University of West Bohemia
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The Fučík curves for problems with integral type boundary conditions

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Prohlášení

Prohlašuji, že jsem tuto diplomovou práci vypracoval pod vedením vedoucího diplomové práce samostatně za použití v práci uvedených pramenů a literatury.

V Plzni, dne 28. července 2022

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podpis

Poděkování

Děkuji mému vedoucímu, panu Ing. Petru Nečesalovi, Ph.D., za dohled nad prací a cenné rady.

Abstrakt

V této práci vyšetřujeme okrajovou úlohu skládající se z diferenciální rovnice druhého řádu, Sturmovy-Liouvilleovy podmínky a podmínky integrálního typu. Popíšeme vlastní čísla příslušné lineární úlohy. Pro okrajovou úlohu představíme implicitní popis Fučíkova spektra v prvním kvadrantu a na základě tohoto popisu sestrojíme parametrizaci spektra pro speciální hodnoty parametrů.

Klíčová slova: Fučíkovo spektrum, nelokální okrajové podmínky, podmínky integrálního typu, Sturmova-Liouvilleova podmínka, vlastní čísla

Abstract

In this thesis we investigate the boundary value problem consisting of a second order differencial equation, Sturm-Liouville condition and integral type condition. We are going to describe eigenvalues of corresponding linear problem. We introduce an implicit description of the Fučík spectrum in the first quadrant for the boundary value problem and based on this description we construct a parametrization of the spectrum for special values of parameters.

Keywords: Fučík spectrum, nonlocal boundary condition, integral type condition, Sturm-Liouville conditon, eigenvalues

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Chapter 1

Introduction

In this thesis, we are going to investigate the boundary value problem with one Sturm-Liouville type condition and one non-local boundary type condition

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,1), \\ u(0) \cdot \sin c = u'(0) \cdot \cos c, & \int_0^1 u(x)\, \mathrm{d}x = \gamma \cdot u'(0), \end{cases} \tag{1.1}$$

where $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\gamma \in \mathbb{R}$. Our goal is to find the Fučík spectrum of (1.1), by which we mean the set

$$\Sigma_c^\gamma := \Big\{(\alpha,\beta) \in \mathbb{R} \times \mathbb{R} : \text{ the problem (1.1) has a non-trivial solution } u\Big\}.$$

In the second chapter, (1.1) is investigated for $\lambda = \alpha = \beta$, which we call a linear version of the problem (1.1). After the introduction of λ, the boundary value problem (1.1) reads

$$\begin{cases} u''(x) + \lambda u(x) = 0, & x \in (0,1), \\ u(0) \cdot \sin c = u'(0) \cdot \cos c, & \int_0^1 u(x)\, \mathrm{d}x = \gamma \cdot u'(0) \end{cases} \tag{1.2}$$

and our goal is to find values λ, so called eigenvalues, for which the problem (1.2) has a non-trivial solution. We are going to prove, that for $c \neq 0$ is λ the eigenvalue of (1.2), if λ is the solution of

$$1 - \cos\sqrt{\lambda} + \sqrt{\lambda} \cdot \cot c \cdot \sin\sqrt{\lambda} = \lambda \cdot \gamma \qquad \text{for } \lambda > 0,$$

or

$$-1 + \cosh\sqrt{-\lambda} + \sqrt{-\lambda} \cdot \cot c \cdot \sinh\sqrt{-\lambda} = -\lambda \cdot \gamma \qquad \text{for } \lambda < 0.$$

Additionaly, $\lambda = 0$ is the eigenvalue for the problem (1.2) if and only if

$$\gamma = \cot c + \frac{1}{2}.$$

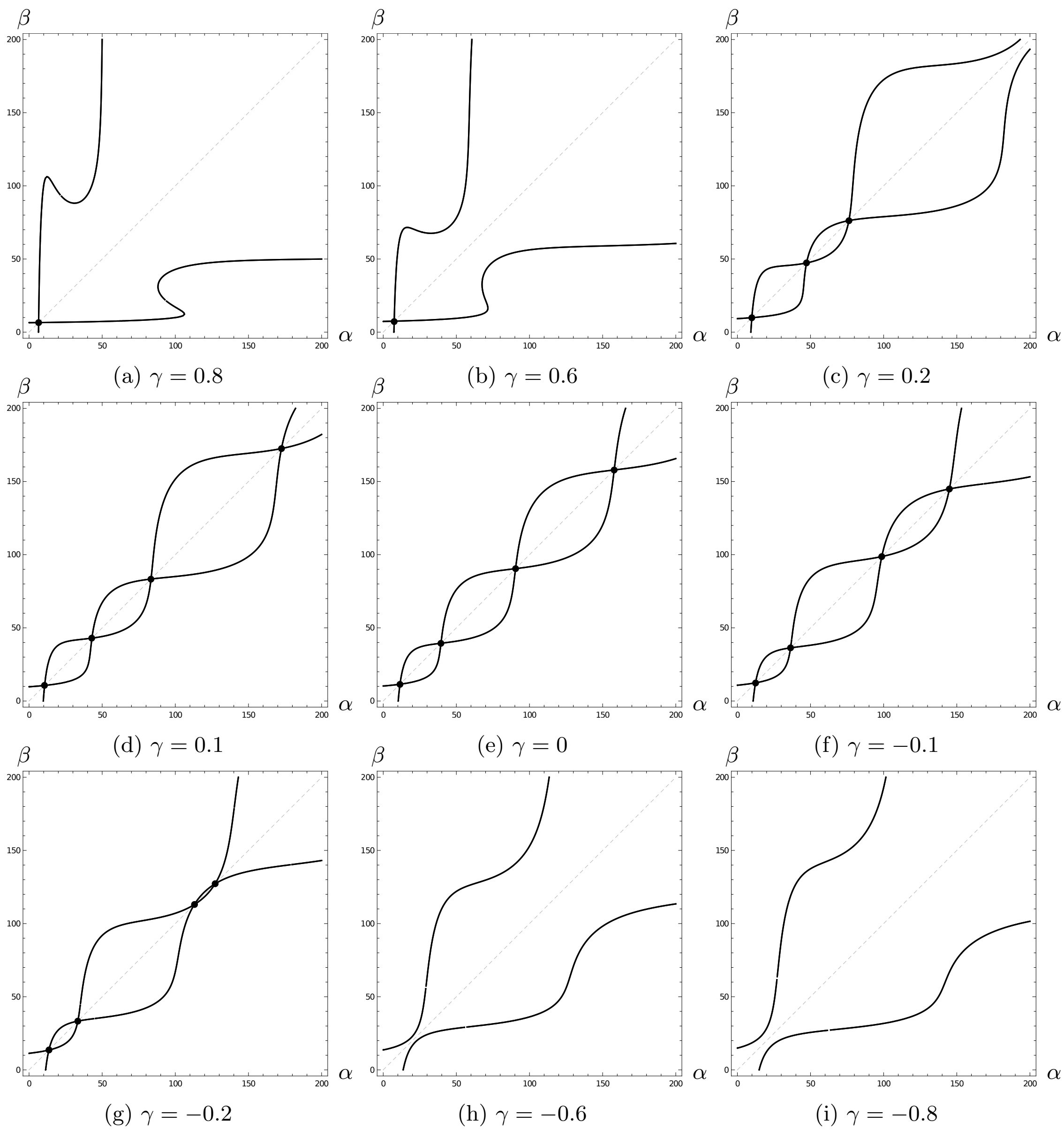
In the third chapter, the problem (1.1) is investigated and we provide an implicit description of the Fučík spectrum Σ_c^γ in the first quadrant of the $\alpha\beta$-plane. For $\alpha, \beta > 0$, we have that $(\alpha,\beta) \in \Sigma_c^\gamma$, if

$$\mathcal{G}\left(\sqrt{\alpha}, \sqrt{\beta}, \frac{2\sqrt{\alpha\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right) = P\left(\sqrt{\alpha}, \sqrt{\beta}, \frac{2\sqrt{\alpha\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right) - \gamma \cdot \sqrt{\alpha\beta} \cdot \cos\left(\sqrt{\alpha} \cdot p(\sqrt{\alpha}, c)\right)$$

or

$$\mathcal{G}\left(\sqrt{\beta}, \sqrt{\alpha}, \frac{2\sqrt{\alpha\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right) = P\left(\sqrt{\beta}, \sqrt{\alpha}, \frac{2\sqrt{\alpha\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right) - \gamma \cdot \sqrt{\alpha\beta} \cdot \cos\left(\sqrt{\beta} \cdot p(\sqrt{\beta}, c)\right),$$

where functions $\mathcal{G}$, P and p are defined in Definition 3.2. Examples of the set Σ_c^γ for different values of parameters are in Figure 1.1.



(a) $\gamma = 0.8$ (b) $\gamma = 0.6$ (c) $\gamma = 0.2$

(d) $\gamma = 0.1$ (e) $\gamma = 0$ (f) $\gamma = -0.1$

(g) $\gamma = -0.2$ (h) $\gamma = -0.6$ (i) $\gamma = -0.8$

Fig. 1.1: The set Σ_c^γ for $\alpha, \beta > 0$, $c = \frac{\pi}{8}$ and different values of γ.

In the fourth chapter, we consider the boundary value problem (1.1) for $c = \frac{\pi}{2}$ and $\gamma \in \left(-\frac{1}{2}, \frac{2}{\pi^2}\right)$, which is

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,1), \\ u(0) = 0, \quad \int_0^1 u(x)\, \mathrm{d}x = \gamma \cdot u'(0). \end{cases} \tag{1.3}$$

We show, that for $\alpha, \beta > 0$, finding the Fučík spectrum $\Sigma^{\gamma}_{\frac{\pi}{2}}$ of (1.3) is equivalent to finding the solution of the quadratic equation (for $s > 0$, given by (4.8))

$$c_{2,\gamma}(s) \cdot k^2 + c_{1,\gamma}(s) \cdot k + c_{0,\gamma}(s) = 0, \tag{1.4}$$

where we denote $k = \frac{b}{a}$ and $c_{2,\gamma}(s)$, $c_{1,\gamma}(s)$ and $c_{0,\gamma}(s)$ are given by (4.10), (4.11) and (4.12) respectively. Additionaly, we prove, that the equation (1.4) has for $s > 0$ and $\gamma \in \left(-\frac{1}{2}, \frac{2}{\pi^2}\right)$ only real solutions.

Lastly, in the fifth chapter, we provide the parametrization of the Fučík spectrum $\Sigma^{\gamma}_{\frac{\pi}{2}}$ in the first quadrant of the $\alpha\beta$-plane, which is a curve $\nu : (0, s^*) \to \mathbb{R}$ and $\nu(s) = (\nu_1(s), \nu_2(s))$, with the description

$$\nu_1(s) := \mu_1^2(s),$$
$$\nu_2(s) := \mu_2^2(s).$$

Functions $\mu_1, \mu_2 : (0, s^*) \to \mathbb{R}$ are defined as

$$\mu_1(s) = \begin{cases} s - n\pi + \pi + n\pi \cdot \dfrac{2 \cdot c_{2,\gamma}(s)}{-c_{1,\gamma}(s) + \sqrt{D(\gamma, s)}} & \text{for } 2n\pi - 2\pi < s \leq 2n\pi - \pi \\ s + \pi + 2n\pi \cdot \dfrac{2 \cdot c_{2,\gamma}(s)}{-c_{1,\gamma}(s) + \sqrt{D(\gamma, s)}} & \text{for } 2n\pi - \pi < s \leq 2n\pi \end{cases}$$

$$\mu_2(s) = \begin{cases} s - n\pi + \pi + n\pi \cdot \dfrac{-c_{1,\gamma}(s) + \sqrt{D(\gamma, s)}}{2 \cdot c_{2,\gamma}(s)} & \text{for } 2n\pi - 2\pi < s \leq 2n\pi - \pi \\ n\pi + (s - n\pi + \pi) \cdot \dfrac{-c_{1,\gamma}(s) + \sqrt{D(\gamma, s)}}{2 \cdot c_{2,\gamma}(s)} & \text{for } 2n\pi - \pi < s \leq 2n\pi \end{cases}$$

where $n \in \mathbb{N}$, $s^* = \Gamma(\gamma)$, Γ is given in (4.33), functions $c_{2,\gamma}$, $c_{1,\gamma}$, $c_{0,\gamma}$ are defined by (4.10), (4.11), (4.12) and $D(\gamma, s)$ is defined by (4.19).

1.1 Literature review

In this section, we provide an overview of papers published previously, which are examining problems related to the focus of this thesis.

1. Paper *The Fučík spectrum for nonlocal BVP with Sturm–Liouville boundary condition [7]* consideres the boundary value problem

$$\begin{cases} u''(x) + \mu u^+(x) - \lambda u^-(x) = 0, & x \in (0,1)\,, \\ \alpha u(0) + (1-\alpha) u'(0) = 0, & \int_0^1 u(s)\, \mathrm{d}s = 0, \end{cases} \tag{1.5}$$

   where μ, $\lambda \in \mathbb{R}$ and $\alpha \in [0, 1]$. The paper [7] provides us with a description of the Fučík spectrum of (1.5) for

   (a) $\alpha = 1$, see [7, Theorem 1, p.506]

   (b) $\alpha = 0$, see [7, Theorem 2, p. 506]

   (c) $\alpha \in (0, 1)$, see [7, Theorem 3, p. 508]

2. The paper *On Fučík type spectrum for problem with integral nonlocal boundary condition [9]* investigates boundary value problems

$$\begin{cases} u''(x) + \mu u^+(x) - \lambda u^-(x) = 0, & x \in (0,1)\,, \\ u(0) = 0, \quad u(1) = \gamma \int_0^{1/2} u(s)\,\mathrm{d}s \end{cases} \tag{1.6}$$

and

$$\begin{cases} u''(x) + \mu u^+(x) - \lambda u^-(x) = 0, & x \in (0,1)\,, \\ u(0) = 0, \quad u(1) = \gamma \int_{1/2}^{1} u(s)\,\mathrm{d}s, \end{cases} \tag{1.7}$$

where μ, λ, $\gamma \in \mathbb{R}$. Analytical descriptions of Fučík spectrums are provided for special values of parameters,

   (a) for $\gamma < 0$ in problems (1.6) and (1.7), see [9, Lemma 1, p. 264],
   (b) for $\gamma \in [0, 8)$ in (1.6) and for $\gamma \in \left[0, \frac{8}{3}\right]$ in (1.7), see [9, Lemma 2, p. 265]
   (c) for $\gamma = 8$ in (1.6) and for $\gamma = \frac{8}{3}$ in (1.7), see [9, Lemma 3, p. 265]
   (d) for $\gamma > 8$ in (1.6) and for $\gamma > \frac{8}{3}$ in (1.7), see [9, Lemma 4, p. 265].

   It also describes branches of the spectrums of the problems (1.6) and (1.7), solution of the problem (1.6) is described in [9, Lemma 5, p. 266] and solution of the problem (1.7) is described in [9, Lemma 6, p. 268].

3. The paper *On the Fučík type problem with integral nonlocal boundary conditions [8]* describes the spectrum of the boundary value problem

$$\begin{cases} u''(x) + \mu u^+(x) - \lambda u^-(x) = 0, & x \in (0,1)\,, \\ u(0) = \gamma \int_0^{1} u(s)\,\mathrm{d}s = u(1), \end{cases} \tag{1.8}$$

where μ, λ, $\gamma \in \mathbb{R}$. Main results of [8] are

   (a) the spectrum of (1.8) does not exists for $\gamma < 0$ (see [8, Lemma 2.1, p. 3])
   (b) location of the branch of the spectrum bolonging to the problem (1.8) (see [8, Lemma 2.2, p. 3]).

   Additionaly some features of branches are provided.

4. The paper *On Some Problems with Nonlocal Integral Condition [6]* studies the boundary value problem

$$\begin{cases} u''(x) + \mu u^+(x) - \lambda u^-(x) = 0, & x \in (0,1)\,, \\ u(0) = 0, \quad u(1) = \gamma \int_0^{1} u(s)\,\mathrm{d}s, \end{cases} \tag{1.9}$$

where μ, λ, $\gamma \in \mathbb{R}$. The paper provides us with a description of the spectrum of the problem (1.9) in [6, Theorem 2, p. 115]. Additionaly, properties of the spectrum are described [6, Section 4, p. 117]. Lastly, the author generalizes (1.9) into the boundary value problem

$$\begin{cases} u''(x) + \mu u^+(x) - \lambda u^-(x) = 0, & x \in (0,1)\,, \\ u(0) = \gamma_1 \int_0^{1} u(s)\,\mathrm{d}s, \quad u(1) = \gamma_2 \int_0^{1} u(s)\,\mathrm{d}s, \end{cases} \tag{1.10}$$

where μ, λ, γ_1, $\gamma_2 \in \mathbb{R}$ and provides several properties and condition of existence for the spectrum of the problem (1.10) (see [6, Section 5, p. 124]).

Chapter 2

Eigenvalues for the linear problem

In this chapter, we investigate the linear case of the boundary value problem (1.1), i.e. the problem

$$\begin{cases} u''(x) + \lambda u(x) = 0, & x \in (0,1), \\ u(0) \cdot \sin c = u'(0) \cdot \cos c, & \int_0^1 u(x)\, \mathrm{d}x = \gamma \cdot u'(0), \end{cases} \tag{2.1}$$

where we denoted $\alpha = \beta = \lambda$. Our goal is to find values λ based on the parameters γ, c, for which the boundary value problem (2.1) has a non-trivial solution u.

2.1 Eigenvalues for $c = \frac{\pi}{2}$ and $c = 0$

Firstly, we are going to investigate the problem (2.1) for special values of parameter c in order to find a simpler conditions for the eigenvalues λ. Let us consider $c = \frac{\pi}{2}$, for which the Sturm-Liouville condition $u(0) \cdot \sin c = u'(0) \cdot \cos c$ changes into the Dirichlet condition, therefore we investigate the boundary value problem

$$\begin{cases} u''(x) + \lambda u(x) = 0, & x \in (0,1), \\ u(0) = 0, & \int_0^1 u(x)\, \mathrm{d}x = \gamma \cdot u'(0). \end{cases} \tag{2.2}$$

Theorem 2.1. *For the boundary value problem* (2.2)*, we have*

1. *no eigenvalues for* $\gamma < 0$,
2. *countably many eigenvalues* $\lambda_k = 4k^2\pi^2$, $k \in \mathbb{N}$, *for* $\gamma = 0$,
3. *finitely many eigenvalues* $\lambda_1, \ldots, \lambda_n$, $n \in \mathbb{N}$, *for* $\gamma > 0$. *Moreover*
   (a) *for* $0 < \gamma < \frac{1}{2}$, *eigenvalues* $\lambda_1, \ldots, \lambda_n$ *are positive solutions of the equation*
   $$\gamma\lambda = 1 - \cos\sqrt{\lambda}, \tag{2.3}$$
   (b) *for* $\gamma = \frac{1}{2}$, *we have exactly one eigenvalue* $\lambda_1 = 0$ *and*
   (c) *for* $\gamma > \frac{1}{2}$, *we have exactly one eigenvalue* $\lambda_1 < 0$ *given as a solution of the equation*
   $$\gamma\lambda = 1 - \cosh\sqrt{-\lambda}. \tag{2.4}$$

Proof. We are going to split the proof according to the sign of λ.

