by Theorem 17). A similar argument applies for the case that condition (8) is satisfied.

Finally, if (9) is satisfied, then we note that the complement of G satisfies (8), and the statement follows. \square

7 Showing ω -colourability

Considering sufficient conditions for ω -colourability, we show the following.

Proposition 18. *Every $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free graph of independence at least 3 is ω -colourable.*

Proposition 19. *Let k, ℓ and n be integers so that $k \geq 3$ and $\ell \geq 2$, and G be a graph on at least n vertices. If some of the following conditions is satisfied, then G is ω -colourable.*

- (1) G is $\{kK_1 \cup K_2, Z_1\}$ -free and n is sufficiently large. For instance, we can choose $n = (k - 1) \left(R(k - 1, 3) + \frac{8k(k^2 - 1)}{2k + 1} + 2k \right) + 2$.
- (2) G is $\{kK_1 \cup K_2, D\}$ -free and n is sufficiently large. For instance, $n = R(2k, R(k, k) + k)$ will do.
- (3) G is $\{(k + 1)K_1, \ell K_1 \cup K_2\}$ -free and n is sufficiently large. For instance, $n = R(k + 1, R(k, k(\ell - 1)) + k(\ell - 1))$ will do.

The proofs of Propositions 18 and 19 are given below. For Proposition 18, the proof follows easily by using Theorem 17 and the ‘only if part’ of Lemma 12 (a similar reasoning and deeper results can be found in the paper of Karthick and Maffray [13]). For each item of Proposition 19, we examine the structure of a major part of a graph and colour it, and we show that the remaining part is small and the colouring extends. For condition (1), the statement follows from Corollary 11. For conditions (2) and (3), we repeatedly apply Theorem 5. The core of the proof is to show the following lemma on a slightly more general class of graphs.

Lemma 20. *Let $k \geq 3$ and $\ell \geq 2$ and G be a $\{kK_1 \cup K_2, \overline{\ell K_1 \cup K_2}\}$ -free graph and ω be its clique number. If ω is sufficiently large, then G is ω -colourable. For instance, $\omega \geq R(k, k(\ell - 1)) + k(\ell - 1)$ will do.*

We start by showing Proposition 18.

Proof of Proposition 18. We let G be a $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free graph of independence at least 3. Clearly, we can assume that G is not perfect. Combining Theorem 17 and the fact that G is $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free, we note that G contains C_5 as an induced subgraph; and we consider a set, say C , of vertices inducing it.

We recall the blow-up process described in Lemma 12 and note that G can be obtained from some of the graphs G_2, \dots, G_{14} (since G is of independence at least 3). To reduce the number of cases, we observe that G can be extended by adding edges so that the resulting graph can, in fact, be obtained from G_5, G_9 or G_{12} (recalling the notation used in the proof of Lemma 12, we add edges so that every vertex of R_e is adjacent to all vertices of O for a particular choice of e). We let G^+ denote the resulting extended graph, and we observe that $\omega(G) = \omega(G^+)$. Hence, it is sufficient to find an ω -colouring of G^+ .

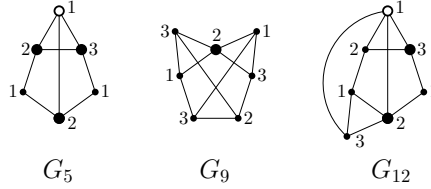


Figure 14: Possible structure of graph G^+ given by graphs G_5, G_9, G_{12} , and particular 3-colourings of these graph.

To this end, we consider a set of vertices of G^+ inducing G_5, G_9 or G_{12} (a subgraph giving the structure of G). We colour the vertices as indicated in Figure 14, and we extend this to a proper colouring of G^+ as follows. We recall that vertex i is blown-up to an independent set and each of vertices c_1, c_2 and c_3 is blown-up to a complete graph, and we refer to the new vertices as clones of i, c_1, c_2 and c_3 , respectively. We colour the clones of i using colour 1, and the clones of c_1 and c_2 using colours $4, \dots, \omega$, and the clones of c_3 (for the case of graph G_5) using colours $3, \dots, \omega$. We conclude that this yields an ω -colouring of G^+ . \square

We show Lemma 20.

Proof of Lemma 20. We let $m = k(\ell - 1)$ and we suppose that $\omega \geq R(k, m) + m$. We consider a set of vertices, say V_1 , inducing a maximum complete subgraph of G (clearly, $|V_1| = \omega$), and we check whether the graph $G - V_1$ contains K_m as a subgraph. If it does, then we consider a set of vertices, say V_2 , inducing a maximum complete subgraph in $G - V_1$ and we proceed by checking the graph $G - V_1 - V_2$ for K_m . We continue this process until we obtain sets V_1, \dots, V_p so that the graph $G - V_1 - \dots - V_p$ is K_m -free (and $|V_p| \geq m$).

We show two claims.

Claim 20.1. *Let i be an integer satisfying $1 \leq i \leq p$. Then every vertex of $V(G) \setminus (V_1 \cup \dots \cup V_i)$ is adjacent to at most $\ell - 1$ vertices of V_i . Furthermore, we get that $p \leq k$ and that the graph $G - V_1 - \dots - V_p$ is $(k - p + 1)K_1$ -free.*

Proof of Claim 20.1. We consider a vertex v of $V(G) \setminus (V_1 \cup \dots \cup V_i)$, and we note that at least one vertex of V_i is not adjacent to v (by the maximality of V_i). Thus, v is adjacent to at most $\ell - 1$ vertices of V_i (since G is $\overline{\ell K_1 \cup K_2}$ -free).