1. Let $\lambda > 0$. Then the differential equation in the problem (2.2) has a general solution $u(x) = c_1 \sin(\sqrt{\lambda}x) + c_0 \cos(\sqrt{\lambda}x)$, where $c_1, c_0 \in \mathbb{R}$. Using the first boundary condition in (2.2), we obtain $u(0) = c_0 \cdot 1 = 0$ and thus $c_0 = 0$.

Using the integral condition in (2.2), we calculate

$$\begin{aligned}
\int_0^1 u(x)\,\mathrm{d}x &= \gamma \cdot u'(0), \\
c_1 \cdot \left[-\frac{\cos(\sqrt{\lambda}x)}{\sqrt{\lambda}} \right]_0^1 &= \gamma c_1 \sqrt{\lambda} \cdot \cos(\sqrt{\lambda} \cdot 0), \\
-\frac{\cos\sqrt{\lambda}}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} &= \gamma\sqrt{\lambda}, \\
\gamma\lambda &= 1 - \cos\sqrt{\lambda},
\end{aligned}$$

Which is the equation (2.3). For $\gamma \in \left(0, \frac{1}{2}\right)$, the equation (2.3) has finitely many solutions λ, since the function $\gamma\lambda$ is linear and strictly increasing, where as the function on the right side $1 - \cos\sqrt{\lambda}$ has the range of $[0, 2]$, as illustrated in Figure 2.1. For $\gamma < 0$ or $\gamma \geq \frac{1}{2}$, the equation (2.3) has no solutions and lastly, for $\gamma = 0$, we have $\gamma\lambda = 0$ and there are countably many values of λ given by $1 - \cos\sqrt{\lambda} = 0$, i.e. $\lambda_k = 4k^2\pi^2$, $k \in \mathbb{N}$.

2. Let $\lambda = 0$. Then the differential equation in (2.2) has the form of $u''(x) = 0$ and its general solution is $u(x) = A \cdot (x - x_0)$, where $A \in \mathbb{R}$ and $x_0 \in \mathbb{R}$. Using $u(0) = 0$ we obtain the relation $Ax_0 = 0$. If the parameter $A = 0$, then the value x_0 is arbitrary and $u(x) \equiv 0$. For $A \neq 0$, we have $x_0 = 0$ and

$$\begin{aligned}
\int_0^1 u(x)\,\mathrm{d}x &= \gamma \cdot u'(0), \\
A \cdot \int_0^1 x\,\mathrm{d}x &= \gamma \cdot A, \\
\left[\frac{x^2}{2}\right]_0^1 &= \gamma, \\
\gamma &= \frac{1}{2}.
\end{aligned}$$

Therefore $\lambda = 0$ is the eigenvalue for the boundary value problem (2.2) if and only if $\gamma = \frac{1}{2}$.

3. Let $\lambda < 0$. The general solution of the differential equation in (2.2) has the form of $u(x) = c_1 \sinh(\sqrt{-\lambda}x) + c_0 \cosh(\sqrt{-\lambda}x)$, where $c_1, c_0 \in \mathbb{R}$. Using the Dirichlet condition, we have $u(0) = c_0 = 0$. Using the integral condition, we obtain

$$\begin{aligned}
\int_0^1 u(x)\,\mathrm{d}x &= \gamma \cdot u'(0), \\
c_1 \cdot \int_0^1 \sinh(\sqrt{-\lambda}x)\,\mathrm{d}x &= \gamma c_1 \sqrt{-\lambda} \cosh(\sqrt{-\lambda} \cdot 0), \\
\left[\frac{\cosh(\sqrt{-\lambda}x)}{\sqrt{-\lambda}}\right]_0^1 &= \gamma\sqrt{-\lambda}, \\
\gamma\lambda &= 1 - \cosh(\sqrt{-\lambda}),
\end{aligned}$$

which is the equation (2.4). The equation (2.4) can be also written as $\gamma = f(\lambda)$, where we denoted $f(\lambda) := \frac{1-\cosh\sqrt{-\lambda}}{\lambda}$. The existence and uniqueness of the solution λ of the equation $\gamma = f(\lambda)$ is guaranteed for $\gamma > \frac{1}{2}$, since the function f is strictly decreasing for $\lambda < 0$ and has the range of $\left(\frac{1}{2}, +\infty\right)$, as illustrated in Figure 2.2.

□

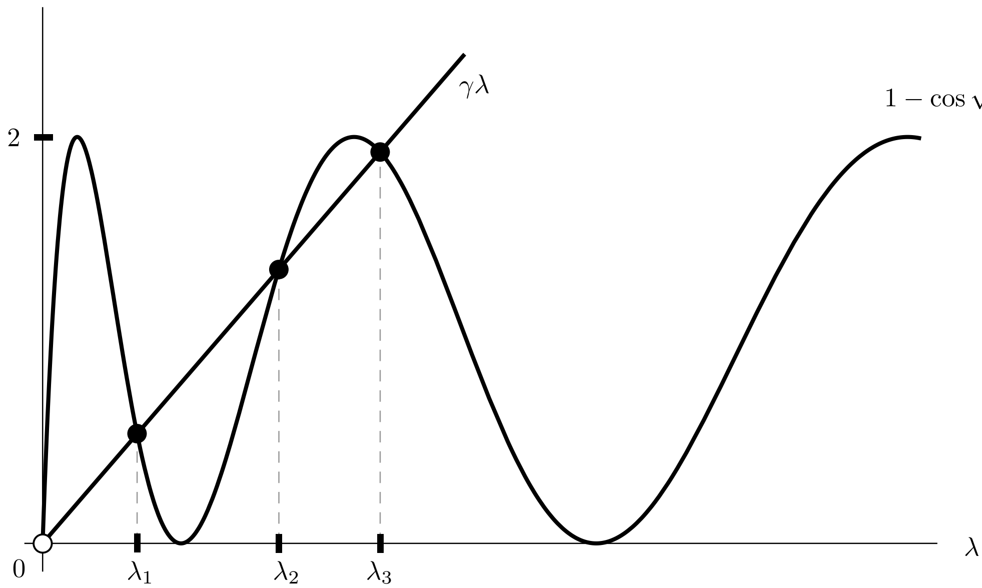


Fig. 2.1: Graphic representation of solutions λ_1, λ_2, λ_3 of the equation (2.3) for $\gamma = 0.02$.

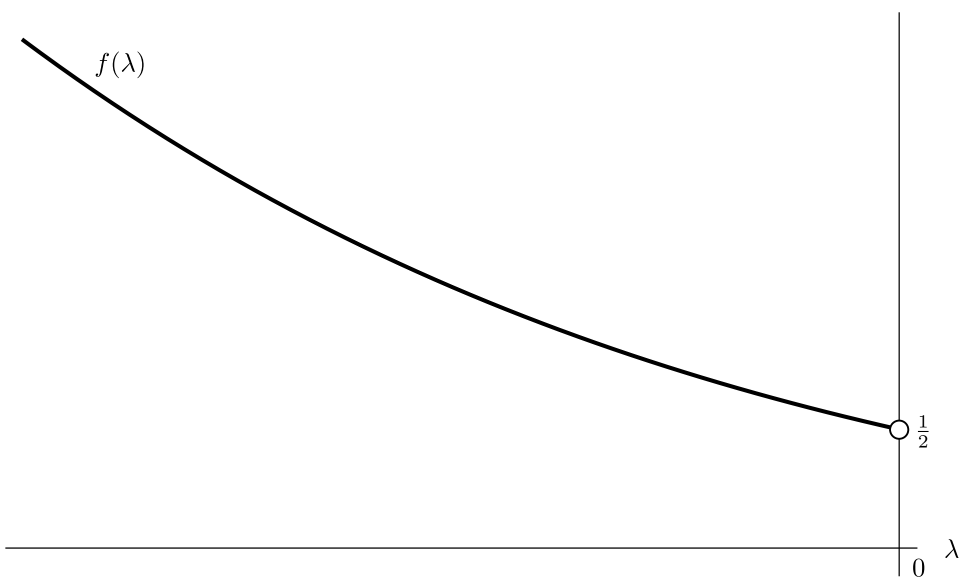


Fig. 2.2: Graph of the function $f(\lambda)$, $\lambda < 0$ defined in part 3 of the proof of Theorem 2.1.

The second special value of c we are going to investigate is $c = 0$. In this case, the Sturm-Liouville condition becomes the Neumann condition $u'(0) = 0$ and thus, the integral condition $\int_0^1 u(x)\,\mathrm{d}x = \gamma \cdot u'(0)$ is independent of the parameter γ, since it has the form of $\int_0^1 u(x)\,\mathrm{d}x = 0$. Therefore we investigate the boundary value problem

$$\begin{cases} u''(x) + \lambda u(x) = 0, & x \in (0,1), \\ u'(0) = 0, \quad \int_0^1 u(x)\,\mathrm{d}x = 0. \end{cases} \tag{2.5}$$

Theorem 2.2. *The eigenvalues λ_k for the boundary value problem* (2.5) *are given as*

$$\lambda_k = k^2\pi^2, \quad k \in \mathbb{N}. \tag{2.6}$$

Proof. Let us split the proof according to the sign of λ.

1. For $\lambda > 0$, the general solution of the differential equation in (2.5) is $u(x) = c_1 \sin(\sqrt{\lambda}x) + c_0 \cos(\sqrt{\lambda}x)$, where $c_1, c_0 \in \mathbb{R}$ and the first derivative is $u'(x) = c_1\sqrt{\lambda}\cos(\sqrt{\lambda}x) - c_0\sqrt{\lambda}\sin(\sqrt{\lambda}x)$. Thus we have $c_1 = 0$ due to the Neumann condition. Using the integral condition, we have

$$\begin{aligned} c_0 \int_0^1 \cos(\sqrt{\lambda}x)\,\mathrm{d}x &= 0, \\ c_0 \cdot \left[\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\right]_0^1 &= 0, \\ c_0 \cdot \sin\sqrt{\lambda} &= 0. \end{aligned}$$

   There are two possible cases. If $c_0 = 0$, then we have only the trivial solution $u(x) \equiv 0$. If $c_0 \neq 0$, then $\sin\sqrt{\lambda} = 0$, and solutions $\lambda_k = k^2\pi^2$, $k \in \mathbb{N}$, are eigenvalues for the boundary value problem (2.5).

2. For $\lambda = 0$, the general solution u of the equation $u''(x) = 0$ in (2.5) is given as $u(x) = A \cdot (x - x_0)$, where $A, x_0 \in \mathbb{R}$. The derivative of u can be calculated as $u'(x) = A$. Therefore using the Neumann condition, we have $A = 0$ and there is only the trivial solution $u(x) \equiv 0$ of the problem (2.5).

3. For $\lambda < 0$, the general solution u of the problem (2.5) is given as $u(x) = c_1 \sinh(\sqrt{-\lambda}x) + c_0 \cosh(\sqrt{-\lambda}x)$, where $c_1, c_0 \in \mathbb{R}$ and the first derivative is $u'(x) = c_1\sqrt{-\lambda}\cosh(\sqrt{-\lambda}x) + c_0\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x)$. Using the Neumann condition, we have $u'(0) = c_1\sqrt{-\lambda} = 0$ and thus $c_1 = 0$.

   If $c_0 = 0$, then $u(x) \equiv 0$, therefore we are going to suppose $c_0 \neq 0$. Then using the integral condition, we obtain

$$\begin{aligned} c_0 \cdot \int_0^1 \cosh(\sqrt{-\lambda}x)\,\mathrm{d}x &= 0 \\ c_0 \cdot \left[\frac{\sinh(\sqrt{-\lambda}x)}{\sqrt{-\lambda}}\right]_0^1 &= 0, \\ c_0 \cdot \frac{\sinh\sqrt{-\lambda}}{\sqrt{-\lambda}} &= 0, \\ \sinh\sqrt{-\lambda} &= 0, \\ \sqrt{-\lambda} &= 0, \end{aligned}$$

   which is a contradiction with the assumption $\lambda < 0$.

□

Remark 2.3. *For $\gamma = 0$, the integral condition $\int_0^1 u(x)\,\mathrm{d}x = \gamma \cdot u'(0)$ in the problem* (2.2) *simplifies to $\int_0^1 u(x)\,\mathrm{d}x = 0$. Let us note, that for $\gamma = 0$, the eigenvalues of* (2.2) *have already been studied in [5] and found eigenvalues λ_k in [5] match findings in Theorem 2.1.*

2.2 Eigenvalues for $c \neq 0$

After investigating special cases, we can shift our attention to the boundary value problem (2.1) with general parameters $c \neq 0$ and $\gamma \in \mathbb{R}$.

Theorem 2.4. *For the boundary value problem* (2.1) *with* $c \neq 0$, *the eigenvalues* λ *are solutions of*

$$\begin{aligned} 1 - \cos\sqrt{\lambda} + \sqrt{\lambda}\cdot\cot c\cdot\sin\sqrt{\lambda} &= \lambda\cdot\gamma && \text{for } \lambda > 0,\\ -1 + \cosh\sqrt{-\lambda} + \sqrt{-\lambda}\cdot\cot c\cdot\sinh\sqrt{-\lambda} &= -\lambda\cdot\gamma && \text{for } \lambda < 0. \end{aligned}$$

Moreover, $\lambda = 0$ *is the eigenvalue for the problem* (2.1) *if and only if*

$$\gamma = \cot c + \frac{1}{2}.$$

Proof. Let us split the proof according to the sign of λ.

1. For $\lambda > 0$, the general solution u of the differential equation in the problem (2.1) is $u(x) = c_1\sin(\sqrt{\lambda}x) + c_0\cos(\sqrt{\lambda}x)$, where $c_1, c_0 \in \mathbb{R}$. Additionaly, we have

$$\begin{aligned} u(0) &= c_0,\\ u'(x) &= c_1\sqrt{\lambda}\cos(\sqrt{\lambda}x) - c_0\sqrt{\lambda}\sin(\sqrt{\lambda}x),\\ u'(0) &= c_1\sqrt{\lambda}. \end{aligned}$$

Using Sturm-Liouville condition in (2.1), we have

$$u(0)\cdot\sin c = u'(0)\cdot\cos c,$$

$$c_0\sin c = c_1\sqrt{\lambda}\cos c, \tag{2.7}$$

$$\frac{c_0}{c_1} = \sqrt{\lambda}\frac{\cos c}{\sin c}, \tag{2.8}$$

where we assume $c_1 \neq 0$, since $c_1 = 0$ implies $c_0 = 0$ due to (2.7) and we have the trivial solution $u(x) \equiv 0$. Using the integral condition in (2.1), we obtain

$$\int_0^1 u(x)\,\mathrm{d}x = \gamma\cdot u'(0),$$

$$c_1\frac{1}{\sqrt{\lambda}}\Big[-\cos(\sqrt{\lambda}x)\Big]_0^1 + c_0\frac{1}{\sqrt{\lambda}}\Big[\sin(\sqrt{\lambda}x)\Big]_0^1 = \gamma\cdot c_1\sqrt{\lambda},$$

$$c_1\cdot(1-\cos\sqrt{\lambda}) + c_0\cdot\sin\sqrt{\lambda} = \gamma\cdot\lambda c_1,$$

$$1 - \cos\sqrt{\lambda} + \frac{c_0}{c_1}\sin\sqrt{\lambda} = \lambda\cdot\gamma. \tag{2.9}$$

By combining (2.8) and (2.9), we obtain the final condition in the following form

$$1 - \cos\sqrt{\lambda} + \sqrt{\lambda}\cdot\cot c\cdot\sin\sqrt{\lambda} = \lambda\cdot\gamma.$$

2. For $\lambda = 0$, the differential equation in the problem (2.1) simplifies to $u''(x) = 0$ and has a general solution $u(x) = A\cdot(x - x_0)$, where $A, x_0 \in \mathbb{R}$. We have $u'(x) = A$, and thus, using the integral condition in (2.1), we get

$$A\cdot\int_0^1 (x - x_0)\,\mathrm{d}x = \gamma\cdot A. \tag{2.10}$$

For $A = 0$, we have only a trivial solution $u(x) \equiv 0$ and thus we assume $A \neq 0$. Then the condition (2.10) reads

$$\left[\frac{x^2}{2} - xx_0\right]_0^1 = \gamma,$$

$$\frac{1}{2} - x_0 = \gamma. \tag{2.11}$$

Now, using the Sturm-Liouville condition in the problem (2.1), we have

$$u\,(0)\cdot\sin c = u'\,(0)\cdot\cos c,$$
$$-Ax_0\cdot\sin c = A\cdot\cos c. \tag{2.12}$$

The condition (2.12) can be simplified using (2.11) as

$$-x_0 = \frac{\cos c}{\sin c},$$
$$\gamma - \frac{1}{2} = \cot c,$$
$$\gamma = \cot c + \frac{1}{2}.$$

3. Lastly, for $\lambda < 0$, the general solution u of the differential equation in the problem (2.1) is $u(x) = c_1\sinh(\sqrt{-\lambda}x) + c_0\cosh(\sqrt{-\lambda}x)$ and additional relations can be calculated based on u as

$$u(0) = c_0,$$
$$u'(x) = c_1\sqrt{-\lambda}\cosh(\sqrt{-\lambda}x) + c_0\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x),$$
$$u'(0) = c_1\cdot\sqrt{-\lambda}.$$

The Sturm-Liouville condition in the problem (2.1) reads

$$u(0)\cdot\sin c = u'(0)\cdot\cos c,$$
$$c_0\cdot\sin c = c_1\cdot\sqrt{-\lambda}\cdot\cos c, \tag{2.13}$$
$$\frac{c_0}{c_1} = \sqrt{-\lambda}\frac{\cos c}{\sin c}, \tag{2.14}$$

where we assume $c_1 \neq 0$. The case of $c_1 = 0$ implies $c_0 = 0$ due to (2.13) and the only solution is $u(x) \equiv 0$. The integral condition in the problem (2.1) can be written in the following form

$$\int_0^1 u(x)\,\mathrm{d}x = \gamma\cdot u'(0),$$
$$c_1\cdot\frac{1}{\sqrt{-\lambda}}\Big[\cosh(\sqrt{-\lambda}x)\Big]_0^1 + c_0\cdot\frac{1}{\sqrt{-\lambda}}\Big[\sinh(\sqrt{-\lambda}x)\Big]_0^1 = \gamma\cdot c_1\sqrt{-\lambda},$$
$$c_1\cdot\frac{1}{\sqrt{-\lambda}}\cdot\Big(\cosh\sqrt{-\lambda} - 1\Big) + c_0\cdot\frac{1}{\sqrt{-\lambda}}\sinh\sqrt{-\lambda} = \gamma\cdot c_1\sqrt{-\lambda},$$
$$\frac{1}{\sqrt{-\lambda}}\cosh\sqrt{-\lambda} - \frac{1}{\sqrt{-\lambda}} + \frac{c_0}{c_1}\cdot\frac{1}{\sqrt{-\lambda}}\sinh\sqrt{-\lambda} = \gamma\cdot\sqrt{-\lambda}. \tag{2.15}$$

Combining (2.14) and (2.15), we obtain the final condition

$$-1 + \cosh\sqrt{-\lambda} + \sqrt{-\lambda}\cdot\cot c\cdot\sinh\sqrt{-\lambda} = -\lambda\cdot\gamma.$$

□

2.3 Observation for $\gamma = \frac{1}{2} + \cot c$

At the end of this chapter, let us introduce a small observation concerning the original problem (1.1) for the second and fourth quadrant of the $\alpha\beta$-plane.

Theorem 2.5. *For $\gamma = \frac{1}{2} + \cot c$ and $\alpha\cdot\beta < 0$, the boundary value problem* (1.1) *has only a trivial solution.*

Proof. Let us take the equation in (1.1), multiply it with the function $v(x) = (x-1)^2$ and then let us integrate the result over the interval $(0,1)$

$$
\begin{aligned}
u''(x) + \alpha u^+(x) - \beta u^-(x) &= 0,\\
\int_0^1 \left(u''(x)\cdot v(x) + \alpha u^+(x)\cdot v(x) - \beta u^-(x)\cdot v(x)\right) \mathrm{d}x &= 0,\\
\left[(x-1)^2 u'(x) - 2(x-1)u(x)\right]_0^1 + \int_0^1 2u(x)\,\mathrm{d}x + \alpha\int_0^1 u^+(x)\cdot v(x)\,\mathrm{d}x - \beta\int_0^1 u^-(x)\cdot v(x)\,\mathrm{d}x &= 0,\\
-u'(0) - 2u(0) + \int_0^1 2u(x)\,\mathrm{d}x + \alpha\int_0^1 u^+(x)\cdot v(x)\,\mathrm{d}x - \beta\int_0^1 u^-(x)\cdot v(x)\,\mathrm{d}x &= 0.
\end{aligned}
$$

Using the condition $\int_0^1 u(x)\,\mathrm{d}x = \gamma\cdot u'(0)$, the last equation transforms into

$$
-u'(0) - 2u(0) + 2\gamma u'(0) + \alpha\int_0^1 u^+(x)\cdot v(x)\,\mathrm{d}x - \beta\int_0^1 u^-(x)\cdot v(x)\,\mathrm{d}x = 0. \tag{2.16}
$$

In order to investigate the values of α and β, additional condition on the first three elements of equation (2.16) is applied together with the condition $u(0)\cdot\sin c = u'(0)\cdot\cos c$ in (1.1). For $c \neq 0$, we obtain

$$
\begin{aligned}
-u'(0) - 2u'(0)\cdot\frac{\cos c}{\sin c} + 2\gamma u'(0) &= 0,\\
u'(0)\cdot(-1 - 2\cot c + 2\gamma) &= 0,
\end{aligned} \tag{2.17}
$$

and the last equality is therefore satisfied for

$$
\frac{1}{2} + \cot c = \gamma. \tag{2.18}
$$

Finally, if the condition (2.18) is satisfied, then the equation (2.16) simplifies to

$$
\alpha\int_0^1 u^+(x)\cdot v(x)\,\mathrm{d}x = \beta\int_0^1 u^-(x)\cdot v(x)\,\mathrm{d}x,
$$

which cannot be satisfied for $\alpha < 0$, $\beta > 0$ or $\alpha > 0$, $\beta < 0$, since both integrals $\int_0^1 u^+(x)\cdot v(x)\,\mathrm{d}x$ and $\int_0^1 u^-(x)\cdot v(x)\,\mathrm{d}x$ are nonnegative and at least one of them is positive, otherwise u would be a trivial solution.

□

Values of c and γ, which satisfy the condition (2.18) are illustrated in Figure 2.3. For $\gamma = \frac{1}{2} + \cot c$, we have due to Theorem 2.5 that the second and the fourth quadrants of the $\alpha\beta$-plane are inadmissible areas for the Fučík spectrum Σ. Moreover, using Theorem 2.4, we obtain that the point $(0,0)$ belongs to the Fučík spectrum Σ for $\gamma = \frac{1}{2} + \cot c$. Thus, if there is a continuous Fučík curve containing the point $(0,0)$, then it has to be located in the first of the third quadrant.

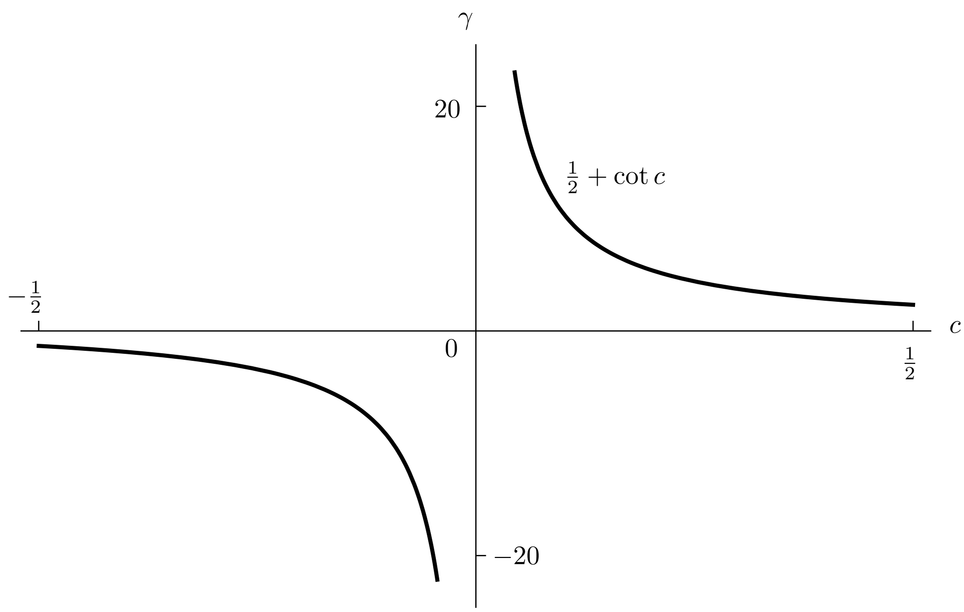


Fig. 2.3: Graph of values c and γ, which satisfy the condition (2.18).

Chapter 3

Implicit description of the Fučík spectrum in the first quadrant

The goal of this chapter is to provide the implicit description of the parts of the Fučík spectrum in the first quadrant of the $\alpha\beta$-plane for the problem

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,1), \\ u(0) \cdot \sin c = u'(0) \cdot \cos c, & \int_0^1 u(x)\,\mathrm{d}x = \gamma \cdot u'(0), \end{cases} \tag{3.1}$$

where $\alpha, \beta > 0$, $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\gamma \in \mathbb{R}$, i.e. to describe the set

$$\hat{\Sigma}_c^\gamma := \Big\{(\alpha,\beta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \text{the problem (3.1) has a non-trivial solution } u\Big\},$$

where $\mathbb{R}^+ = (0, +\infty)$. Let us note, that the problem (3.1) has already been studied for some values of parameters γ and c. For $\gamma = 0$, $c = 0$, the problem is investigated in [4]. Results of [4] were further generalized for $\gamma = 0$ and $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in [5]. We are going to generalize procedures described in these texts in order to find an implicit description of the set $\hat{\Sigma}_c^\gamma$.

In the first quadrant of the $\alpha\beta$-plane, i.e. for α, $\beta > 0$, let us denote $a = \sqrt{\alpha}$ and $b = \sqrt{\beta}$. The problem (1.1) then transforms to

$$\begin{cases} u''(x) + a^2 u^+(x) - b^2 u^-(x) = 0, & x \in (0,1), \\ u(0) \cdot \sin c = u'(0) \cdot \cos c, & \int_0^1 u(x)\,\mathrm{d}x = \gamma \cdot u'(0), \end{cases} \tag{3.2}$$

and the problem of finding pairs (α, β), such that (3.1) is satisfied, transforms into the problem of finding pairs $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$, such that (3.2) is satisfied. Using the approach justified in [4], we are going to introduce the initial value problem, solutions u of which include also solutions of the boundary value problem (3.2). The initial value problem is defined

$$\begin{cases} u''(x) + a^2 u^+(x) - b^2 u^-(x) = 0, & x \in \mathbb{R}, \\ u(p(a,c)) = 0, & u'(p(a,c)) = a \cdot b > 0, \end{cases} \tag{3.3}$$

where $p = p(a,c)$ ensures the fulfillment of the $u(0) \cdot \sin c = u'(0) \cdot \cos c$ condition and is a greatest non-positive value of x, such that $u(x) = 0$. Our goal is to find a function u which satisfies the problem (3.3) and also satisfies the integral condition

$$\int_0^1 u(x)\,\mathrm{d}x = \gamma \cdot u'(0). \tag{3.4}$$

Similarly to the set $\hat{\Sigma}_c^\gamma$ we can define set of all acceptable pairs in the first quadrant of the a, b-plane

$$\mathcal{M}_c^\gamma := \Big\{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : \text{the solution } u \text{ of the initial value problem (3.3) satisfies } \int_0^1 u(x)\,\mathrm{d}x = \gamma \cdot u'(0)\Big\}. \tag{3.5}$$

Our goal is to find the description of the set $\mathcal{M}_c^\gamma$, since the set $\hat{\Sigma}_c^\gamma$ can then be described based on the following lemma.

Lemma 3.1 (Pokorný [5, p. 15]). *We have a connection between $\mathcal{M}_c^\gamma$ and $\hat{\Sigma}_c^\gamma$ in the following way, $(a,b) \in \mathcal{M}_c^\gamma$ if and only if $(a^2,b^2) \in \hat{\Sigma}_c^\gamma$ and $(b^2,a^2) \in \hat{\Sigma}_c^\gamma$.*

3.1 Known results for $\gamma = 0$

In this section, let us recall some known results, the goal is to provide the reader with a basic knowledge of results from [5] necessary to follow the results in the next sections.

The problem (3.2) is examined for $\gamma = 0$, i.e. we investigate the boundary value problem

$$\begin{cases} u''(x) + a^2u^+(x) - b^2u^-(x) = 0, & x \in (0,1)\,, \\ u(0)\cdot \sin c = u'(0)\cdot\cos c, & \int_0^1 u(x)\,\mathrm{d}x = 0. \end{cases} \tag{3.6}$$

This can be formulated as an initial value problem (3.3) and our goal is to find a solution u which also satisfies the integral condion

$$\int_0^1 u(x)\,\mathrm{d}x = 0. \tag{3.7}$$

Additional functions can be derived based on the initial value problem (3.3) and the integral condition (3.7), which will help us to describe the set $\mathcal{M}_c^0$.

Definition 3.2 (Pokorný [5, p. 17]). *Let us define $p: \mathbb{R}^+ \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ as*

$$p(a,c) := \begin{cases} -\frac{1}{a}\operatorname{arccot}\left(\frac{1}{a}\tan c\right) & \text{for } c \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right), \\ 0 & \text{for } c = \frac{\pi}{2}. \end{cases} \tag{3.8}$$

Then the function $P: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ as

$$P\left(a,b,t\right) := \left(\frac{b}{a} - \frac{a}{b}\right)\frac{t}{\pi} + 1, \quad a > 0, \quad b > 0, \quad t \in \mathbb{R}, \tag{3.9}$$

and the function $\mathcal{G}:\ \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, which is 2π-periodic in the third variable

$$\forall a > 0\ \forall b > 0\ \forall t \in \mathbb{R}:\ \mathcal{G}\left(a,b,t+2\pi\right) = \mathcal{G}\left(a,b,t\right),$$

and is defined for $a > 0,\ b > 0,\ t \in (0, 2\pi]$ as

$$\mathcal{G}\left(a,b,t\right) := \begin{cases} \frac{b}{a}\cos\left(\frac{a+b}{2b}t - ap\right) - \frac{b}{a}\cos\left(ap\right) + P\left(a,b,t\right) & \text{for } t \in I_1, \\ \frac{a}{b}\cos\left(\frac{a+b}{2a}\left(t-2\pi\right) - bp\right) - \frac{b}{a}\cos\left(ap\right) + P\left(a,b,t-\pi\right) & \text{for } t \in I_2, \\ \frac{b}{a}\cos\left(\frac{a+b}{2b}\left(t-2\pi\right) - ap\right) - \frac{b}{a}\cos\left(ap\right) + P\left(a,b,t-2\pi\right) & \text{for } t \in I_3, \end{cases} \tag{3.10}$$

where

$$I_1 := \left(0, \frac{2b\left(\pi+ap\right)}{a+b}\right], \quad I_2 := \left(\frac{2b\left(\pi+ap\right)}{a+b}, 2\pi + \frac{2abp}{a+b}\right),$$

$$I_3 := \left[2\pi + \frac{2abp}{a+b}, 2\pi\right],$$

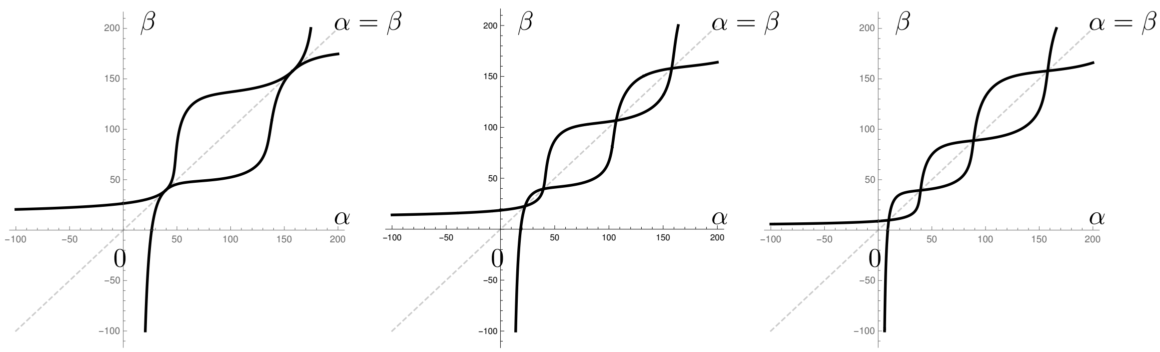
and p stands for the value of $p(a,c)$ defined in (3.8).

This definition then allows us to formulate the following Corollary introduced in [5].

Corollary 3.3 (Pokorný [5, p. 19]). *We have that $(a,b) \in \mathcal{M}_c^0$ if and only if $a,b > 0$ and*

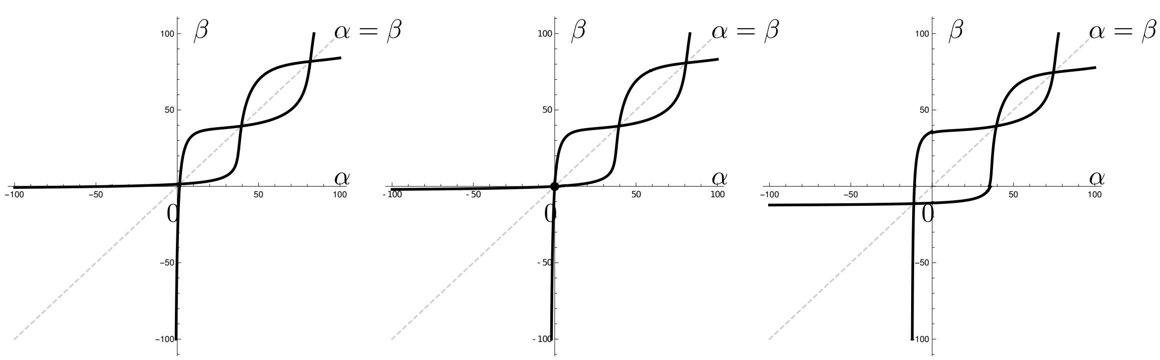
$$\mathcal{G}\left(a,b,\frac{2ab}{a+b}\right) = P\left(a,b,\frac{2ab}{a+b}\right). \tag{3.11}$$

The equation (3.11) with the Definition 3.2 of $\mathcal{G}$ and P provides an easy and straightforward way of generating the implicit description of the Fučík spectrum of the problem (3.6) for a given parameter c. Examples of spectra can be found in Figures 3.1 and 3.2.



(a) $c = \frac{\pi}{2}$ (b) $c = \frac{7\pi}{16}$ (c) $c = 0$

Fig. 3.1: The Fučík spectrum Σ_c^0 for different non-negative values of the parameter c, sourced from [5].



(a) $c = -1.05$ (b) $c = \arctan(-2)$ (c) $c = -1.3$

Fig. 3.2: The Fučík spectrum Σ_c^0 for different negative values of the parameter c, sourced from [5].

3.2 The problem for $c = \frac{\pi}{2}$ and $\gamma \in \mathbb{R}$

Let us consider the boundary value problem (1.1) for $c = \frac{\pi}{2}$, i.e. the problem

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,1), \\ u(0) = 0, \quad \int_0^1 u(x)\,\mathrm{d}x = \gamma \cdot u'(0), \end{cases} \tag{3.12}$$

where $\alpha, \beta > 0$ and $\gamma \in \mathbb{R}$. Similarly to the problem (3.3), let us move to the initial value problem

$$\begin{cases} u''(x) + a^2 u^+(x) - b^2 u^-(x) = 0, & x \in \mathbb{R}, \\ u(0) = 0, \quad u'(0) = 1 \end{cases} \tag{3.13}$$

where $a, b, > 0$ and $\gamma \in \mathbb{R}$ and we choose the value of the first derivative $u'(0) = 1$. Our goal is to find the solution u, for which there exists a pair (a, b) belonging to the set $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$, a special case of the set (3.5)

$$\mathcal{M}^{\gamma}_{\frac{\pi}{2}} = \Big\{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ \colon \text{the solution } u \text{ of the initial value problem (3.13) satisfies } \int_0^1 u(x)\,\mathrm{d}x = \gamma\Big\}. \tag{3.14}$$

Let u be the solution of the initial value problem (3.13). Then according to [5, p. 15] is u a T-periodic function, where $T = \frac{\pi}{a} + \frac{\pi}{b}$ and we have

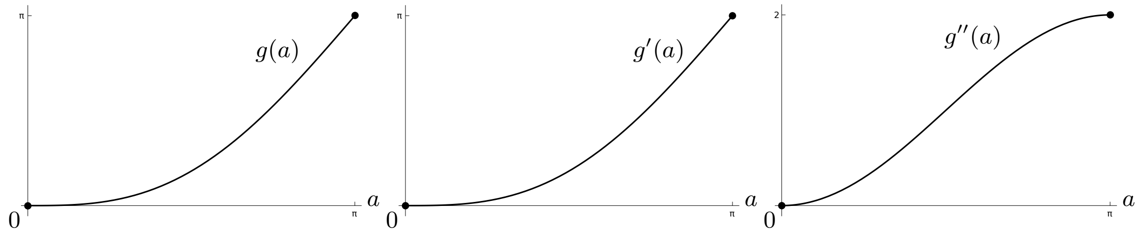
$$u(x) = \begin{cases} \frac{1}{a}\sin(ax) & \text{for } 0 < x \le \frac{\pi}{a}, \\ -\frac{1}{b}\sin\left(b\left(x - \frac{\pi}{b}\right)\right) & \text{for } \frac{\pi}{a} < x \le T. \end{cases} \tag{3.15}$$

Now, let us investigate some properties of the set $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$.

Lemma 3.4. *For $\gamma \ge \frac{1}{2}$, the set $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$ is empty.*

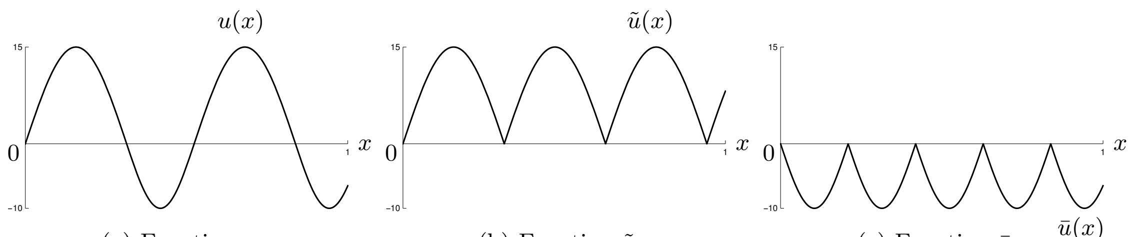
Proof. Let u be the solution of the initial value problem (3.13), then u is of the form in (3.15) and we are going to split the proof according to the value of a.