We prove the second part of the claim by contradiction. To this end, we consider a maximum independent set (possibly empty), say I , of $G - V_1 - \dots - V_p$ and suppose that $p > k$ or $|I| > k - p$. We will repeatedly use the first statement of the claim. First, we show that there is an independent set of size k in $G - V_1$ (this is clearly satisfied if $|I| \geq k$). We suppose that $|I| < k$, and we let $j = k - |I|$. In particular, we have $j < p$ (since $p > k$ or $|I| > k - p$). We note that I can be extended to a larger independent set by adding (in sequence) one vertex from each of V_{j+1}, V_j, \dots, V_2 (since $j < p$), and the resulting independent set is of size k . Now, we use that there is an independent set of size k in $G - V_1$, and we note that it can be extended by adding two vertices of V_1 so that the resulting set induces $kK_1 \cup K_2$, a contradiction. ■

Claim 20.2. *Let i be an integer satisfying $2 \leq i \leq p$. If X is a subset of $V_1 \cup \dots \cup V_{i-1}$ such that each vertex of X has at least ℓ neighbours in V_i , then $X \cup V_i$ induces a complete graph. In particular, we have $\omega - |X| \geq |V_i|$.*

Proof of Claim 20.2. Since G is $\overline{\ell K_1 \cup K_2}$ -free, we get that each vertex of X is adjacent to all vertices of V_i , and consequently that all vertices of X are adjacent. Thus, $X \cup V_i$ induces a complete graph and the inequality follows. ■

Finally, we show that G is ω -colourable. We start by colouring the subgraph induced by $V_1 \cup \dots \cup V_p$. Clearly, the complete subgraph given by V_1 is ω -colourable. We suppose that there is an ω -colouring of the subgraph induced by $V_1 \cup \dots \cup V_{i-1}$, and we show that it can be extended to $V_1 \cup \dots \cup V_i$. To this end, we construct an auxiliary bipartite graph (A, B) such that the vertices of A encode the vertices of V_i and the vertices of B encode the colours, and two vertices are adjacent if and only if the corresponding member of V_i has no neighbour in $V_1 \cup \dots \cup V_{i-1}$ coloured by the corresponding colour. We use Claims 20.1 and 20.2 and show that set A satisfies Hall's condition (that is, $|N(S)| \geq |S|$ for every subset S of A). Claim 20.1 implies that every vertex of V_i has at most $\ell - 1$ neighbours in each of V_1, \dots, V_{i-1} , and hence every vertex of A has degree at least $\omega - (i - 1)(\ell - 1)$. In particular, we get $|N(S)| \geq \omega - (i - 1)(\ell - 1)$ for every subset S of A , and thus $|N(S)| \geq (i - 1)(\ell - 1)$ (since $\omega \geq 2k(\ell - 1)$ and $k \geq i$). On the other hand, every vertex of $B \setminus N(S)$ encodes a colour such that each vertex of V_i belonging to S is adjacent to a vertex of $V_1 \cup \dots \cup V_{i-1}$ coloured by this colour. In particular if $|S| > (i - 1)(\ell - 1)$, then one of these vertices has at least ℓ neighbours in V_i (since there are at most $i - 1$ vertices of

the same colour in $V_1 \cup \dots \cup V_{i-1}$). We note that the inequality from Claim 20.2 translates to $|N(S)| \geq |A|$ (since $|A| = |V_i|$ and $|B| = \omega$). Thus, we can apply Hall's theorem and obtain a matching covering A , and we note that the matching translates back to the desired extension of the colouring.

Hence, there is an ω -colouring of the subgraph induced by $V_1 \cup \dots \cup V_p$, and we extend it to G as follows. We recall that the graph $G - V_1 - \dots - V_p$ is $\{kK_1, K_m\}$ -free, and so it has fewer than $R(k, m)$ vertices (by Theorem 5). By Claim 20.1, each of these vertices has at most $p(\ell - 1)$ neighbours in $V_1 \cup \dots \cup V_p$, and thus they can be readily coloured (since $\omega \geq R(k, m) + m \geq R(k, m) + p(\ell - 1)$). \square

Finally, we show Proposition 19.

Proof of Proposition 19. We suppose that condition (1) is satisfied, and we choose n as suggested. We apply Corollary 11 and we view G as a disjoint union of graphs M and N (defined in Corollary 11). Clearly, M is ω -colourable, and we get that N is bipartite or small. For the latter case, we note that N has at most $n' - 1$ vertices where

$$n' = R(k - 1, 3) + \frac{8k(k^2 - 1)}{2k + 1} + 2k + 2,$$

and so M has at least $n'(k - 2) - 2k + 5$ vertices (since $n = n'(k - 1) - 2k + 4$), and hence M contains a complete subgraph of order $n' - 1$. We conclude that the disjoint union of M and N is ω -colourable.

We suppose that (2) is satisfied. We choose n as suggested, and we consider a $\{kK_1 \cup K_2, D\}$ -free graph G on at least n vertices. By Theorem 5, G contains an independent set of size $2k$ or a complete subgraph of order $R(k, k) + k$, and we discuss these two cases.

First, we consider a maximum independent set, say I , and suppose that $|I| \geq 2k$. Clearly, every vertex of $V(G) \setminus I$ is adjacent to a vertex of I (by definition), and so it is adjacent to more than $|I| - k$ vertices of I (since G is $kK_1 \cup K_2$ -free). Consequently, if u and v are vertices of $V(G) \setminus I$, then u and v have at least two common neighbours in I ; and hence u and v are non-adjacent (since G is D -free). It follows that G is bipartite, and thus ω -colourable.

Next, we suppose that $\omega \geq R(k, k) + k$. We note that graph D can be viewed as $\ell\overline{K_1} \cup \overline{K_2}$ for $\ell = 2$, and the ω -colourability follows by Lemma 20.