1. For $0 < a \le \pi$, we show that $\int_0^1 u(x)\,\mathrm{d}x < \frac{1}{2}$ and therefore the condition $\int_0^1 u(x)\,\mathrm{d}x = \gamma$ is not satisfied. Indeed, we obtain $\frac{\pi}{a} \ge 1$ and only the first part in (3.15) needs to be considered, thus
$$\int_0^1 u(x)\,\mathrm{d}x = \int_0^1 \frac{\sin(ax)}{a}\,\mathrm{d}x = \frac{1}{a^2}\left[-\cos(ax)\right]_0^1 = \frac{1-\cos(a)}{a^2}. \tag{3.16}$$
To show, that $\frac{1-\cos(a)}{a^2} < \frac{1}{2}$ or after a simple manipulation
$$0 < \frac{a^2}{2} + \cos(a) - 1,$$
we will define a new function
$$g(a) := \frac{a^2}{2} + \cos(a) - 1 \quad \text{for } 0 \le a \le \pi$$
and investigate its properties (see Figure 3.3). It holds, that $g(0) = 0$, $g(\pi) = \frac{\pi^2}{2} - 2 > 0$ and g is also strictly increasing, since we have
$$g'(a) = a - \sin(a),$$
$$g''(a) = 1 - \cos(a).$$
For the second derivative g'' holds, that $g''(a) > 0$ for $a \in (0, \pi)$. Thus the function g' is strictly incrasing on $(0, \pi)$. Because we also have $g'(0) = 0$, the function g' is positive on $(0, \pi)$. Therefore the function g is stictly increasing on $(0, \pi)$.



(a) Function g (b) Function g' (c) Function g''

Fig. 3.3: Graphs of the function g and its derivatives, as defined in Lemma 3.4.



(a) Function u (b) Function $\tilde{u}$ (c) Function $\bar{u}$

Fig. 3.4: Graphs of the function u and coresponding functions $\tilde{u}$ and $\bar{u}$, as defined in proofs of Lemma 3.5 and Lemma 3.6 respectively.

2. Let us assume $a > \pi$. An upper estimate of the integral $\int_0^1 u(x)\,\mathrm{d}x$ can be calculated as

$$\int_0^1 u(x)\,\mathrm{d}x \le \max_{x\in[0,1]} u(x) = \frac{1}{a} < \frac{1}{\pi} < \frac{1}{2}.$$

Therefore $\int_0^1 u(x)\,\mathrm{d}x < \frac{1}{2}$ and the condition $\int_0^1 u(x)\,\mathrm{d}x = \gamma$ is not satisfied, since $\gamma \ge \frac{1}{2}$.

□

Another special case is for $\frac{2}{\pi^2} \le \gamma < \frac{1}{2}$, as illustrated in the following lemma.

Lemma 3.5. *For* $\frac{2}{\pi^2} \le \gamma < \frac{1}{2}$, *the set* $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$ *forms a line* $a = a_0$, *where* a_0 *is the unique solution of the equation* $1 - \cos(a) = a^2\gamma$ *on the interval* $(0,\,\pi]$.

Proof. Let u be a solution of the initial value problem (3.13) such that

$$\int_0^1 u(x)\,\mathrm{d}x = \gamma \quad \text{and} \quad \frac{2}{\pi^2} \le \gamma < \frac{1}{2}.$$

Then u is in the form of (3.15) and we are going to split the proof according to the value of a.

1. For $0 < a \le \pi$, the function u is positive on the interval $(0,\,1)$ and we obtain (as in (3.16))

$$\gamma = \int_0^1 u(x)\,\mathrm{d}x = \frac{1-\cos(a)}{a^2}.$$

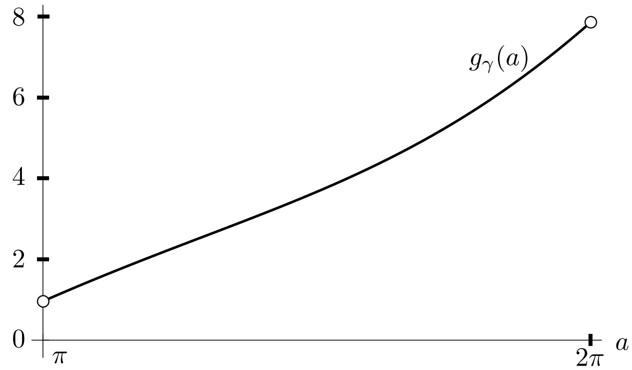


Fig. 3.5: Graph of the function g_γ, $\gamma = 0.3$, as defined in part 2 of the proof of Lemma 3.5.

Thus for $a \in (0, \pi]$, parameters have to satisfy the relation

$$1 - \cos(a) = a^2\gamma.$$

2. For $\pi < a < 2\pi$ we are going to construct an upper estimate $\tilde{u}$ of the function u in the form of continuous, $\frac{\pi}{a}$-periodic function $\tilde{u}$, defined as

$$\tilde{u}(x) := \frac{1}{a}\sin(ax), \quad \text{for } x \in \left(0, \frac{\pi}{a}\right]. \tag{3.17}$$

The function $\tilde{u}$ is illustrated in Figure 3.4. Therefore we have

$$\begin{aligned}\gamma = \int_0^1 u(x)\,\mathrm{d}x \leq \int_0^1 \tilde{u}\,\mathrm{d}x \\ = \int_0^{\frac{\pi}{a}} \tilde{u}\,\mathrm{d}x + \int_{\frac{\pi}{a}}^1 \tilde{u}\,\mathrm{d}x \\ =\frac{2}{a^2} + \int_{\frac{\pi}{a}}^1 b \cdot \sin\left(a \cdot \left(x - \frac{\pi}{a}\right)\right)\,\mathrm{d}x \\ =\frac{2}{a^2} - \frac{1}{a^2}\left[\cos\left(a \cdot \left(x - \frac{\pi}{a}\right)\right)\right]_{\frac{\pi}{a}}^1 \\ =\frac{3 + \cos(a)}{a^2},\end{aligned}$$

which implies

$$\gamma a^2 - \cos(a) - 3 \leq 0.$$

This inequality cannot be satisfied, since the function $g_\gamma(a) := \gamma a^2 - \cos(a) - 3$, $a \in (\pi, 2\pi)$ reaches only a positive values (see Figure 3.5). This can be shown, since

$$\inf_{\gamma \in \left[\frac{2}{\pi^2}, \frac{1}{2}\right)} g_\gamma(a) = \inf_{\gamma \in \left[\frac{2}{\pi^2}, \frac{1}{2}\right)} \left(\gamma a^2 - \cos a - 3\right) = \frac{2}{\pi^2}a^2 - \cos a - 3.$$

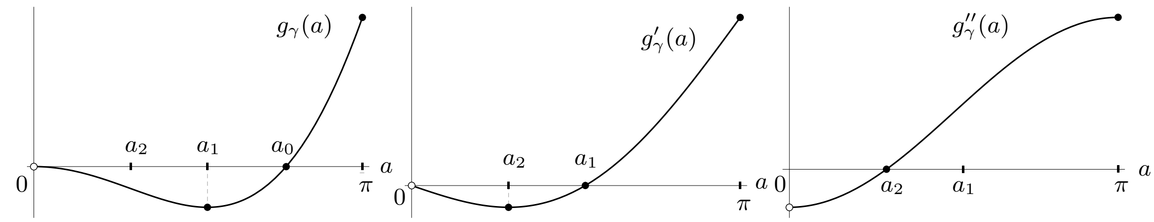
Also for $\tilde{\gamma} := \frac{2}{\pi^2}$ we have

$$\lim_{a \to 0+} g_{\tilde{\gamma}}(s) = 0 \quad \text{and} \quad \frac{\mathrm{d}g_{\tilde{\gamma}}(a)}{\mathrm{d}a} = 2a \cdot \frac{2}{\pi^2} + \sin(a) > 0.$$

3. For $2\pi \leq a$, we use an upper estimate of the function u as

$$\gamma = \int_0^1 u(x)\,\mathrm{d}x \leq \max_{x \in [0,1]} u(x) = \frac{1}{a} \leq \frac{1}{2\pi} < \frac{2}{\pi^2}$$

and therefore $\gamma < \frac{2}{\pi^2}$, which is a contradiction with an assumption $\gamma \geq \frac{2}{\pi^2}$.



(a) Function g_γ (b) Function g'_γ (c) Function g''_γ

Fig. 3.6: Graphs of the function g_γ and its derivatives, as defined in part 3 of the proof of Lemma 3.5.

Finally, to prove that the equation $1 - \cos(a) = a^2\gamma$ has only a unique solution on $(0, \pi]$, let us define the function $g_\gamma(a) := \gamma a^2 + \cos(a) - 1$, $a \in (0, \pi]$, and investigate its properties (see Figure 3.6). It holds, that

$$g'_\gamma(a) = 2\gamma a - \sin(a),$$
$$g''_\gamma(a) = 2\gamma - \cos(a),$$

and g''_γ has a zero point $a_2 := \arccos(2\gamma)$. The function g''_γ is negative for $0 < a < a_2$ and positive for $a_2 < a \le \pi$. Therefore the function g'_γ is strictly decreasing for $0 < a < a_2$ and strictly increasing for $a_2 < a \le \pi$. Since

$$\lim_{a\to 0+} g'_\gamma(a) = 0, \qquad g'_\gamma(\pi) = 2\gamma\pi > 0,$$

it implies that there exists exactly one zero point a_1 of g'_γ on the interval $(0, \pi)$. Therefore g'_γ is negative for $0 < a < a_1$ and positive for $a_1 < a < \pi$ and the inequalities $0 < a_2 < a_1 < \pi$ also hold. Finally, g_γ is strictly decreasing for $0 < a < a_1$ and strictly increasing for $a_1 < a \le \pi$. Since $\lim_{a\to 0+} g_\gamma(a) = 0$ and $g_\gamma(\pi) = \gamma\pi^2 - 2 \ge 0$, there exists exactly one point $a_0 \in (0, \pi]$ such that $g_\gamma(a_0) = 0$, which finishes the proof.

□

Additionaly, for $\gamma \le -\frac{1}{2}$, we have a similar result as in the Lemma 3.4.

Lemma 3.6. *For $\gamma \le -\frac{1}{2}$, the set $\mathcal{M}^\gamma_{\frac{\pi}{2}}$ is empty.*

Proof. The proof is similar to the proof of the Lemma 3.4. The function u, as the solution of the initial value problem (3.13) is in the form of (3.15) and the proof can be split according to the value of b.

1. For $0 < b \le \pi$, we define the function $\bar{u}$ as an lower estimate of the function u, in the form of continuous, $\frac{\pi}{b}$-periodic function

$$\bar{u}(x) := -\frac{1}{b}\sin(bx), \quad \text{for } x \in \left(0, \frac{\pi}{b}\right], \tag{3.18}$$

as illustrated in Figure 3.4. Then we obtain

$$\begin{aligned}\gamma = \int_0^1 u(x)\,\mathrm{d}x &\geq \int_0^1 \bar{u}(x)\,\mathrm{d}x \\ &= -\frac{1}{b}\int_0^1 \sin(bx)\,\mathrm{d}x \\ &= \frac{1}{b^2}\left[\cos(bx)\right]_0^1 \\ &= -\frac{1-\cos(b)}{b^2} \\ &> -\frac{1}{2},\end{aligned}$$

which contradicts the condition $\int_0^1 u(x)\,\mathrm{d}x = \gamma$.

2. For $\pi < b$, we construct a lower estimate of $\int_0^1 u(x)\,\mathrm{d}x$ as

$$\int_0^1 u(x)\,\mathrm{d}x \geq \min_{x\in[0,1]} u(x) = -\frac{1}{b} > -\frac{1}{\pi} > -\frac{1}{2},$$

which again contradicts the condition $\int_0^1 u(x)\,\mathrm{d}x = \gamma$.

□

At the end of this section, we are going to sum up our findings conserning the boundary value problem (3.12) with the theorem.

Theorem 3.7. *If u is a non-trivial solution of the boundary value problem (3.12) then $\gamma \in \left(-\frac{1}{2},\, \frac{1}{2}\right)$. Moreover, for $\alpha, \beta > 0$ and $\gamma \in \left[\frac{2}{\pi^2},\, \frac{1}{2}\right)$, the Fučík spectrum for (3.12) consists of two lines*

$$\left(\alpha - \lambda_0^2\right)\left(\beta - \lambda_0^2\right) = 0,$$

where λ_0 is the unique solution of the equation $1 - \cos\lambda = \gamma\lambda^2$ on the interval $(0,\, \pi]$.

Proof. The Theorem 3.7 is a direct consequence of Lemmas 3.4, 3.5 and 3.6. □

Remark 3.8. *For $\alpha, \beta > 0$ and $\gamma = \frac{2}{\pi^2}$, the Fučík spectrum for (3.12) consists of two lines*

$$\left(\alpha - \pi^2\right)\left(\beta - \pi^2\right) = 0.$$

According to Lemmas 3.4, 3.5 and 3.6 it remains to investigate the set $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$ for $\gamma \in \left(-\frac{1}{2},\, \frac{2}{\pi^2}\right)$.

3.3 Description in the first quadrant

Now, we are going to focus on the first quadrant of the ab-plane (i.e. first quadrant of the $\alpha\beta$-plane) and generalize the findings of Section 3.2 for $c \in \left(-\frac{\pi}{2},\, \frac{\pi}{2}\right]$.

Theorem 3.9. *We have that $(a, b) \in \mathcal{M}^{\gamma}_{c}$ if and only if $a, b > 0$ and*

$$\mathcal{G}\left(a, b, \frac{2ab}{a+b}\right) = P\left(a, b, \frac{2ab}{a+b}\right) - \gamma \cdot ab \cdot \cos\left(ap(a, c)\right). \tag{3.19}$$

Functions $\mathcal{G}$, P and p are defined in the previous section, Definition 3.2.

Proof. Let u be the solution of the initial value problem (3.3). Then using (3.11) in [5], we have for all $x \in \mathbb{R}$ that

$$G(x) = 1 + \frac{2}{\pi}(b-a)x - F(x), \tag{3.20}$$

where $F(x) = \int_0^x u(t)\,\mathrm{d}t$ and $G(x) = \mathcal{G}\left(a, b, \frac{2ab}{a+b}x\right)$ due to (3.17) in [5]. Let us note that the function $\mathcal{G}$ is given in (3.10) in Definition 3.2. Now, the equation (3.20) can be also written in the following form

$$\begin{aligned}\int_0^x u(t)\,\mathrm{d}t &= 1 + \frac{2}{\pi}(b-a)x - \mathcal{G}\left(a, b, \frac{2ab}{a+b}x\right),\\ \int_0^x u(t)\,\mathrm{d}t &= P\left(a, b, \frac{2ab}{a+b}x\right) - \mathcal{G}\left(a, b, \frac{2ab}{a+b}x\right),\end{aligned} \tag{3.21}$$

where the function P is given in (3.9) in Definition 3.2. Indeed, we have that

$$P\left(a, b, \frac{2ab}{a+b}x\right) = \frac{b^2-a^2}{ab} \cdot \frac{1}{\pi} \cdot \frac{2ab}{a+b} \cdot x + 1 = \frac{2(b-a)}{\pi} \cdot x + 1.$$

Finally, using (3.21) for $x = 1$, the integral condition in (3.2)

$$\int_0^1 u(x)\,\mathrm{d}x = \gamma \cdot u'(0)$$

reads as

$$\begin{aligned}P\left(a, b, \frac{2ab}{a+b}\right) - \mathcal{G}\left(a, b, \frac{2ab}{a+b}\right) &= \gamma \cdot u'(0),\\ P\left(a, b, \frac{2ab}{a+b}\right) - \mathcal{G}\left(a, b, \frac{2ab}{a+b}\right) &= \gamma \cdot ab \cdot \cos(ap(a, c)),\end{aligned} \tag{3.22}$$

where we determined $u'(0)$ using (3.6) in [5]. The equation (3.22) is exactly the implicit equation (3.19), which finishes the proof.

□

Theorem 3.9 provides us with a way of numerically generating the set $\mathcal{M}_c^\gamma$, example of a code used can be found in Appendix A. Example of the set $\mathcal{M}_c^\gamma$ can be found in Figure 3.7.

Remark 3.10. *Theorem 3.9 provides us with a way of generating pairs (a, b) which belong to the set $\mathcal{M}_c^\gamma$. The Fučík spectrum $\hat{\Sigma}_c^\gamma$ is then easy to obtain based on the Lemma 3.1.*

Remark 3.11. *Let us note, that for $\lambda > 0$ and $a = b = \sqrt{\lambda}$, we have*

$$\mathcal{G}\left(\sqrt{\lambda}, \sqrt{\lambda}, \sqrt{\lambda}\right) = \cos\left(\sqrt{\lambda} - \sqrt{\lambda}p\right) - \cos\left(\sqrt{\lambda}p\right) + 1$$

and

$$P\left(\sqrt{\lambda}, \sqrt{\lambda}, \sqrt{\lambda}\right) = 1.$$

The condition of the Theorem 3.9 can therefore be manipulated as

$$\begin{aligned}\cos\left(\sqrt{\lambda} - \sqrt{\lambda}p\right) + \cos\left(\sqrt{\lambda}p\right)(-1+\gamma\lambda) &= 0,\\ \cos\sqrt{\lambda}\frac{\frac{1}{\sqrt{\lambda}}\tan c}{\sqrt{\frac{1}{\sqrt{\lambda}}\tan^2 c + 1}} + \sin\sqrt{\lambda}\frac{1}{\sqrt{\frac{1}{\sqrt{\lambda}}\tan^2 c + 1}} + \frac{\frac{1}{\sqrt{\lambda}}\tan c}{\sqrt{\frac{1}{\sqrt{\lambda}}\tan^2 c + 1}}(-1+\gamma\lambda) &= 0,\\ -\sqrt{\lambda}\sin\sqrt{\lambda} \cdot \cot c - \cos\sqrt{\lambda} + 1 &= \gamma\lambda.\end{aligned}$$

This is exactly the relation for the eigenvalues $\lambda > 0$ described in Theorem 2.4.