Lastly, we suppose that (3) is satisfied. By the choice of n and by Theorem 5, we get $\omega \geq R(k, k(\ell - 1)) + k(\ell - 1)$. Thus, the statement follows by Lemma 20. \square

8 Discussing all remaining pairs

In order to prove the ‘only if part’ of Theorems 2 and 3, we exclude all possible remaining pairs of forbidden subgraphs. To this end, we consider families $\mathcal{F}_1, \dots, \mathcal{F}_{13}$ of graphs depicted in Figures 15, 17 and 18. In addition, we use

Lemma 13 and Observations 14 and 15, and Theorem 4 which resolves the pairs containing $K_{1,3}$. As the main result of this section, we show the following.

Proposition 21. *Let \mathcal{X} be a pair of graphs, and consider the class notation of Definition 2 and the collections described in Definition 4. For each of the following choices of a class, if \mathcal{X} does not belong to the corresponding collection, then the class contains infinitely many \mathcal{X} -free graphs which are not ω -colourable.*

- (1) For $\mathcal{G}_{c,o,\alpha}$, the collection is Ω_4^+ .
- (2) For $\mathcal{G}_{c,o}$, it is Ω_3^+ .
- (3) For $\mathcal{G}_{c,\alpha}$, it is Ω_{2c}^+ .
- (4) For $\mathcal{G}_{o,\alpha}$, it is Ω_2^+ .
- (5) For each of $\mathcal{G}_c, \mathcal{G}_o$, it is Ω_1^+ .

The proof of Proposition 21 is given below. In addition, we show the following two observations.

Observation 22. *Each of the following conditions is satisfied by infinitely many non-perfect graphs of independence 3 which are distinct from an odd cycle.*

- (1) The graphs are $\{4K_1, Z_1\}$ -free.
- (2) The graphs are connected and $\{4K_1, D\}$ -free.
- (3) The graphs are connected and $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$ -free.

Proof. We consider the graphs belonging to $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_3 (depicted in Figure 15), and we note that the statement is satisfied subject to condition (1), (2) and (3), respectively. \square

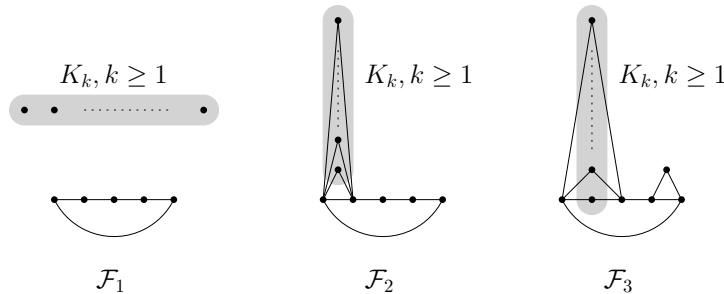


Figure 15: Families $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_3 of graphs. The grey ovals depict complete subgraphs K_k , and every choice of a positive integer k gives a graph belonging to the family. The graphs of $\mathcal{F}_1, \mathcal{F}_2$ and \mathcal{F}_3 are non-perfect and of independence 3.

Observation 23. *Each of the following conditions is satisfied by some connected graph which is not ω -colourable and distinct from an odd cycle.*

- (1) The graph is $\{3K_1, K_4\}$ -free.
- (2) The graph is $\{4K_1, K_3\}$ -free and of independence 3.

Proof. For instance, we consider graphs F_6 and F_{10} depicted in Figure 16, and we note that they have the desired properties and F_6 satisfies condition (1) and F_{10} satisfies (2). \square

We remark that there are precisely 13, 14 graphs satisfying Observation 23 subject to condition (1), (2), respectively.



Figure 16: Graphs F_6 and F_{10} . (In fact, they are the smallest members of families \mathcal{F}_6 and \mathcal{F}_{10} , respectively.)

In the remainder of the section, we show Proposition 21.

Proof of Proposition 21. We consider families $\mathcal{F}_4, \dots, \mathcal{F}_{13}$ of graphs (depicted in Figures 17 and 18), and we observe that no graph of $\mathcal{F}_4, \dots, \mathcal{F}_{13}$ is ω -colourable. We let X and Y denote the graphs of \mathcal{X} , and we show statements (1) ... (5).

We show statement (1), that is, if \mathcal{X} does not belong to Ω_4^+ , then $\mathcal{G}_{c,o,\alpha}$ contains infinitely many \mathcal{X} -free graphs which are not ω -colourable. For the sake of clarity and efficiency of the proof, we show several claims.

Claim 21.1. *No member of \mathcal{X} is $3K_1$ or an induced subgraph of P_4 , and we can assume that no member is $K_{1,3}$.*

Proof of Claim 21.1. The first part of the claim follows from the assumption that \mathcal{X} does not belong to Ω_4^+ . For the case that $K_{1,3}$ belongs to \mathcal{X} , we recall that the other member of \mathcal{X} is not an induced subgraph of P_5 or of Z_2 (since \mathcal{X} does not belong to Ω_4^+), and so the statement of the proposition follows by Theorem 4. Thus, we can assume that $K_{1,3}$ does not belong to \mathcal{X} . \blacksquare

Claim 21.2. *We can assume that X is a forest and Y is not. Furthermore, we can assume that X is $K_{1,4}$ -free.*

Proof of Claim 21.2. We suppose that either both members of \mathcal{X} are forests or none of them is, and we show that the statement of the proposition is satisfied. For the case that both are forests, we note that each member of \mathcal{X} has at least four vertices and is distinct from P_4 and $K_{1,3}$ (by Claim 21.1). Hence, each member of \mathcal{X} contains some of $4K_1, 2K_1 \cup K_2, K_1 \cup P_3, 2K_2$ as an induced subgraph (by item (1) of Lemma 13). We consider family \mathcal{F}_4 and note that it consists of $\{4K_1, 2K_1 \cup K_2, K_1 \cup P_3, 2K_2\}$ -free graphs. Thus, every graph of \mathcal{F}_4 is \mathcal{X} -free, and the statement is satisfied. In the latter case, we have that both members of \mathcal{X} contain a cycle, and we consider family \mathcal{F}_5 and observe that it contains infinitely many \mathcal{X} -free graphs. Consequently, we can assume that X is a forest and Y is not. The ‘furthermore part’ of the claim follows similarly by considering \mathcal{F}_5 . \blacksquare

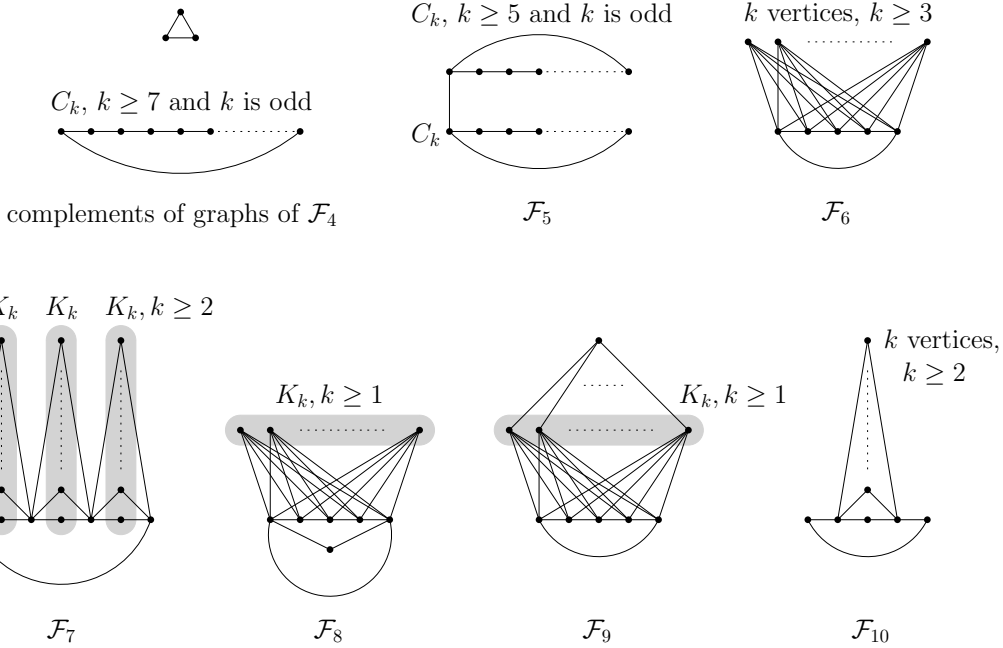


Figure 17: Family \mathcal{F}_4 of graphs (given by complements) and families $\mathcal{F}_5, \dots, \mathcal{F}_{10}$. Similarly to Figure 15, each grey oval depicts a complete subgraph. The graphs of $\mathcal{F}_4, \dots, \mathcal{F}_{10}$ are connected and of independence at least 3 and not ω -colourable.

Claim 21.3. *In addition, we can assume that X is either $K_1 \cup P_3$ or contains some of $4K_1, 2K_1 \cup K_2, 2K_2$ as an induced subgraph, and that Y is $\{4K_1, 2K_1 \cup K_2, K_1 \cup P_3, 2K_2, K_1 \cup K_3, C_5\}$ -free.*

Proof of Claim 21.3. The first part of the claim follows by Claims 21.1 and 21.2 and item (1) of Lemma 13. For the case that Y contains some of $4K_1, 2K_1 \cup K_2, K_1 \cup P_3, 2K_2, K_1 \cup K_3, C_5$ as an induced subgraph, we consider family \mathcal{F}_4 and observe that the statement is satisfied. \blacksquare

Claim 21.4. *In addition, we can assume that X is distinct from $K_1 \cup K_{1,3}$ and $K_{1,3}^+$ and that Y is K_4 -free.*

Proof of Claim 21.4. We suppose that X is either $K_1 \cup K_{1,3}$ or $K_{1,3}^+$. We recall that Y is not a forest (by Claim 21.2) and that Y is $K_1 \cup K_3$ -free (by Claim 21.3) and distinct from K_3 and from Z_1 (since X is $K_1 \cup K_{1,3}$ or $K_{1,3}^+$ and \mathcal{X} does not belong to Ω_4^+). Thus, Y is not an induced subgraph of P_5 or of Z_2 , and the statement follows by Theorem 4.

Next, we suppose that Y contains K_4 as a subgraph. If X contains some of $2K_1 \cup K_2, K_1 \cup P_3, 2K_2$ as an induced subgraph, then we consider family \mathcal{F}_6 and note that the statement is satisfied. Hence, we can assume that X is $\{2K_1 \cup K_2, K_1 \cup P_3, 2K_2\}$ -free, and so it contains induced $4K_1$ (by Claim 21.3). Furthermore, X is a forest and it is $K_{1,4}$ -free (by Claim 21.2). Consequently, we

get that X contains no edge. In other words, X is kK_1 (for some $k \geq 4$). Thus, Y has at least two non-edges (since \mathcal{X} does not belong to Ω_4^+). Furthermore, we recall that Y is $K_1 \cup K_3$ -free (by Claim 21.3). We consider the graphs of \mathcal{F}_7 and observe that they are \mathcal{X} -free. ■