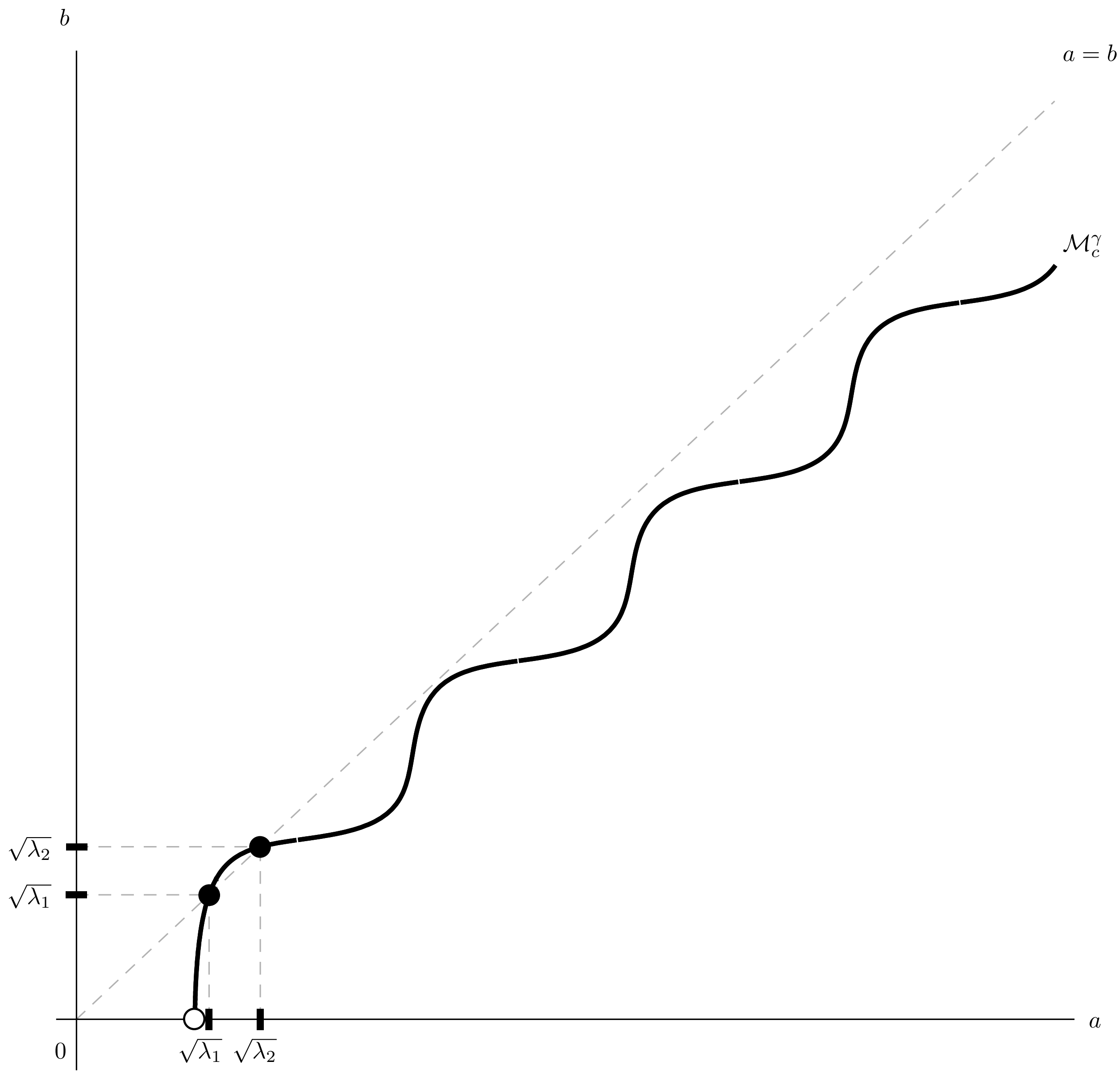


Fig. 3.7: The set $\mathcal{M}_c^\gamma$ for $c = \frac{\pi}{4}$ and $\gamma = -0.1$.

Chapter 4

Solvability of the problem with the Dirichlet condition

In this chapter, we consider the following boundary value problem

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, \quad x \in (0,1), \\ u(0) = 0, \quad \int_0^1 u(x)\ \mathrm{d}x = \gamma \cdot u'(0), \end{cases} \tag{4.1}$$

where $\alpha, \beta > 0$ and $\gamma \in \left(-\frac{1}{2}, \frac{2}{\pi^2}\right)$. Our goal is to study the structure of the Fučík spectrum $\hat{\Sigma}_c^\gamma$ for $c = \frac{\pi}{2}$. Due to Lemma 3.1, it is enough to investigate the set $\mathcal{M}_c^\gamma$ for $c = \frac{\pi}{2}$. Using Theorem 3.9 for $c = \frac{\pi}{2}$, we get that $(a,b) \in \mathcal{M}_{\frac{\pi}{2}}^\gamma$ if and only if $a, b > 0$ and (note that $p(a, \frac{\pi}{2}) = 0$)

$$\mathcal{G}\left(a, b, \frac{2ab}{a+b}\right) = P\left(a, b, \frac{2ab}{a+b}\right) - \gamma \cdot ab, \tag{4.2}$$

where $\mathcal{G}$ and P are given by (3.10) and (3.9). Moreover, since $p(a, \frac{\pi}{2}) = 0$, the function $\mathcal{G}$ simplifies as

$$\mathcal{G}(a,b,t) = \begin{cases} \frac{b}{a}\cos\left(\frac{a+b}{2b}t\right) - \frac{b}{a} + P(a,b,t) & \text{for } t \in \left(0, \frac{2b\pi}{a+b}\right], \\ \frac{a}{b}\cos\left(\frac{a+b}{2a}(t-2\pi)\right) - \frac{b}{a} + P(a,b,t-\pi) & \text{for } t \in \left(\frac{2b\pi}{a+b}, 2\pi\right]. \end{cases} \tag{4.3}$$

At first, let us denote (as in [4])

$$k := \frac{b}{a} > 0. \tag{4.4}$$

Then $P(a,b,t) = \tilde{P}(k,t) := \left(k - \frac{1}{k}\right)\frac{t}{\pi} + 1$ and $\mathcal{G}(a,b,t) = \tilde{\mathcal{G}}(k,t)$, where

$$\tilde{\mathcal{G}}(k,t) = \begin{cases} k\cos\left(\frac{1+k}{2k}t\right) - k + \tilde{P}(k,t) & \text{for } t \in \left(0, \frac{2k\pi}{k+1}\right], \\ \frac{1}{k}\cos\left(\frac{1+k}{2}(t-2\pi)\right) - k + P(k,t-\pi) & \text{for } t \in \left(\frac{2k\pi}{k+1}, 2\pi\right]. \end{cases} \tag{4.5}$$

At second, let us denote (in the same manner as in [4])

$$t := \frac{2ab}{a+b} > 0. \tag{4.6}$$

Then the equation (4.2) can be written as

$$\tilde{\mathcal{G}}(k,t) = \tilde{P}(k,t) - \gamma \cdot \frac{(1+k)^2}{4k}t^2. \tag{4.7}$$

Indeed, we have that

$$\frac{(1+k)^2}{4k}t^2 = \frac{(1+\frac{b}{a})^2}{4\frac{b}{a}}\frac{4a^2b^2}{(a+b)^2} = \frac{(a+b)^2}{4ab}\frac{4a^2b^2}{(a+b)^2} = ab.$$

Finally, let us introduce the last third substitution (based on the variable s in [4]) in order to transform (4.7) to a polynomial equation with respect to k ($n \in \mathbb{N}_0$)

$$s := \begin{cases} \frac{1+k}{2}(t - 2n\pi) + 2n\pi - \pi & \text{for } t \in \left(2n\pi - \frac{2\pi}{1+k}, 2n\pi\right], \\ \frac{1+k}{2k}(t - 2n\pi) + 2n\pi - \pi & \text{for } t \in \left(2n\pi, 2(n+1)\pi - \frac{2\pi}{1+k}\right]. \end{cases} \tag{4.8}$$

See Figure 4.1 for the graph of s with respect to t. Let us note that for $t \in (0, +\infty)$, we have that $s \in (-\pi, +\infty)$.

Lemma 4.1. *For $k, t > 0$, the equation (4.7) can be equivalently written as*

$$c_{2,\gamma}(s) \cdot k^2 + c_{1,\gamma}(s) \cdot k + c_{0,\gamma}(s) = 0, \quad s > -\pi, \tag{4.9}$$

where s is given by (4.8) and ($n \in \mathbb{N}_0$)

$$c_{2,\gamma}(s) := \begin{cases} 2n - n^2\pi^2\gamma & \text{for } 2n\pi - 2\pi < s \le 2n\pi - \pi, \\ 1 + \cos s + 2n - (s - n\pi + \pi)^2\gamma & \text{for } 2n\pi - \pi < s \le 2n\pi, \end{cases} \tag{4.10}$$

$$c_{1,\gamma}(s) := -2n\pi(s - n\pi + \pi)\gamma \qquad \text{for } 2n\pi - 2\pi < s \le 2n\pi, \tag{4.11}$$

$$c_{0,\gamma}(s) := \begin{cases} 1 + \cos s - 2n - (s - n\pi + \pi)^2\gamma & \text{for } 2n\pi - 2\pi < s \le 2n\pi - \pi, \\ -2n - n^2\pi^2\gamma & \text{for } 2n\pi - \pi < s \le 2n\pi. \end{cases} \tag{4.12}$$

Proof. Let us split the proof according to the value of $t > 0$.

1. At first, let us assume that $t \in \left(2n\pi - \frac{2\pi}{1+k}, 2n\pi\right]$, $n \in \mathbb{N}$. In this case, we have for s given by (4.8) that

$$2n\pi - 2\pi < s \le 2n\pi - \pi.$$

   Due to 2π-periodicity of $\tilde{\mathcal{G}}$ in the second variable t, the equation (4.7) can be written as

$$\begin{aligned} \tilde{\mathcal{G}}(k, t - 2n\pi + 2\pi) &= \tilde{P}(k,t) - \gamma \cdot \tfrac{(1+k)^2}{4k} t^2, \\ \tfrac{1}{k}\cos\left(\tfrac{1+k}{2}(t - 2n\pi)\right) - k + \tilde{P}(k, t - 2n\pi + \pi) &= \tilde{P}(k,t) - \gamma \cdot \tfrac{(1+k)^2}{4k} t^2, \\ \tfrac{1}{k}\cos(s - 2n\pi + \pi) - k + (k - \tfrac{1}{k}) \cdot \tfrac{\pi - 2n\pi}{\pi} &= -\gamma \cdot \tfrac{(1+k)^2}{4k} t^2, \\ -\tfrac{1}{k}\cos(s) - k - (k - \tfrac{1}{k})(2n-1) &= -\gamma \cdot \tfrac{(1+k)^2}{4k} t^2, \\ \cos(s) + k^2 + (k^2 - 1)(2n-1) &= \gamma \cdot \tfrac{(1+k)^2}{4} t^2. \end{aligned} \tag{4.13}$$

   Now, since we have that

$$t = \tfrac{2}{1+k}(s - 2n\pi + \pi) + 2n\pi$$

   the equation (4.13) reads

$$\cos(s) + k^2 + (k^2 - 1)(2n-1) = \gamma \cdot \tfrac{(1+k)^2}{4}\left(\tfrac{2}{1+k}(s - 2n\pi + \pi) + 2n\pi\right)^2. \tag{4.14}$$

   Using simple manipulations, the equation (4.14) can be rewritten into the following form

$$(2n - n^2\pi^2\gamma) \cdot k^2 - 2n\pi(s - n\pi + \pi)\gamma \cdot k + 1 + \cos s - 2n - (s - n\pi + \pi)^2\gamma = 0,$$

   which is exactly the equation (4.9) for $s \in (2n\pi - 2\pi, 2n\pi - \pi]$.

2. At second, let us assume that $t \in \left(2n\pi, 2(n+1)\pi - \frac{2\pi}{1+k}\right]$, $n \in \mathbb{N}_0$. In this case, we have for s given by (4.8) that

$$2n\pi - \pi < s \le 2n\pi.$$

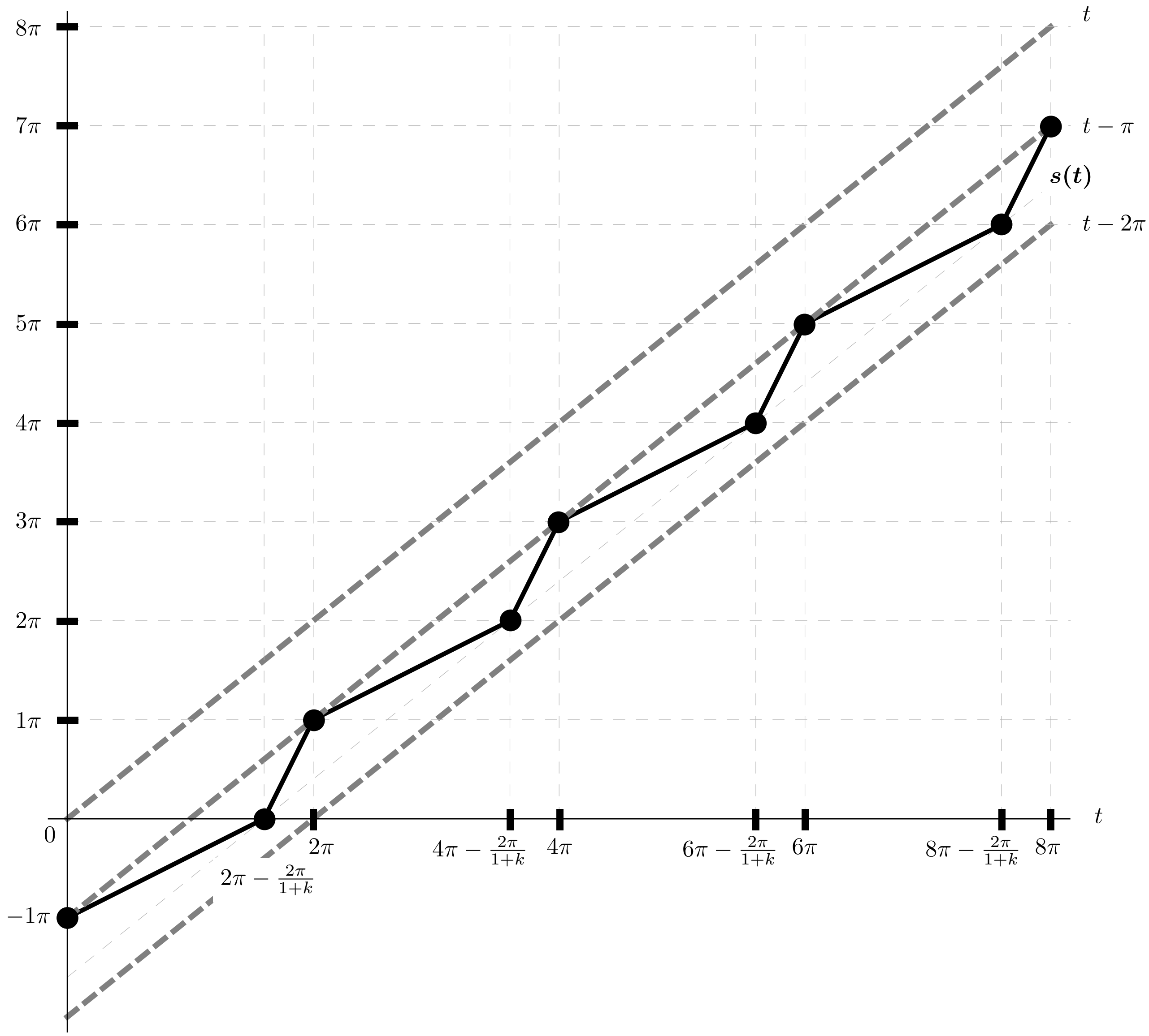


Fig. 4.1: Graph of the function s, defined by (4.8).

Due to 2π-periodicity of $\tilde{\mathcal{G}}$ in the second variable t, the equation (4.7) can be written as

$$\begin{aligned} \tilde{\mathcal{G}}(k, t-2n\pi) &= \tilde{P}(k,t) - \gamma \cdot \tfrac{(1+k)^2}{4k} t^2, \\ k\cos\left(\tfrac{1+k}{2k}(t-2n\pi)\right) - k + \tilde{P}(k,t-2n\pi) &= \tilde{P}(k,t) - \gamma \cdot \tfrac{(1+k)^2}{4k} t^2, \\ k\cos(s-2n\pi+\pi) - k - (k-\tfrac{1}{k}) \cdot \tfrac{2n\pi}{\pi} &= -\gamma \cdot \tfrac{(1+k)^2}{4k} t^2, \\ k\cos(s) + k + 2n(k-\tfrac{1}{k}) &= \gamma \cdot \tfrac{(1+k)^2}{4k} t^2, \\ k^2\cos(s) + k^2 + 2n(k^2-1) &= \gamma \cdot \tfrac{(1+k)^2}{4} t^2. \end{aligned} \tag{4.15}$$

Now, since we have that

$$t = \tfrac{2k}{1+k}(s-2n\pi+\pi) + 2n\pi$$

the equation (4.15) reads

$$k^2\cos(s) + k^2 + 2n(k^2-1) = \gamma \cdot \tfrac{(1+k)^2}{4} \left(\tfrac{2k}{1+k}(s-2n\pi+\pi) + 2n\pi\right)^2. \tag{4.16}$$

Using simple manipulations, the equation (4.16) can be rewritten into the following form

$$(1+\cos s + 2n - (s-n\pi+\pi)^2\gamma) \cdot k^2 - 2n\pi(s-n\pi+\pi)\gamma \cdot k - 2n - n^2\pi^2\gamma = 0,$$

which is exactly the equation (4.9) for $s \in (2n\pi-\pi, 2n\pi]$.

□

Now, before proceeding any further, we are going to use the estimate from below of the function cos on the interval $(-\pi, 0)$ (see Figure 4.2) and prove a lemma, which will become useful in the following text.

Lemma 4.2. *For $s \in (-\pi, 0)$, we have that $\frac{2}{\pi^2}(s+\pi)^2 - 1 < \cos(s)$.*

Proof. Let us define the function $f(s) := -\frac{2}{\pi^2}(s+\pi)^2 + 1 + \cos(s)$ for $-\pi \le s \le 0$. Our goal is to show that $f(s) > 0$ for $s \in (-\pi,\, 0)$.

First of all, we have $f'(s) = -\frac{2}{\pi^2} \cdot 2 \cdot (s+\pi) - \sin(s)$ and $f''(s) = -\frac{4}{\pi^2} - \cos(s)$. For the function f'' follows that $f''(-\pi) = -\frac{4}{\pi^2} + 1 > 0$ and $f''(0) = -\frac{4}{\pi^2} - 1 < 0$. Since the function $f''(s)$ is strictly decreasing, there is exactly one point s_0, for which $f''(s_0) = 0$, as ilustrated in Figure 4.3c. The value s_0 can be expressed as $s_0 = -\pi + \arccos\left(\frac{4}{\pi^2}\right)$. The function f'' is therefore positive for $s \in [-\pi,\, s_0)$ and negative for $s \in (s_0,\, 0]$.

Based on the values of the function f'', we know that the function f' is strictly increasing for $s \in [-\pi,\, s_0)$ and strictly decreasing for $s \in (s_0,\, 0]$. Since $f'(-\pi) = 0$, we know that $f'(s_0) > 0$ and because $f'(0) = -\frac{4}{\pi} < 0$ and the function f' is decreasing for $s \in (s_0,\, 0]$, there is exactly one point $s_1 \in (s_0,\, 0)$, where $f'(s_1) = 0$, as ilustrated in Figure 4.3b.

Finally, we know based on the values of the function f', that the function f is increasing for $s \in [-\pi,\, s_1)$ and decreasing for $s \in (s_1,\, 0]$ (see Figure 4.3a). Due to $f(-\pi) = 0$ and $f(0) = 0$, we have that $f(s) > 0$ for $\forall s \in (-\pi,\, 0)$, which finishes the proof. □

Now we can use Lemma 4.2 to prove the following statement.