Claim 21.5. *In addition, we can assume that X is distinct from $K_1 \cup P_3$ and that Y is C_4 -free and that Y contains K_3 as a subgraph.*

Proof of Claim 21.5. We suppose that X is $K_1 \cup P_3$. In particular Y is not an induced subgraph of $\overline{K_3 \cup P_4}$ (since \mathcal{X} does not belong to Ω_4^+). We recall that Y is $\{4K_1, 2K_1 \cup K_2, K_1 \cup P_3, 2K_2, K_1 \cup K_3, K_4, C_5\}$ -free (by Claims 21.3 and 21.4). Consequently, Y contains some of $\overline{P_5}, K_{3,3}, K_{2,2,2}$ as an induced subgraph (by Observation 15), and we consider family \mathcal{F}_6 . Thus, we can assume that X is distinct from $K_1 \cup P_3$. In particular, X contains some of $4K_1, 2K_1 \cup K_2, 2K_2$ as an induced subgraph (by Claim 21.3), and we use this for showing the second part of the claim.

We suppose that Y contains induced C_4 , and we note that the statement is satisfied by discussing two cases. For the case that X contains induced $2K_1 \cup K_2$, we consider family \mathcal{F}_8 . Otherwise, we consider family \mathcal{F}_9 .

Hence, we can assume that Y is C_4 -free. We recall that Y is $\{2K_2, C_5\}$ -free (by Claim 21.3), and so it is, in fact, $\{C_4, C_5, C_6, \dots\}$ -free. Furthermore, Y is not a forest (by Claim 21.2), and thus it contains K_3 . ■

Claim 21.6. *In addition, we can assume that X is $(k+2)K_1$ or $kK_1 \cup K_2$ (for some $k \geq 2$) and that Y is $3K_1$ -free.*

Proof of Claim 21.6. We recall that Y contains K_3 (by Claim 21.5). If X contains some of $2K_1 \cup P_3, K_1 \cup P_4, 2K_2$ as an induced subgraph, then we consider family \mathcal{F}_{10} . Hence, we can assume that X is $\{2K_1 \cup P_3, K_1 \cup P_4, 2K_2\}$ -free. Furthermore, we recall that X is $K_{1,4}$ -free (by Claim 21.2) and that X is distinct from $K_1 \cup P_3, K_1 \cup K_{1,3}$ and $K_{1,3}^+$ (by Claims 21.4 and 21.5). Consequently, we get that X is $(k+2)K_1$ or $kK_1 \cup K_2$ for some $k \geq 2$ (by item (2) of Lemma 13).

We suppose that Y contains induced $3K_1$. We note that Y contains at least one additional vertex (by Claim 21.1), and hence it contains induced $K_{1,3}$ (since Y is $\{4K_1, 2K_1 \cup K_2, K_1 \cup P_3\}$ -free by Claim 21.3). We conclude that the statement follows by Theorem 4. ■

Claim 21.7. *Consequently, we get that Y is $\overline{K_1 \cup P_4}$.*

Proof of Claim 21.7. We recall that Y is $\{3K_1, 2K_2, C_4, C_5\}$ -free (by Claims 21.3, 21.5 and 21.6), and we consider the complement of Y . In particular, we get that \overline{Y} is a forest. Furthermore, \overline{Y} is $\{4K_1, 2K_2, K_{1,3}\}$ -free (since Y is $\{K_1 \cup K_3, K_4, C_4\}$ -free by Claims 21.3, 21.4 and 21.5). Also, \overline{Y} is distinct from P_4 (since Y is distinct from P_4 by Claim 21.1) and distinct from $3K_1$ and $K_1 \cup P_3$ (since Y is distinct

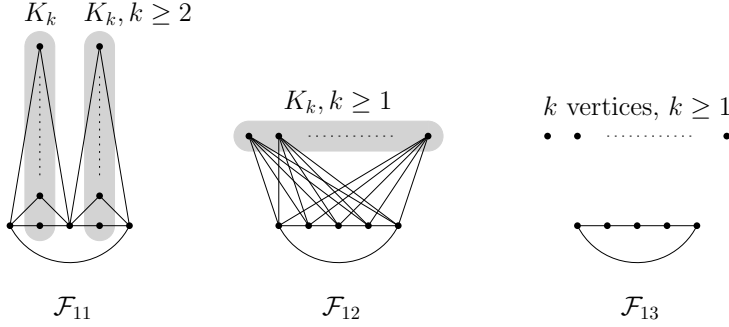


Figure 18: Families \mathcal{F}_{11} , \mathcal{F}_{12} and \mathcal{F}_{13} of graphs. (The grey ovals depict complete subgraphs.) The graphs are not ω -colourable.

from K_3 and Z_1 by the assumptions on \mathcal{X}). We note that we can apply item (2) of Lemma 13 and conclude that \bar{Y} is $K_1 \cup P_4$. \blacksquare

In particular, X is distinct from $2K_1 \cup K_2$ (since \mathcal{X} does not belong to Ω_4^+). Thus, X contains induced $4K_1$ (by Claim 21.6). The desired statement follows by considering family \mathcal{F}_7 .

Next, we show (2). We note that if \mathcal{X} does not belong to Ω_4^+ , then the statement follows from (1). Hence, we can assume that \mathcal{X} belongs to Ω_4^+ but not to Ω_3^+ , that is, we assume that at least one of the following is satisfied.