Lemma 4.3. *For $\gamma \in \left(-\frac{1}{2}, \frac{2}{\pi^2}\right)$, the equation (4.9) is not solvable for $s \in (-\pi, 0]$.*

Proof. For $s \in (-\pi, 0]$, the equation (4.9) has the following form

$$\begin{aligned} \left(1 + \cos(s) - (s+\pi)^2\gamma\right) \cdot k^2 &= 0, \\ 1 + \cos(s) &= (s+\pi)^2\gamma. \end{aligned} \tag{4.17}$$

For $\gamma \le 0$, the equation (4.17) cannot be satisfied since $1 + \cos(s) > 0$. For $0 < \gamma < \frac{2}{\pi^2}$, we get using Lemma 4.2 that

$$1 + \cos(s) \ge \tfrac{2}{\pi^2}(s+\pi)^2 > \gamma(s+\pi)^2$$

and thus, the equation (4.17) cannot be satisfied. □

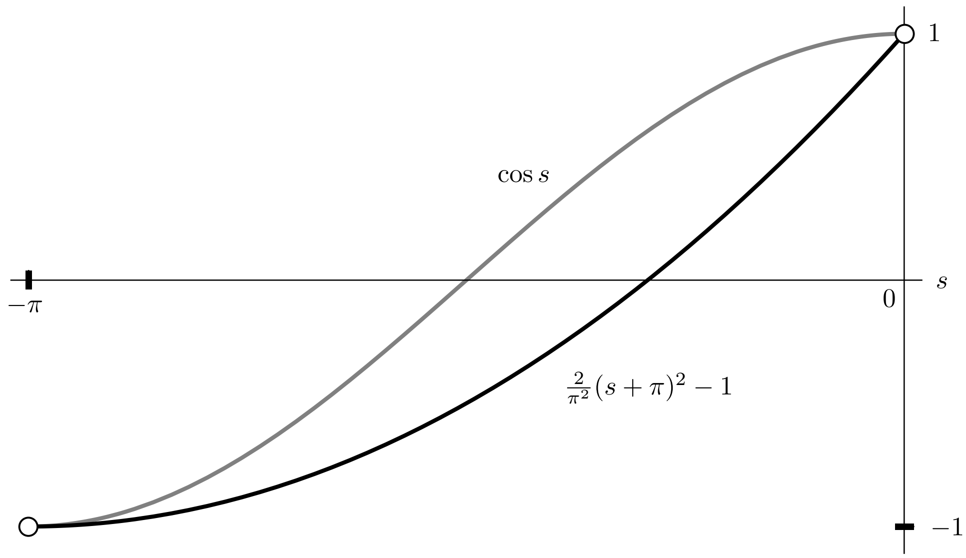
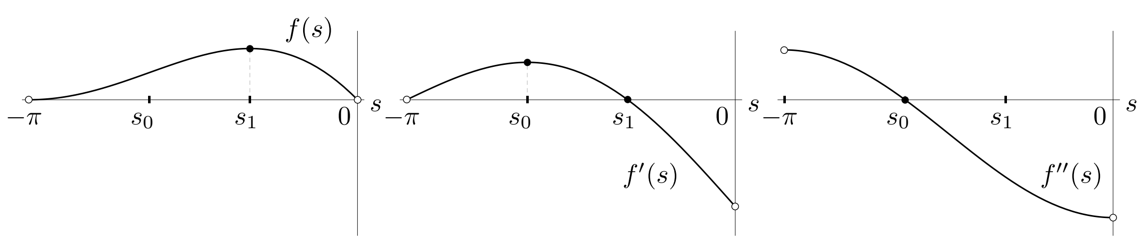


Fig. 4.2: Graph of an lower estimate (black curve) of the function cos (gray curve) constructed in Lemma 4.2.



(a) Graph of the function $f(s)$ for $s \in [-\pi,\, 0]$.

(b) Graph of the function $f'(s)$ for $s \in [-\pi,\, 0]$.

(c) Graph of the function $f''(s)$ for $s \in [-\pi,\, 0]$.

Fig. 4.3: Graphs of the functions f, f' and f'' used in the proof of the Lemma 4.2.

Theorem 4.4. *Let* $\gamma \in \left(-\frac{1}{2}, \frac{2}{\pi^2}\right)$*. Then* $(a,b) \in \mathcal{M}^{\gamma}_{\frac{\pi}{2}}$ *if and only if* $a, b > 0$ *and*

$$c_{2,\gamma}(s) \cdot k^2 + c_{1,\gamma}(s) \cdot k + c_{0,\gamma}(s) = 0, \tag{4.18}$$

where $k = \frac{b}{a}$, $s > 0$ *is given by* (4.8) *with* $t = \frac{2ab}{a+b}$, *and* $c_{2,\gamma}(s)$, $c_{1,\gamma}(s)$, $c_{0,\gamma}(s)$ *are given by* (4.10), (4.11), (4.12), *respectively.*

Proof. Let us recall that the equation (4.2) can be equivalently written as (4.7). The statement now follows directly using Lemmas 4.1 and 4.3. □

4.1 Discriminant of the quadratic equation (4.18)

The goal of this section is to show, that the quadratic equation (4.18) for $s > 0$ has a real solution, i.e. that the discriminant of the corresponding polynomial is positive. Let us denote

$$D(\gamma, s) := c^2_{1,\gamma}(s) - 4 \cdot c_{2,\gamma}(s) \cdot c_{0,\gamma}(s) \quad \text{for } -\frac{1}{2} \le \gamma \le \frac{2}{\pi^2} \text{ and } s \ge 0, \tag{4.19}$$

which represents the discriminant of the quadratic polynomial in (4.18). Let us note that $D(\gamma, 0) = 0$. Indeed, we have

$$D(\gamma, 0) = c^2_{1,\gamma}(0) - 4 \cdot c_{2,\gamma}(0) \cdot c_{0,\gamma}(0) = 0 - 4 \cdot c_{2,\gamma}(0) \cdot 0.$$

For the discriminant D, we have the following representation

$$D(\gamma, s) = \begin{cases} 4n \cdot D_{1,n}(\gamma, s) & \text{for } 2n\pi - 2\pi < s \le 2n\pi - \pi, \\ 4n \cdot D_{2,n}(\gamma, s) & \text{for } 2n\pi - \pi < s \le 2n\pi, \end{cases}$$

where

$$\begin{aligned} D_{1,n}(\gamma, s) &:= 4n - 2 - n\gamma\pi(3\pi + 4s) + 2\gamma(s+\pi)^2 + (n\gamma\pi^2 - 2)\cos(s), \\ D_{2,n}(\gamma, s) &:= 4n + 2 + n\gamma\pi(5\pi + 4s) - 2\gamma(s+\pi)^2 + (n\gamma\pi^2 + 2)\cos(s). \end{aligned}$$

The representation of D can be verified, for example for $s \in (2n\pi - 2\pi,\, 2n\pi - \pi]$, as

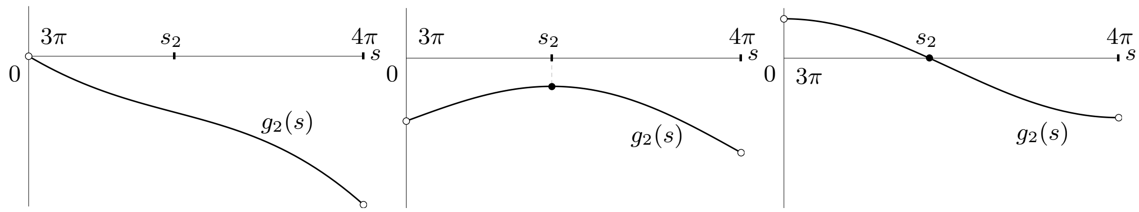
$$\begin{aligned} D(\gamma, s) &= c^2_{1,\gamma}(s) - 4 \cdot c_{2,\gamma}(s) \cdot c_{0,\gamma}(s) \\ &= \left(-2n\pi(s - n\pi + \pi)\gamma\right)^2 - 4\left(2n - n^2\pi^2\gamma\right)\left(1 + \cos s - 2n - (s - n\pi + \pi)^2\gamma\right) \\ &= 4n \cdot \left(-2 - 2\cos s + 4n + 2\gamma s^2 + 2\gamma\pi^2 - 4\gamma s n\pi + \gamma s\pi - 3\gamma n\pi^2 + n\pi^2\gamma \cdot \cos s\right) \\ &= 4n \cdot \left(4n - 2 - n\gamma\pi(3\pi + 4s) + 2\gamma(s+\pi)^2 + (n\gamma\pi^2 - 2)\cos(s)\right). \end{aligned}$$

Our goal for the next part is to find the lowest value of the function $D(\gamma, s)$ on rectangles $\left[-\frac{1}{2}, \frac{2}{\pi^2}\right] \times [2n\pi - 2\pi,\, 2n\pi - \pi]$ and $\left[-\frac{1}{2}, \frac{2}{\pi^2}\right] \times [2n\pi - \pi,\, 2n\pi]$, $n \in \mathbb{N}$. Firstly, we are going to show, that there are no stationary points of $D = D(\gamma, s)$ on open rectangles $\left(-\frac{1}{2}, \frac{2}{\pi^2}\right) \times (2n\pi - 2\pi,\, 2n\pi - \pi)$ and $\left(-\frac{1}{2}, \frac{2}{\pi^2}\right) \times (2n\pi - \pi,\, 2n\pi)$, $n \in \mathbb{N}$. See following Lemmas 4.5, 4.6, 4.7 and 4.8.

Lemma 4.5. *The function* $D_{2,n} = D_{2,n}(\gamma, s)$, $n \in \mathbb{N}$, *has no stationary point in the open rectangle* $\left(-\frac{1}{2}, \frac{2}{\pi^2}\right) \times (2n\pi - \pi,\, 2n\pi)$.

Proof. To investigate stationary points of the function $D_{2,n}$, let us denote the partial derivative $\frac{\partial D_{2,n}}{\partial \gamma}$ as g_n and calculate its first two derivatives as

$$\begin{aligned} g_n(s) :&= \frac{\partial D_{2,n}(\gamma, s)}{\partial \gamma} = n\pi(5\pi + 4s) - 2(s+\pi)^2 + n\pi^2\cos(s), \\ g_n'(s) &= 4n\pi - 4(s+\pi) - n\pi^2\sin(s), \\ g_n''(s) &= -4 - n\pi^2\cos(s). \end{aligned}$$



(a) Function g_2 (b) Function g_2' (c) Function g_2''

Fig. 4.4: Graph of the function g_n, $n = 2$ defined in the proof of Lemma 4.5.

The function g_n and its derivatives are illustrated in Figure 4.4. Then for $s \in (2n\pi - \pi,\, 2n\pi)$ the function g_n'' has a zero point $s_n := 2n\pi - \arccos\left(-\frac{4}{n\pi^2}\right)$. Moreover, the function g_n'' is positive on $(2n\pi - \pi,\, s_n)$ and negative on $(s_n,\, 2n\pi)$ and therefore the point s_n is a strict maximum of the function g_n' on the interval $(2n\pi - \pi,\, 2n\pi)$. We also claim, that $g_n'(s_n) < 0$. Indeed, we have that

$$\begin{aligned} g_n'(s_n) &= 4n\pi - 4 \cdot \left(2n\pi - \arccos\left(-\frac{4}{n\pi^2}\right) + \pi\right) + n\pi^2 \sin\left(\arccos\left(-\frac{4}{n\pi^2}\right)\right) \\ &= -4(n+1)\pi + 4\arccos\left(-\frac{4}{n\pi^2}\right) + n\pi^2 \cdot \sqrt{1 - \left(\frac{4}{n\pi^2}\right)^2} \\ &= -4(n+1)\pi + 4\arccos\left(-\frac{4}{n\pi^2}\right) + \sqrt{n^2\pi^4 - 16} \\ &< -4(n+1)\pi + 4\pi + \sqrt{n^2\pi^4 - 16} < 0 \end{aligned}$$

and the last inequality holds, since we have

$$\begin{aligned} \sqrt{n^2\pi^4 - 16} &< 4(n+1)\pi - 4\pi, \\ \sqrt{n^2\pi^4 - 16} &< 4n\pi, \\ n^2\pi^4 - 16 &< 16n^2\pi^2, \\ n^2\pi^4 - 16n^2\pi^2 &< 16, \\ n^2\pi^2(\pi^2 - 16) &< 16. \end{aligned}$$

Finally, g_n is strictly decreasing on the interval $(2n\pi - \pi,\, 2n\pi)$ and also

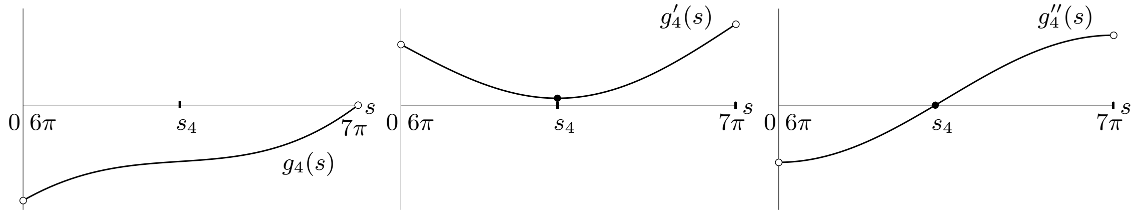
$$g_n(2n\pi - \pi) = n\pi\,(5\pi + 8n\pi - 4\pi) - 2\,(2n\pi - \pi + \pi)^2 + n\pi^2 \cos(2n\pi - \pi) = 0,$$

therefore the function g_n is negative on $(2n\pi - \pi,\, 2n\pi)$ and the function $D_{2,n}$ has no stationary point in $\left(-\frac{1}{2},\, \frac{2}{\pi^2}\right) \times (2n\pi - \pi,\, 2n\pi)$. □

Lemma 4.6. *The function $D_{1,n} = D_{1,n}(\gamma, s)$, $n \in \mathbb{N}$, $n \geq 3$, has no stationary point in the open rectangle $\left(-\frac{1}{2},\, \frac{2}{\pi^2}\right) \times (2n\pi - 2\pi,\, 2n\pi - \pi)$.*

Proof. Similarly to the proof of Lemma 4.5, let us define a partial derivative of the function $D_{1,n}$ and calculate its derivatives as

$$\begin{aligned} g_n(s) &:= \frac{\partial D_{1,n}(\gamma, s)}{\partial \gamma} = -n\pi(3\pi + 4s) + 2(s + \pi)^2 + n\pi^2 \cos(s), \\ g_n'(s) &= -4n\pi + 4(s + \pi) - n\pi^2 \sin(s), \\ g_n''(s) &= 4 - n\pi^2 \cos(s). \end{aligned}$$



(a) Function g_4 (b) Function g_4' (c) Function g_4''

Fig. 4.5: Graphs of the function g_n, $n = 4$ and its derivatives, defined in Lemma 4.6.

An example of the function g_n and its derivatives is in Figure 4.5. The zero point of the function g_n'' is $s_n := 2n\pi - 2\pi + \arccos\left(\frac{4}{n\pi^2}\right)$, $s_n \in (2n\pi - 2\pi,\, 2n\pi - \pi)$. Moreover, the function g'' is negative on $(2n\pi - 2\pi,\, s_n)$ and positive on $(s_n,\, 2n\pi - \pi)$ and therefore the point s_n is a strict minimum of the function g_n' on the interval $(2n\pi - 2\pi,\, 2n\pi - \pi)$. The inequality $g_n'(s_n) > 0$ also holds, since

$$\begin{aligned} g_n'(s_n) &= -4n\pi + 4\left(2n\pi - \pi + \arccos\left(\frac{4}{n\pi^2}\right)\right) - n\pi^2 \sin\left(\arccos\left(\frac{4}{n\pi^2}\right)\right) \\ &= 4(n-1)\pi + 4\arccos\left(\frac{4}{n\pi^2}\right) - n\pi^2\sqrt{1 - \frac{16}{n^2\pi^4}} \\ &= 4(n-1)\pi + 4\arccos\left(\frac{4}{n\pi^2}\right) - \sqrt{n^2\pi^4 - 16} \\ &> 4(n-1)\pi + 4\cdot 1.1 - \sqrt{n^2\pi^4 - 16} > 0 \end{aligned}$$

and the last step is justified by

$$\begin{aligned} \sqrt{n^2\pi^4 - 16} &< 4(n-1)\pi + 4.4, \\ n^2\pi^4 - 16 &< 16(n-1)^2\pi^2 + 4.4^2 + 8(n-1)\cdot 4.4\pi, \\ n^2\cdot(\pi^4 - 16\pi^2) + n\cdot(32\pi^2 - 8\pi\cdot 4.4) &< 16 + 16\pi^2 + 4.4^2 - 8\pi\cdot 4.4, \\ n\cdot(\pi^4 - 16\pi^2) + 32\pi^2 - 8\pi\cdot 4.4 &< \frac{16 + 16\pi^2 + 4.4^2 - 8\pi\cdot 4.4}{n}. \end{aligned} \tag{4.20}$$

For $n = 3$, we can numerically verify the inequality (4.20) and for $n \geq 4$, the lefthand side of (4.20) is always negative while the righthand side is positive.

Finally, g_n is strictly increasing on the interval $(2n\pi - 2\pi,\, 2n\pi - \pi)$ and also

$$\begin{aligned} g_n(2n\pi - \pi) &= -n\pi(3\pi + 8n\pi - 4\pi) + 2(2n\pi - \pi + \pi)^2 + n\pi^2\cos(2n\pi - \pi) \\ &= -8n^2\pi^2 + n\pi^2 + 8n^2\pi^2 - n\pi^2 \\ &= 0, \end{aligned}$$

therefore the function g_n is negative and the function $D_{1,n}$ has no stationary point on $\left(-\frac{1}{2},\, \frac{2}{\pi^2}\right) \times (2n\pi - 2\pi,\, n\pi - \pi)$.

□

Note that in Lemma 4.6, the assumption of $n \geq 3$ is crucial for proving the inequality (4.20). Cases of $n = 1$ and $n = 2$ will be solved separetly in the following Lemmas 4.7 and 4.8.

Lemma 4.7. *The function $D_{1,1} = D_{1,1}(\gamma, s)$ has no stationary point on the rectangle $\left(-\frac{1}{2},\, \frac{2}{\pi^2}\right) \times (0,\, \pi)$.*

Proof. Once again, we are going to define the partial derivative of the function $D_{1,1}$ and its derivatives as

$$\begin{aligned} g_1(s) &:= \frac{\partial D_{1,1}(\gamma, s)}{\partial \gamma} = -\pi(3\pi + 4s) + 2(s+\pi)^2 + \pi^2 \cos(s), \\ g_1'(s) &= -4\pi + 4(s+\pi) - \pi^2 \sin(s), \\ g_1''(s) &= 4 - \pi^2 \cos(s). \end{aligned}$$

The function g_1'' has a zero point $s_2 := \arccos\left(\frac{4}{\pi^2}\right)$, $s_2 \in (0,\, \pi)$ and g_1'' is negative on $(0,\, s_2)$ and positive on $(s_2,\, \pi)$. Thus g_1' is strictly decreasing on $(0,\, s_2)$ and strictly increasing on $(s_2,\, \pi)$. At the same time, $g_1'(0) = 0$ and $g_1'(\pi) = 4\pi > 0$, therefore there is exactly one point $s_1 \in (0,\, \pi)$ such that $g_1'(s_1) = 0$.