- X is $2K_1 \cup K_2$ and Y is $\overline{K_1 \cup P_4}$.
- X is $K_{1,3}$ and Y is either $K_1 \cup P_3$ or $2K_2$ or P_5 or $K_1 \cup K_3$ or Z_2 .
- X is $K_1 \cup P_3$ and Y is an induced subgraph of $\overline{K_3 \cup P_4}$ such that Y is not an induced subgraph of P_4 and is distinct from K_3 and Z_1 and D .
- X is $3K_1$ and Y is not an induced subgraph of P_4 and is distinct from K_k (for every $k \geq 3$) and from $\overline{\ell K_1 \cup K_2}$ (for every $\ell \geq 2$) and from Z_1 .

We note that X contains $3K_1$ as an induced subgraph, and we discuss Y in more detail. We observe that if the statement of the third item is satisfied, then Y contains some of $C_4, \overline{K_1 \cup P_4}, K_{1,3}$ as an induced subgraph (by Observation 14). Furthermore, we show that if the statement of the last item is satisfied, then Y contains some of $3K_1, 2K_2, K_1 \cup K_3, C_4, C_5, \overline{K_1 \cup P_4}, \overline{2K_1 \cup P_3}$ as an induced subgraph. For the sake of a contradiction, we suppose that Y is $\{3K_1, 2K_2, K_1 \cup K_3, C_4, C_5, \overline{K_1 \cup P_4}, \overline{2K_1 \cup P_3}\}$ -free. In other words, \bar{Y} is $\{K_3, C_4, K_{1,3}, 2K_2, C_5, K_1 \cup P_4, 2K_1 \cup P_3\}$ -free, and so it is a $\{K_{1,3}, 2K_2, K_1 \cup P_4, 2K_1 \cup P_3\}$ -free forest. Furthermore, \bar{Y} is not an induced subgraph of P_4 and it is distinct from kK_1 (for every $k \geq 3$) and from $\ell K_1 \cup K_2$ (for every $\ell \geq 2$) and from $K_1 \cup P_3$. A contradiction follows by item (2) of Lemma 13.

We recall that X contains induced $3K_1$, and we discuss three cases for Y . For the case that Y contains induced $\overline{K_1 \cup P_4}$ or $\overline{2K_1 \cup P_3}$, we consider family \mathcal{F}_{11} and observe that the statement is satisfied. If Y contains C_5 as an induced subgraph, then we consider the family of all graphs whose complement is an odd cycle of length at least 7. Otherwise, we conclude that Y contains some of

$3K_1, 2K_2, K_1 \cup K_3, C_4$ as an induced subgraph and consider family \mathcal{F}_{12} .

We show (3). Similarly as above, we can assume that \mathcal{X} belongs to Ω_4^+ but not to Ω_{2c}^+ . Hence, X contains $K_{1,3}$ as an induced subgraph and Y contains K_3 , and the desired statement follows by considering the family of all odd cycles of length at least 7.

We show (4). We can assume that \mathcal{X} belongs to Ω_4^+ but not to Ω_2^+ , and hence X contains induced $K_{1,3}$ and Y contains induced $2K_2$ or K_3 , and we consider family \mathcal{F}_{13} .

Lastly, we show the two statements given by (5). We note that for \mathcal{G}_c , the statement follows from (2) and (3) (since if \mathcal{X} does not belong to Ω_1^+ , then it does not belong to Ω_{2c}^+ or Ω_3^+). Similarly, for \mathcal{G}_o the statement follows from (2) and (4). \square

9 Proving the main results

Finally, we put together the statements shown in Sections 6, 7 and 8 and Theorem 4, and we prove Theorems 2 and 3.

Proof of Theorem 2. For the sake of efficiency, we shall first show the ‘if part’ subject to items (1), (2), \dots , (10), and then the ‘only if part’ subject to items (10), (9), \dots , (1).

For the ‘if part’ of item (1), we need to show that if \mathcal{X} belongs to \mathcal{P}_1 , then every \mathcal{X} -free graph of \mathcal{G}_5 is perfect. We recall that every P_4 -free graph is perfect (for instance, by Theorem 17). Hence, we can consider a graph of \mathcal{G}_5 and assume that it is $\{2K_1 \cup K_2, Z_1\}$ -free or $\{K_1 \cup P_3, Z_1\}$ -free or $\{K_1 \cup P_3, D\}$ -free, and the perfectness follows by item (8), (9), (10) of Proposition 16, respectively.

For the ‘if part’ of item (2), we need to show that if \mathcal{X} belongs to \mathcal{P}_2 , then every \mathcal{X} -free graph of \mathcal{G}_α is perfect. We note that if \mathcal{X} belongs to \mathcal{P}_1 , then the statement follows by the ‘if part’ of (1) shown above. Also, if \mathcal{X} contains $3K_1$, then the considered restricted class is empty and the statement is satisfied trivially. Hence, we can assume that \mathcal{X} belongs to \mathcal{P}_2 but neither to \mathcal{P}_1 nor to \mathcal{I} . In particular, one member of \mathcal{X} is $K_1 \cup P_3$ and the other is an induced subgraph of $\overline{K_3} \cup \overline{P_4}$, and the statement follows by item (2) of Proposition 16.

We show the ‘if part’ of (3). Similarly as above, we can assume that \mathcal{X} belongs to \mathcal{P}_{2c} but not to \mathcal{P}_2 . Hence, \mathcal{X} is either $\{K_{1,3}, 2K_2\}$ or $\{K_{1,3}, P_5\}$. We note that every connected \mathcal{X} -free graph of independence at least 3 is, in fact, distinct from an odd cycle, and the perfectness follows by Theorem 4.

For the ‘if part’ of (4), we can assume that \mathcal{X} belongs to \mathcal{P}_3 but not to \mathcal{P}_1 . Thus, \mathcal{X} is either $\{K_{1,3}^+, K_3\}$ or $\{K_{1,3}^+, Z_1\}$ or $\{K_{1,3}, K_3\}$ or $\{K_{1,3}, Z_1\}$, and the perfectness follows by item (3) of Proposition 16.

Similarly considering the ‘if part’ of (5), we can assume that \mathcal{X} is either $\{K_{1,3}, K_1 \cup K_3\}$ or $\{K_{1,3}, Z_2\}$, and the statement follows by Theorem 4.

Next, we show the ‘if part’ of items (6), \dots , (10). We note that for most pairs \mathcal{X} , the ω -colourability follows by the ‘if part’ of items (1), \dots , (5). In particular considering (6), we can assume that \mathcal{X} is $\{2K_1 \cup K_2, D\}$. We note that the statement follows by item (1) of Proposition 16 (since each of the graphs E_1, \dots, E_4 depicted in Figure 12 is ω -colourable). The ‘if part’ of (9) follows by the same argument. Consequently for (7), (8) and (10), we can assume that \mathcal{X} is $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$. We conclude that the statement follows by Proposition 18.

In order to show the ‘only if part’ of (10), we need to show that if \mathcal{X} does not belong to Ω_4 , then $\mathcal{G}_{c,o,\alpha}$ contains an \mathcal{X} -free graph which are not ω -colourable. We note that for every pair \mathcal{X} not belonging to Ω_4^+ , the statement follows by item (1) of Proposition 21. Hence, we can assume that \mathcal{X} is either $\{kK_1, K_\ell\}$ (for some $k \geq 4$ and $\ell \geq 3$), or $\{kK_1 \cup K_2, K_3\}$ or $\{(k+1)K_1, Z_1\}$ or $\{kK_1 \cup K_2, Z_1\}$ or $\{(k+1)K_1, D\}$ or $\{kK_1 \cup K_2, D\}$ (for some $k \geq 3$), or $\{kK_1, \overline{\ell K_1 \cup K_2}\}$ (for some $k \geq 4$ and $\ell \geq 3$), or $\{K_1 \cup K_{1,3}, K_3\}$ or $\{K_1 \cup K_{1,3}, Z_1\}$. In all cases, the statement follows by item (2) of Observation 23.

Consequently for the ‘only if part’ of (9), we can assume that \mathcal{X} is none of the aforementioned pairs, and so \mathcal{X} belongs to Ω_4 . In addition, we can assume that \mathcal{X} belongs to Ω_3^+ (by item (2) of Proposition 21). Hence, \mathcal{X} is either $\{3K_1, K_{k+1}\}$ or $\{3K_1, \overline{kK_1 \cup K_2}\}$ (for some $k \geq 3$) and the statement follows by item (1) of Observation 23.

Similarly for (8), (7) and (6), the statement follows by using item (3), (4) and (5) of Proposition 21, respectively.

Regarding the ‘only if part’ of (5), \dots , (1), we note that for most pairs \mathcal{X} , the existence of a desired non-perfect graph follows by the ‘only if part’ of items (10), \dots , (6). In particular, we can assume that \mathcal{X} is either $\{2K_1 \cup K_2, D\}$ or $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$. We consider graphs E_1, \dots, E_4 depicted in Figure 12, and we note that they have the desired properties. \square

We show Theorem 3.

Proof of Theorem 3. We show the ‘if part’ of the statement subject to items (1), (2), \dots , (11), and then the ‘only if part’ subject to items (11), (10), \dots , (1).

In order to show the ‘if part’ of (1), we need to show that if \mathcal{X} belongs to \mathcal{P}_1^+ , then the set of all non-perfect \mathcal{X} -free graphs is finite. We recall that if \mathcal{X} belongs to \mathcal{P}_1 , then all \mathcal{X} -free graphs (except for C_5) are perfect by item (1) of Theorem 2. Also, if \mathcal{X} belongs to \mathcal{R} or \mathcal{X} is $\{3K_1, K_k\}$ (for some $k \geq 4$), then there are only finitely many \mathcal{X} -free graphs by Theorem 5, and so the statement is satisfied trivially. Hence, we can assume that \mathcal{X} is either $\{2K_1 \cup K_2, D\}$ or $\{kK_1 \cup K_2, K_3\}$ or $\{3K_1, \overline{kK_1 \cup K_2}\}$ (for some $k \geq 3$), and the statement follows by items (1), (6) and (7) of Proposition 16, respectively.

For the ‘if part’ of (2), we can assume that \mathcal{X} belongs to \mathcal{P}_{1c}^+ but not to \mathcal{P}_1^+ . Hence, \mathcal{X} is either $\{(k+1)K_1, Z_1\}$ or $\{kK_1 \cup K_2, Z_1\}$ (for some $k \geq 3$), and the statement follows by item (5) of Proposition 16.

For the ‘if part’ of (3) and (4), we observe that the statement follows similarly as for (1) and (2).

Consequently for (5) and (6), we can assume that \mathcal{X} is either $\{K_1 \cup K_{1,3}, K_3\}$ or $\{K_1 \cup K_{1,3}, Z_1\}$. The statement follows by item (4) of Proposition 16.

For the ‘if part’ of (7), we need to show that if \mathcal{X} belongs to Ω_1^+ , then the set of all \mathcal{X} -free graphs which are not ω -colourable is finite. We can assume that \mathcal{X} does not belong to \mathcal{P}_1^+ , and hence \mathcal{X} is either $\{(k+1)K_1, Z_1\}$ or $\{kK_1 \cup K_2, Z_1\}$ or $\{(k+1)K_1, D\}$ or $\{kK_1 \cup K_2, D\}$ (for some $k \geq 3$), or $\{kK_1, \overline{\ell K_1 \cup K_2}\}$ (for some $k \geq 4$ and $\ell \geq 3$), and the statement follows by Proposition 19.