Finally, g_1 is strictly decreasing on $(0,\, s_1)$, strictly increasing on $(s_1,\, \pi)$ and also $g_1(0) = g_1(\pi) = 0$. Therefore $g_1(s) < 0$ for $s \in (0,\, \pi)$. □

Lemma 4.8. *The function $D_{1,2} = D_{1,2}(\gamma, s)$ has no stationary point in the open rectangle $\left(-\frac{1}{2},\, \frac{2}{\pi^2}\right) \times (2\pi,\, 3\pi)$.*

Proof. We are going to investigate the partial derivative of the function $D_{1,2}$ in the form of

$$g_2(s) := \frac{\partial D_{1,2}(\gamma, s)}{\partial \gamma} = -2\pi(3\pi + 4s) + 2(s+\pi)^2 + 2\pi^2 \cos(s).$$

Using Lemma 4.2, it is straightforward to show that

$$g_2(s) < -2\pi(3\pi + 4s) + 2(s+\pi^2) + 2\pi^2 \left(1 - \frac{2}{\pi^2}(s - 3\pi + \pi)\right) = -2(s-3\pi)^2 < 0$$

and thus the function $D_{1,2}$ does not have a stationary point in the open rectangle $\left(-\frac{1}{2},\, \frac{2}{\pi^2}\right) \times (2\pi,\, 3\pi)$. □

We have described the behaviour of the function $D = D(\gamma, s)$ in the interior of rectangles $\left(-\frac{1}{2},\, \frac{2}{\pi^2}\right) \times (2n\pi - 2\pi,\, 2n\pi - \pi)$ and $\left(-\frac{1}{2},\, \frac{2}{\pi^2}\right) \times (2n\pi - \pi,\, 2n\pi)$, now we are going to shift our attention to the border.

Lemma 4.9. *On the closed rectangle $\left[-\frac{1}{2},\, \frac{2}{\pi^2}\right] \times [2n\pi - \pi,\, 2n\pi]$, $n \in \mathbb{N}$, the function $D_{2,n} = D_{2,n}(\gamma, s)$ is non-negative and attains its minimum 0 only at the corner point $(\gamma, s) = (\frac{2}{\pi^2},\, 2n\pi)$.*

Proof. Let us define the rectangle R_n, on which we are going to investigate properties of the function $D_{2,n}$ as $R_n := \left[-\frac{1}{2},\, \frac{2}{\pi^2}\right] \times [2n\pi - \pi,\, 2n\pi]$. According to Lemma 4.5, $D_{2,n}$ has no stationary point inside the rectangle R_n, therefore $D_{2,n}$ restricted to R_n attains its extremes on the boundary ∂R_n. On the border of this rectangle, it holds

$$\begin{aligned} D_{2,n}(\gamma, 2n\pi - \pi) &= 4n > 0, \\ D_{2,n}(\gamma, 2n\pi) &= 2(n+1)(2 - \gamma\pi^2) \geq 0, \\ D_{2,n}\left(-\frac{1}{2}, s\right) &= 2 + (s+\pi)^2 + 4n - \frac{n\pi}{2}(5\pi + 4s) + \left(2 - \frac{n\pi^2}{2}\right)\cos(s) > 0, \\ D_{2,n}\left(\frac{2}{\pi^2}, s\right) &= 2 - \frac{4}{\pi^2}(s+\pi)^2 + 4n + \frac{2n}{\pi}(5\pi + 4s) + (2+2n)\cos(s) \geq 0. \end{aligned}$$

It remains to justify last two inequalities.

1. Using Lemma 4.2, we get

$$\begin{aligned} D_{2,n}\left(-\frac{1}{2}, s\right) &= 2 + (s+\pi)^2 + 4n - \frac{n\pi}{2}(5\pi + 4s) + \left(2 - \frac{n\pi^2}{2}\right)\cos(s) \\ &\geq 2 + (s+\pi)^2 + 4n - \frac{n\pi}{2}(5\pi + 4s) + \left(2 - \frac{n\pi^2}{2}\right)\left(1 - \frac{2}{\pi^2}(s - 2n\pi)^2\right) \\ &=: g_n(s). \end{aligned}$$

Moreover, g_n is a quadratic polynomial

$$g_n(s) = \left(1 + n - \frac{4}{\pi^2}\right) s^2 + 2\left(\pi - n\pi - 2n^2\pi + \frac{8n}{\pi}\right) s + \pi^2\left(4n^3 - 3n + 1\right) + 4 + 4n - 16n^2$$

and its stationary point is

$$s_n = 2n\pi - \pi - \frac{4\pi}{(n+1)\pi^2 - 4} < 2n\pi - \pi.$$

This means that g_n is strictly increasing on $[2n\pi - \pi,\, 2n\pi]$ and thus

$$D_{2,n}\left(-\frac{1}{2}, s\right) \geq g_n(s) \geq g_n(2n\pi - \pi) = 4n > 0.$$

2. Using Lemma 4.2, we get

$$\begin{aligned} D_{2,n}\left(\frac{2}{\pi^2}, s\right) &= 2 - \frac{4}{\pi^2}(s+\pi)^2 + 4n + \frac{2n}{\pi}(5\pi + 4s) + (2+2n)\cos(s) \\ &\geq 2 - \frac{4}{\pi^2}(s+\pi)^2 + 4n + \frac{2n}{\pi}(5\pi + 4s) + (2+2n)\left(\frac{2}{\pi^2}\left(s - 2n\pi + \pi\right)^2 - 1\right) \\ &=: g_n(s). \end{aligned}$$

Moreover, we have

$$g_n(s) = \frac{4n}{\pi^2}(s - 2n\pi)^2$$

and thus g_n is strictly decreasing on $[2n\pi - \pi,\, 2n\pi]$. Finally, we obtain

$$D_{2,n}\left(\frac{2}{\pi^2}, s\right) \geq g_n(s) \geq g_n(2n\pi) = 0.$$

□

Lemma 4.10. *On the closed rectangle* $\left[-\frac{1}{2},\, \frac{2}{\pi^2}\right] \times \left[2n\pi - 2\pi,\, 2n\pi - \pi\right]$, $n \in \mathbb{N}$, $n \geq 2$, *the function* $D_{1,n} = D_{1,n}(\gamma, s)$ *is non-negative and attains minimum* 0 *only at the corner point* $(\gamma, s) = (\frac{2}{\pi^2}, 2n\pi - 2\pi)$.

Proof. Using Lemmas 4.6 and 4.8, we obtain that $D_{1,n}$ restricted to the closed rectangle $R_n := \left[-\frac{1}{2},\, \frac{2}{\pi^2}\right] \times [2n\pi - 2\pi,\, 2n\pi - \pi]$ attains its extremes on the boundary ∂R_n. We have

$$\begin{aligned} D_{1,n}(\gamma, 2n\pi - \pi) &= 4n > 0, \\ D_{1,n}(\gamma, 2n\pi - 2\pi) &= 2(n-1)(2 - \gamma\pi^2) \geq 0, \\ D_{1,n}\left(-\frac{1}{2}, s\right) &= -2 - (s+\pi)^2 + 4n + \frac{n\pi}{2}(3\pi + 4s) - \left(2 + \frac{n\pi^2}{2}\right)\cos(s) > 0, \\ D_{1,n}\left(\tfrac{2}{\pi^2}, s\right) &= -2 + \frac{4}{\pi^2}(s+\pi)^2 + 4n - \frac{2n}{\pi}(3\pi + 4s) + (-2+2n)\cos(s) \geq 0. \end{aligned}$$

It remains to justify last two inequalities.

1. Using Lemma 4.2, we get

$$\begin{aligned} D_{1,n}\left(-\frac{1}{2}, s\right) &= -2 - (s+\pi)^2 + 4n + \frac{n\pi}{2}(3\pi + 4s) - \left(2 + \frac{n\pi^2}{2}\right)\cos(s) \\ &\geq -2 - (s+\pi)^2 + 4n + \frac{n\pi}{2}(3\pi + 4s) - \left(2 + \frac{n\pi^2}{2}\right)\left(1 - \frac{2}{\pi^2}\left(s - 2n\pi + 2\pi\right)^2\right) \\ &=: g_n(s). \end{aligned}$$

Now, g_n is a quadratic polynomial

$$g_n(s) = \left(n - 1 + \frac{4}{\pi^2}\right) s^2 - 2\left(\pi - 3n\pi + 2n^2\pi + \frac{8(n-1)}{\pi}\right) s + \pi^2(4n^3 - 8n^2 + 5n - 1) + 12 - 28n + 16n^2$$

and has a stationary point

$$s_n = 2\pi n - \pi - \frac{4\pi}{(n-1)\pi^2 + 4}.$$

Let us note that $s_n \in [2n\pi - 2\pi,\, 2n\pi - \pi)$. Finally, we get

$$D_{1,n}\left(-\frac{1}{2}, s\right) \geq g_n(s) \geq g_n(s_n) = \frac{(4n\pi^2 + 16)(n-1)}{(n-1)\pi^2 + 4} > 0.$$

2. Using Lemma 4.2, we get

$$\begin{aligned} D_{1,n}\left(\tfrac{2}{\pi^2}, s\right) &= -2 + \frac{4}{\pi^2}(s+\pi)^2 + 4n - \frac{2n}{\pi}(3\pi + 4s) + (-2+2n)\cos(s) \\ &\geq -2 + \frac{4}{\pi^2}(s+\pi)^2 + 4n - \frac{2n}{\pi}(3\pi + 4s) + (-2+2n)\left(\frac{2}{\pi^2}(s - 2n\pi + \pi)^2 - 1\right) \\ &=: g_n(s). \end{aligned}$$

Moreover, we have that

$$g_n(s) = \frac{4n}{\pi^2}(s - (2n\pi - 2\pi))^2$$

and thus g_n is strictly increasing on $[2n\pi - 2\pi,\, 2n\pi - \pi]$. Finally, we obtain

$$D_{1,n}\left(\tfrac{2}{\pi^2}, s\right) \geq g_n(s) \geq g_n(2n\pi - 2\pi) = 0.$$

□

Lemma 4.11. *On the closed rectangle* $\left[-\frac{1}{2}, \frac{2}{\pi^2}\right] \times [0, \pi]$, *the function* $D_{1,1} = D_{1,1}(\gamma, s)$ *is non-negative and attains its minimum only on the boundary line segment* $s = 0$ *and* $\gamma \in \left[-\frac{1}{2}, \frac{2}{\pi^2}\right]$.

Proof. Using Lemma 4.7, we obtain that $D_{1,1}$ restricted to the closed rectangle $R_1 := \left[-\frac{1}{2}, \frac{2}{\pi^2}\right] \times [0, \pi]$ attains its extremes on the boundary ∂R_1. We have

$$\begin{aligned} D_{1,1}(\gamma, \pi) &= 4 > 0, \\ D_{1,1}(\gamma, 0) &= 0, \\ D_{1,1}\left(-\frac{1}{2}, s\right) &= 2 - (s+\pi)^2 + \frac{\pi}{2}(3\pi + 4s) - \left(2 + \frac{\pi^2}{2}\right)\cos(s) \geq 0, \qquad (4.21) \\ D_{1,1}\left(\frac{2}{\pi^2}, s\right) &= 2 + \frac{4}{\pi^2}(s+\pi)^2 - \frac{2}{\pi}(3\pi + 4s) = \frac{4}{\pi^2}s^2 \geq 0. \qquad (4.22) \end{aligned}$$

It remains to justify the inequality in (4.21). Using Lemma 4.2, we get

$$\begin{aligned} D_{1,1}\left(-\frac{1}{2}, s\right) &= 2 - (s+\pi)^2 + \frac{\pi}{2}(3\pi + 4s) - \left(2 + \frac{\pi^2}{2}\right)\cos(s) \\ &\geq 2 - (s+\pi)^2 + \frac{\pi}{2}(3\pi + 4s) - \left(2 + \frac{\pi^2}{2}\right)\left(1 - \frac{2}{\pi^2}s^2\right) =: g_n(s). \end{aligned}$$

We have

$$g_n(s) = \frac{4}{\pi^2}s^2$$

amd thus, we get

$$D_{1,1}\left(-\frac{1}{2}, s\right) \geq g_n(s) \geq 0.$$

□

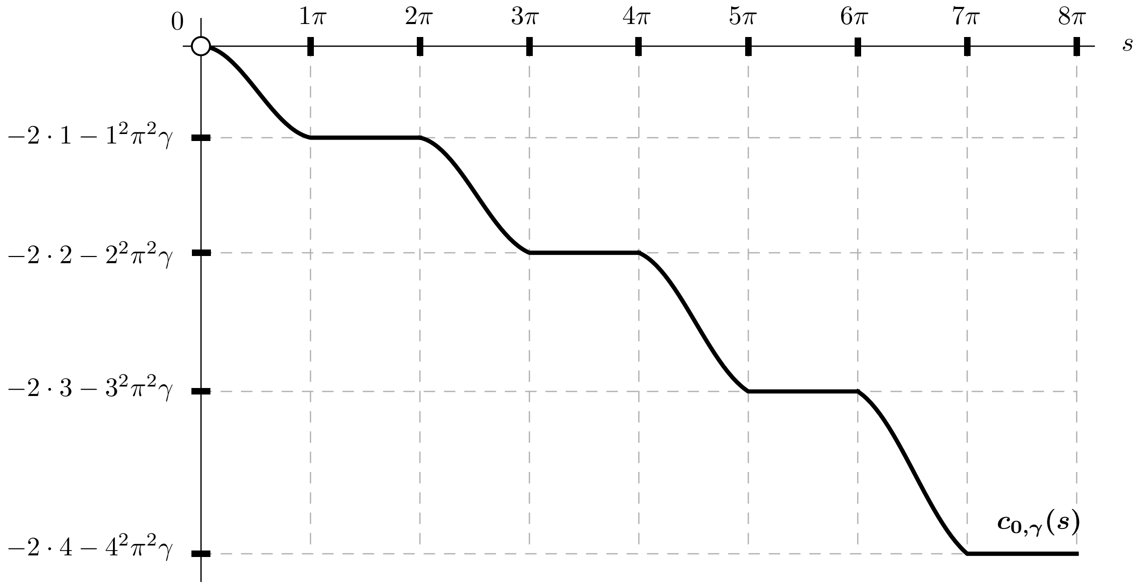


Fig. 4.6: Graph of the function $c_{0,\gamma}$ for $\gamma = 0.03$.

We have all necessary lemmas to justify the final theorem of this section.

Theorem 4.12. *The discriminant $D(\gamma, s)$ of the guadratic polynomial in* (4.18) *is positive for $s > 0$ and $-\frac{1}{2} < \gamma < \frac{2}{\pi^2}$.*

Proof. The statement is a direct consequence of Lemmas 4.9, 4.10 and 4.11. □

4.2 Solvability of the quadratic equation (4.18)

Due to Theorem 4.12, we know that roots of the quadratic polynomial in (4.18) are real. Now we can take a look on their other properties by investigating functions $c_{0,\gamma}$, $c_{1,\gamma}$ and $c_{2,\gamma}$. First of all, we are going to assume only $\gamma > 0$. The case of $\gamma \leq 0$ will be discussed at the end of this section.

Lemma 4.13. *For $0 < \gamma < \frac{2}{\pi^2}$, the function $c_{0,\gamma}$ is continuous, decreasing and its range is $(-\infty, 0)$.*

The function $c_{0,\gamma}$ is illustrated in Figure 4.6.

Proof. Firstly, let us prove the continuity of the function $c_{0,\gamma}$. It is easy to verify, that both parts of the function $c_{0,\gamma}$ are continuous with respect to s, therefore only the continuity in the points of connection remains to be examined[1]. For given $n \in \mathbb{N}$, we obtain

$$\lim_{s \to 2n\pi -} c_{0,\gamma}(s) = c_{0,\gamma}(2n\pi) = -2n - n^2\pi^2\gamma,$$

$$\lim_{s \to 2n\pi +} c_{0,\gamma}(s) = 1 + \cos(2n\pi) - 2(n+1) - (2n\pi - (n+1)\pi + \pi)^2 \gamma = -2n - n^2\pi^2\gamma,$$

and also

$$\lim_{s \to (2n\pi - \pi)-} c_{0,\gamma}(s) = c_{0,\gamma}(2n\pi - \pi) = 1 + \cos(2n\pi - \pi) - 2n - (2n\pi - \pi - n\pi + \pi)^2\gamma = -2n - n^2\pi^2\gamma,$$

$$\lim_{s \to (2n\pi - \pi)+} c_{0,\gamma}(s) = \lim_{s \to (2n\pi - \pi)+} -2n - n^2\pi^2\gamma = -2n - n^2\pi^2\gamma.$$

[1]The function f is said to be continuous at the point x_0, if there exist $f(x_0)$, $\lim_{x \to x_0+} f(x)$, $\lim_{x \to x_0-} f(x)$ and at the same time $f(x_0) = \lim_{x \to x_0+} f(x) = \lim_{x \to x_0-} f(x)$.

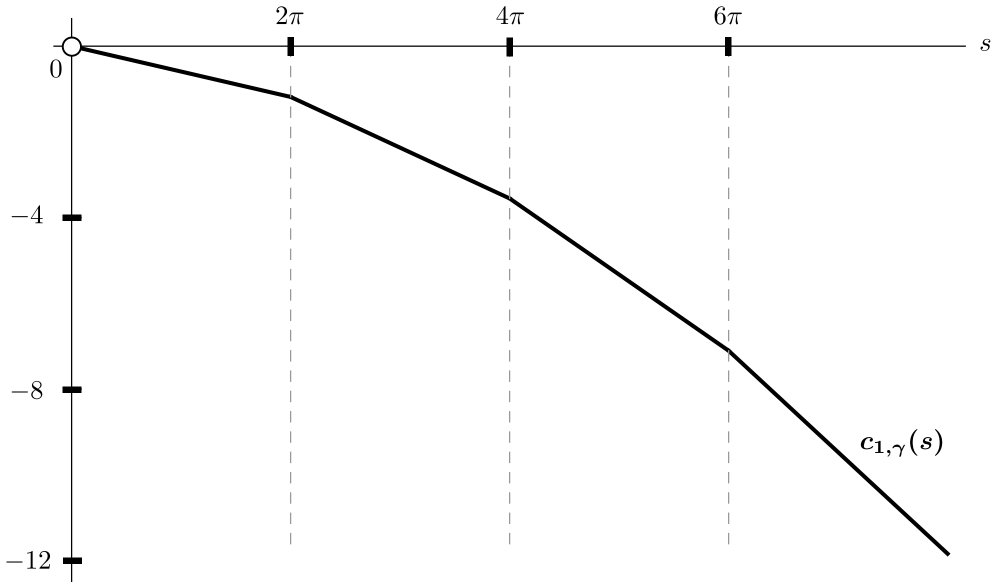


Fig. 4.7: Graph of the function $c_{1,\gamma}$ for $\gamma = 0.03$.

The function $c_{0,\gamma}$ is therefore continuous for $s \in (0, +\infty)$.

The function $c_{0,\gamma}$ is also decreasing. Since the function is given piecewise and it is constant for $s \in (2n\pi - \pi,\, 2n\pi]$, we are going to only examine $c_{0,\gamma}$ for $s \in (2n\pi - 2\pi,\, 2n\pi - \pi]$. We have $c'_{0,\gamma}(s) = -\sin(s) - 2\gamma\,(s - n\pi + \pi)$. Since $-\sin(s) \le 0$, $-2\gamma < 0$ and $s - n\pi + \pi > 0$ for $s \in (2n\pi - 2\pi,\, 2n\pi - \pi]$, we obtain $c'_{0,\gamma}(s) \le 0$, and $c_{0,\gamma}$ is indeed decreasing.