For (8), \dots , (11), we note that the statement follows similarly (in addition, using item (7) of Theorem 2 for the pair $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$).

Regarding the ‘only if part’ of (11), \dots , (7), we note that the statement follows by item (1), \dots , (5) of Proposition 21, respectively.

We show the ‘only if part’ of (6). We can assume that \mathcal{X} belongs to Ω_4^+ (otherwise, the statement is satisfied by the ‘only if part’ of (11) discussed above). Hence, we can assume that \mathcal{X} is either $\{(k+1)K_1, D\}$ or $\{kK_1 \cup K_2, D\}$ (for some $k \geq 3$), or $\{kK_1, \overline{\ell K_1 \cup K_2}\}$ (for some $k \geq 4$ and $\ell \geq 3$), or $\{2K_1 \cup K_2, \overline{K_1 \cup P_4}\}$, and we note that the statement follows by items (2) and (3) of Observation 22.

We conclude that for items (5), \dots , (1), the statement follows similarly (using items (1), (2) and (3) of Observation 22). \square

Acknowledgements

We thank the anonymous referee for their helpful comments. The work of the first author was partially supported by DMS-EPSRC grant DMS-2120644. The work of the second and fourth author was supported by projects 17-04611S and 20-09525S of the Czech Science Foundation. The second author was also supported by the MUNI Award in Science and Humanities of the Grant Agency of Masaryk University. The work of the third author was supported by the National Natural Science Foundation of China (Nos. 12171393, 12071370) and the Natural Science Basic Research Program of Shaanxi (Nos. 2021JM-040, 2020JQ-099).

References

- [1] C. Berge: Färbung von Graphen deren sämtliche bzw. deren ungerade Kreise starr sind (Zusammenfassung), *Wissenschaftliche Zeitschrift, Martin Luther Universität HalleWittenberg, Mathematisch-Naturwissenschaftliche Reihe* (1961), 114–115.
- [2] M. Bonamy and M. Pilipczuk: Graphs of bounded cliquewidth are polynomially χ -bounded, *Advances in Combinatorics* (2020).
- [3] J. A. Bondy and U.S.R. Murty: *Graph Theory*, Springer, 2008.

- [4] C. Brause, P. Holub, A. Kabela, Z. Ryjáček, I. Schiermeyer and P. Vrána: On forbidden induced subgraphs for $K_{1,3}$ -free perfect graphs, *Discrete Mathematics* 342 (2019), 1602–1608.
- [5] C. Brause, B. Randerath, I. Schiermeyer and E. Vumar: On the chromatic number of $2K_2$ -free graphs, *Discrete Applied Mathematics* 253 (2019), 14–24.
- [6] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas: The strong perfect graph theorem, *Annals of Mathematics* 164 (2006), 51–229.
- [7] M. Chudnovsky and P. Seymour: Claw-free graphs VI. Colouring, *Journal of Combinatorial Theory, Series B* 100 (2010), 560–572.
- [8] P. Erdős: Graph theory and probability, *Canadian Journal of Mathematics* 11 (1959), 34–38.
- [9] A. Gyárfás: Problems from the world surrounding perfect graphs, *Applicationes Mathematicae* 19 (1987), 413–441.
- [10] P. A. Golovach, M. Johnson, D. Paulusma and J. Song: A Survey on the computational complexity of coloring graphs with forbidden subgraphs, *Journal of Graph Theory* 84 (2017), 331–363.
- [11] M. Grötschel, L. Lovász and A. Schrijver: The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* 1 (1981), 169–197.
- [12] S. Hougardy: Classes of perfect graphs, *Discrete Mathematics* 306 (2006), 2529–2571.
- [13] T. Karthick, F. Maffray: Coloring (gem, co-gem)-free graphs, *Journal of Graph Theory* 89 (2018), 288–303.
- [14] D. Král', J. Kratochvíl, Zs. Tuza and G. Woeginger: Complexity of coloring graphs without forbidden induced subgraphs, *Proceedings 27th International Workshop on Graph-Theoretic Concepts in Computer Science, Lecture Notes in Computer Science* 2204 (2001), 254–262.
- [15] L. Lovász: A characterization of perfect graphs, *Journal of Combinatorial Theory, Series B* 13 (1972), 95–98.
- [16] S. Olariu: Paw-free graphs, *Information Processing Letters* 28 (1988), 53–54.
- [17] J. L. Ramírez-Alfonsín and B. A. Reed: *Perfect Graphs*, Wiley, 2001.
- [18] F. P. Ramsey: On a problem of formal logic, *Proceedings of the London Mathematical Society* 30 (1930), 264–286.
- [19] M. Rao: Decomposition of (gem, co-gem)-free graphs. Unpublished manuscript, Laboratoire d'Informatique, de Robotique et de Microélectronique de Montpellier, France, 2007.

- [20] F. Roussel, I. Rusu and H. Thuillier: The strong perfect graph conjecture: 40 years of attempts, and its resolution, *Discrete Mathematics* 309 (2009), 6092–6113.
- [21] I. Schiermeyer and B. Randerath: Polynomial χ -binding functions and forbidden induced subgraphs: a survey, *Graphs and Combinatorics* 35 (2019), 1–31.
- [22] A. Scott and P. Seymour: A survey of χ -boundedness, *Journal of Graph Theory* 95 (2020), 473–504.
- [23] D. Seinsche: On a property of the class of n -colorable graphs, *Journal of Combinatorial Theory, Series B* 16 (1974), 191–193.
- [24] D. P. Sumner: Subtrees of a graph and chromatic number, *Theory and Applications of Graphs* (1981), 557–576.