And finally, to determine the range of the function $c_{0,\gamma}$, we calculate

$$\lim_{s\to 0+} c_{0,\gamma}(s) = \lim_{s\to 0+} \left(1 + \cos s - 2 - s^2\gamma\right) = 0$$

and we also claim, that

$$\lim_{s\to +\infty} c_{0,\gamma}(s) = -\infty.$$ [2]

Let us set arbitrary $K < 0$. Then let $n_1 \in \mathbb{N}$, such that $c_{0,\gamma}(2n_1\pi) = -2n_1 - n_1^2\pi^2\gamma < K$. We can set $s_0 = 2n_1\pi$ and since the function $c_{0,\gamma}$ is decreasing, then $\forall s > s_0 : \; c_{2,\gamma}(s) \le c_{2,\gamma}(s_0) < K$, which finishes the proof.

□

The function $c_{1,\gamma}$ (see Figure 4.7) has similar properties as the function $c_{0,\gamma}$.

Lemma 4.14. *For $0 < \gamma < \frac{2}{\pi^2}$, the function $c_{1,\gamma}$ is continuous, decreasing and its range is $(-\infty, 0)$.*

Proof. First, let us examine the continuity of the function $c_{1,\gamma}$. The continuity on the interval $(2n\pi - 2\pi, 2n\pi]$ is clear, points of connection remain to be examined. Indeed, we have

$$\lim_{s\to 2n\pi-} c_{1,\gamma}(s) = c_{1,\gamma}(2n\pi) = -2n\pi\,(n\pi + \pi)\,\gamma = -2n^2\pi^2\gamma - 2n\pi^2\gamma,$$

$$\lim_{s\to 2n\pi+} c_{1,\gamma}(s) = -2(n+1)\pi\,(2n\pi - (n+1)\pi + \pi)\,\gamma = -2n^2\pi^2\gamma - 2n\pi^2\gamma,$$

and the function $c_{1,\gamma}$ is therefore continuous for $s > 0$.

The function $c_{1,\gamma}$ is also decreasing, since

$$c'_{1,\gamma}(s) = -2n\pi\gamma < 0 \quad \text{for } 2n\pi - 2\pi < s < 2n\pi.$$

[2]The function f diverges to $-\infty$ for $s \to +\infty$ by definition if $\forall K < 0\, \exists s_0 \in \mathbb{R}^+\, \forall s > s_0 : \; f(s) < K$.

And finally, let us examine the range of the function $c_{1,\gamma}$. We have

$$\lim_{s\to 0+} c_{1,\gamma}(s) = \lim_{s\to 0+} -2\pi\gamma s = 0$$

and

$$\lim_{n\to +\infty} c_{1,\gamma}(2n\pi) = \lim_{n\to +\infty} -2n\pi(n+1)\pi\gamma = -\infty.$$

Therefore $\lim_{s\to +\infty} c_{1,\gamma}(s) = -\infty$ due to the monotony of the function $c_{1,\gamma}$ and thus the range of $c_{1,\gamma}$ is $(-\infty, 0)$. □

The investigation of properties of the function $c_{2,\gamma}$ is going to be more challenging than for $c_{0,\gamma}$ and $c_{1,\gamma}$ and will be therefore split into several lemmas, however, this knowledge is critial as shown in the following lemma.

Lemma 4.15. *For $0 < \gamma < \frac{2}{\pi^2}$ and $s > 0$, the quadratic equation (4.18) is solvable for $k > 0$ if and only if $c_{2,\gamma}(s) > 0$. Moreover, if $c_{2,\gamma}(s) > 0$ then the positive solution k of (4.18) is unique.*

Proof. For $c_{2,\gamma}(s) = 0$, the quadratic equation (4.18) simplifies to the linear one and we have

$$k = -\frac{c_{0,\gamma}(s)}{c_{1,\gamma}(s)},$$

since $c_{1,\gamma}(s) \neq 0$ due to Lemma 4.14. The solution k is therefore negative, since $c_{0,\gamma}(s) < 0$ and $c_{1,\gamma}(s) < 0$ according to Lemmas 4.13 and 4.14.

Now for $c_{2,\gamma}(s) \neq 0$, let the quadratic equation (4.18) have solutions k_1 and k_2. We have according to Vieta's formulas in [3], that

$$k_1 \cdot k_2 = \frac{c_{0,\gamma}(s)}{c_{2,\gamma}(s)}, \qquad k_1 + k_2 = -\frac{c_{1,\gamma}(s)}{c_{2,\gamma}(s)}. \tag{4.23}$$

The rest of the proof is going to be split according to the sign of $c_{2,\gamma}(s)$.

1. Let $c_{2,\gamma}(s) < 0$. Then, since $c_{0,\gamma}(s)$ is negative for $s > 0$ due to Lemma 4.13, we have $k_1, k_2 < 0$ or $k_1, k_2 > 0$ from the first equation in (4.23). At the same time, using the second equation in (4.23) and Lemma 4.14, we obtain that $k_1 + k_2 < 0$. Therefore $k_1 < 0$ and $k_2 < 0$ and the equation (4.18) does not have a positive solution k.
2. Let $c_{2,\gamma}(s) > 0$. Then, using Lemma 4.13 and the first equation in (4.23), we get $k_1 < 0$, $k_2 > 0$ or $k_1 > 0$, $k_2 < 0$. Therefore there is exactly one positive solution k of the equation (4.18).

□

Lemma 4.16. *For $0 < \gamma < \frac{2}{\pi^2}$, the function $c_{2,\gamma}$ is continuous.*

Proof. The continuity of the function $c_{2,\gamma}$ is easy to verify, since

$$\lim_{s\to(2n\pi-\pi)-} c_{2,\gamma}(s) = c_{2,\gamma}(2n\pi - \pi) = 2n - n^2\pi^2\gamma,$$

which is the same value as

$$\begin{aligned}\lim_{s\to(2n\pi-\pi)+} c_{2,\gamma}(s) &= 1 - 1 + 2n - (2n\pi - \pi - n\pi + \pi)^2\gamma \\ &= 2n - (n\pi)^2\gamma.\end{aligned}$$

In the second point of connection, we have

$$\begin{aligned}\lim_{s\to 2n\pi-} c_{2,\gamma}(s) &= c_{2,\gamma}(2n\pi) \\ &= 1 + 1 + 2n - (2n\pi - n\pi + \pi)^2\gamma \\ &= 2 + 2n - (n\pi + \pi)^2\gamma,\end{aligned}$$

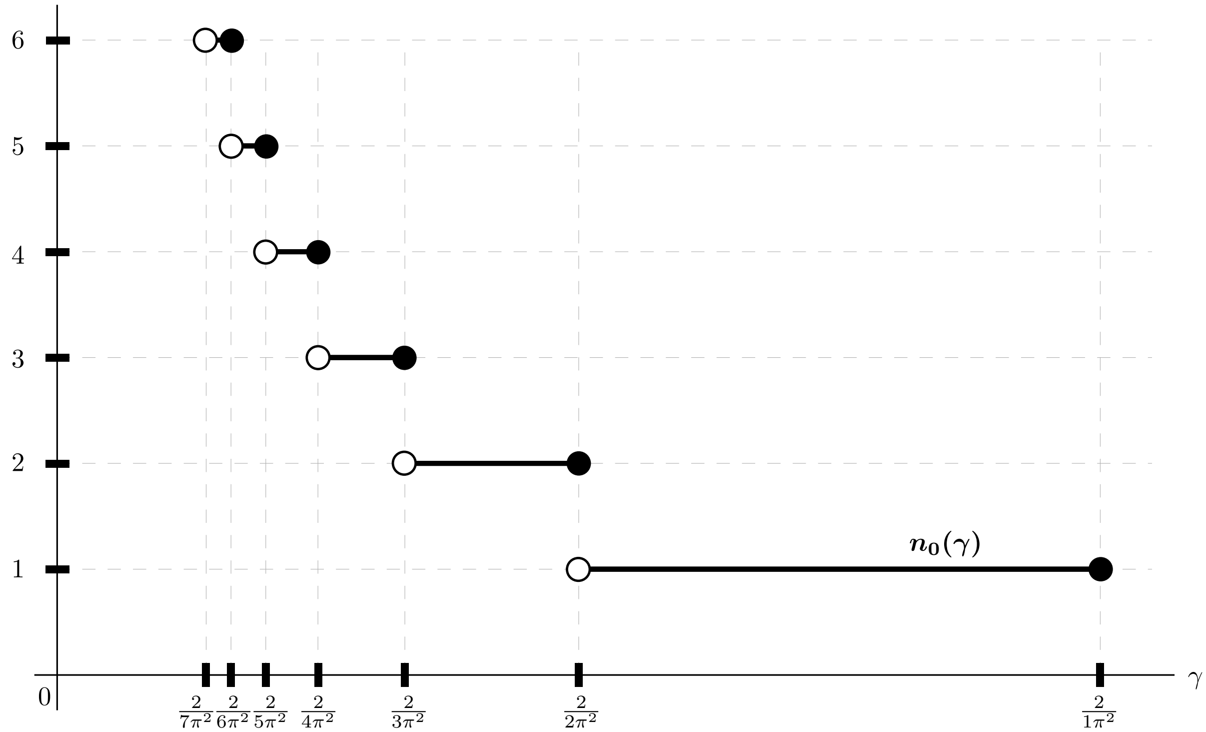


Fig. 4.8: Graph of the function $n_0(\gamma)$, introduced in Definition 4.17.

and

$$\lim_{s\to 2n\pi+} c_{2,\gamma}(s) = 2(n+1) - (n+1)^2\pi^2\gamma$$
$$= 2 + 2n - (n\pi+\pi)^2\,\gamma.$$

□

In the following definition, we introduce the important natural number n_0, that depends on γ. We will show later, that this value n_0 determines the interval for s, on which the function $c_{2,\gamma}$ changes sign.

Definition 4.17. *For $0 < \gamma < \frac{2}{\pi^2}$, let us define the value*

$$n_0 := \left\lfloor \frac{2}{\pi^2\gamma} \right\rfloor .$$

The relation between the value of γ and n_0 is illustrated on Figure 4.8. By a simple manipulation of the definition of n_0, we obtain

$$n_0 \le \frac{2}{\pi^2\gamma} < n_0 + 1,$$
$$\frac{2}{(n_0+1)\pi^2} < \gamma \le \frac{2}{n_0\pi^2}. \tag{4.24}$$

In the following four lemmas, we are going to investigate the sign and the monotony of the function $c_{2,\gamma}$.

Lemma 4.18. *For $0 < \gamma \le \frac{2}{2\pi^2}$ we have*

$$c_{2,\gamma}(s) > 0 \qquad \textit{for } s \in (0,\, 2(n_0-1)\pi)\,.$$

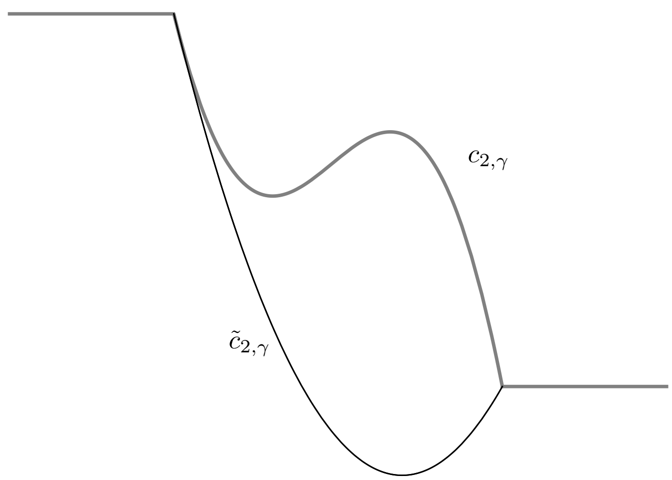


Fig. 4.9: Detail of the graph of functions $c_{2,\gamma}$ (gray curve) and its lower estimate $\tilde{c}_{2,\gamma}$ (black curve).

Proof. For $0 < \gamma \leq \frac{2}{2\pi^2}$, we have $n_0 \geq 2$ according to Definition 4.17. Since the function $c_{2,\gamma}$ is given piecewice, we are going to split the proof into two sections.

At first, let $s \in (2n\pi - 2\pi,\, 2n\pi - \pi]$ and $n \in \{1, 2, \ldots, n_0 - 1\}$. Then we have

$$\begin{aligned} c_{2,\gamma}(s) = 2n - n^2\pi^2\gamma > 0, \\ \frac{2}{n\pi^2} > \gamma. \end{aligned}$$

The last inequality is true due to (4.24) and $n \leq n_0 - 1$.

In the second part, let us assume $s \in (2n\pi - \pi,\, 2n\pi)$ and $n \in \{1, 2, \ldots, n_0 - 1\}$. Then, using the Lemma 4.2, we obtain a lower estimate of $c_{2,\gamma}(s)$ in the following way

$$c_{2,\gamma}(s) = 1 + \cos(s) - 2n - (s - n\pi + \pi)^2\,\gamma > \frac{2}{\pi^2}(s + \pi)^2 - 2n - (s - n\pi + \pi)^2\,\gamma =: \tilde{c}_{2,\gamma}(s).$$

See Figure 4.9 for a comparison between functions $c_{2,\gamma}$ and $\tilde{c}_{2,\gamma}$. It holds, that $\tilde{c}'_{2,\gamma}(s) = \frac{4}{\pi^2}(s + \pi) - 2\gamma(s - n\pi + \pi)$ and the stationary point s_0 of the function $\tilde{c}_{2,\gamma}$ can be determined as

$$\begin{aligned} \tilde{c}'_{2,\gamma}(s_0) &= 0, \\ \tfrac{4}{\pi^2}(s_0 + \pi) - 2\gamma(s_0 - n\pi + \pi) &= 0, \\ s_0\left(\frac{4}{\pi^2} - 2\gamma\right) &= -\frac{4}{\pi} - 2\gamma n\pi + 2\gamma\pi, \\ s_0 &= \frac{\pi\left(2 - \pi^2\gamma + n\pi^2\gamma\right)}{-2 + \pi^2\gamma}. \end{aligned}$$

For the minimum of the function $\tilde{c}_{2,\gamma}$, we have

$$\min_{s\in\mathbb{R}} \tilde{c}_{2,\gamma}(s) = \tilde{c}_{2,\gamma}(s_0) = 2n\left(1 + n + \frac{2n}{\pi^2\gamma - 2}\right) > 0 \quad \text{for } \gamma \neq \frac{2}{n_0\pi^2} \text{ or } n < n_0 - 1. \tag{4.25}$$

It remains to justify, that $\min_{s\in\mathbb{R}} \tilde{c}_{2,\gamma}(s)$ is positive. Let us manipulate

$$\begin{aligned} 1+n+\frac{2n}{\pi^2\gamma-2} &> 0,\\ 2n &< (-1-n)(\pi^2\gamma-2),\\ \pi^2\gamma &< \frac{2n}{-n-1}+\frac{-2n-2}{-n-1},\\ \gamma &< \frac{2}{(n+1)\pi^2}. \end{aligned}$$

The last inequality holds due to (4.24).

And finally, for the case not covered by the (4.25), i.e. for $n = n_0 - 1$ and $\gamma = \frac{2}{n_0\pi^2}$, we can proceed in the similar manner as in the previous case and we get the stationary point s_0 of the function $\tilde{c}_{2,\gamma}$ as

$$\begin{aligned} \tilde{c}'_{2,\gamma}(s_0) &= 0,\\ \frac{4}{\pi^2}(s_0+\pi)-\frac{4}{n_0\pi^2}(s_0-(n_0-1)\pi+\pi) &= 0,\\ s_0 &= -2\pi. \end{aligned}$$

$$\min_{s\in\mathbb{R}} \tilde{c}_{2,\gamma}(s) = \tilde{c}_{2,\gamma}(s_0) = 2(n_0-1)\left(1+(n_0-1)+\frac{2(n_0-1)}{\pi^2\frac{2}{n_0\pi^2}-2})\right) = 2(n_0-1)(n_0-n_0) = 0$$

Since $\tilde{c}_{2,\gamma}$ is the quadratic function with the minimum 0 in -2π, the value $\tilde{c}_{2,\gamma}(s)$ is positive for every point $s \neq -2\pi$.

□

Lemma 4.19. *For $0 < \gamma < \frac{2}{\pi^2}$, we have*

$$c_{2,\gamma}(s) < 0 \qquad \text{for } s > 2n_0\pi.$$

Proof. Firstly, in the case of $s \in (2n\pi - 2\pi,\, 2n\pi - \pi]$, $n \in \mathbb{N}$, $n > n_0$, the function $c_{2,\gamma}$ is constant and we have

$$c_{2,\gamma}(s) = 2n - n^2\pi^2\gamma.$$

For this value to be negative, we require that

$$2n - n^2\pi^2\gamma < 0,$$

$$\frac{2}{n\pi^2} < \gamma.$$

This is true according to (4.24). This specificaly means, that on the first interval for $s > 2n_0\pi$, i.e. on the interval $2n_0\pi < s \leq 2n_0\pi + \pi$, the function $c_{2,\gamma}$ is already negative.

In the case of $s \in (2n\pi - \pi,\, 2n\pi)$, $n \in \mathbb{N}$, $n > n_0$, we obtain

$$c'_{2,\gamma}(s) = -\sin(s) - 2\gamma\,(s - n\pi + \pi)\,.$$

Since $0 < -\sin(s) < 1$ and our intent is to show that the function $c_{2,\gamma}$ is strictly decreasing, we require $2\gamma\,(s - n\pi + \pi) > 1$ or after a short manipulation $s - n\pi + \pi > \frac{1}{2\gamma}$ in order to $c'_{2,\gamma}$ to be negative. Based on the inequality (4.24) and the relation $n > n_0$ we have

$$\frac{1}{2n\pi} \leq \frac{1}{2(n_0+1)\pi} < \frac{1}{2(n_0+1)\pi}\cdot\frac{4}{\pi} < \gamma$$

and using the interval of s, we also have $n\pi < s - n\pi + \pi$. Let us summarize

$$\frac{1}{2\gamma} < n\pi < s - n\pi + \pi,$$

and thus $c'_{2,\gamma}(s) < 0$ for $s \in (2n\pi - \pi,\, 2n\pi)\,,\, n \in \mathbb{N},\, n > n_0$.

To sum up, the function $c_{2,\gamma}$ has negative values for $2n_0\pi < s \le 2n_0\pi + \pi$ and is always decreasing for $s > 2n_0\pi$. The assertion of this lemma is therefore justified. □

Lemma 4.20. *For $0 < \gamma \le \frac{2}{2\pi^2}$, the function $c_{2,\gamma}$ is strictly decreasing on $(2n_0\pi - \pi, 2n_0\pi)$.*

Proof. First of all, let us recall that n_0 is given in Definition 4.17 and that for $0 < \gamma \le \frac{2}{2\pi^2}$, we have that $n_0 \ge 2$. Now, for $s \in (2n_0\pi - \pi, 2n_0\pi)$, we have that

$$\begin{aligned} c_{2,\gamma}(s) &= 1 + 2n_0 - (s - n_0\pi + \pi)^2\gamma + \cos(s), \\ c'_{2,\gamma}(s) &= -2(s - n_0\pi + \pi)\gamma - \sin(s), \\ c''_{2,\gamma}(s) &= -2\gamma - \cos(s). \end{aligned}$$

Let us denote $s_0 := 2n_0\pi - \arccos(-2\gamma)$. Then $c''_{2,\gamma}(s_0) = 0$ and moreover, we have that

$$\begin{aligned} \max_{s\in(2n_0\pi-\pi,2n_0\pi)} c'_{2,\gamma}(s) &= c'_{2,\gamma}(s_0) \\ &= -2(n_0\pi - \arccos(-2\gamma) + \pi)\cdot\gamma - \sin\left(-\arccos(-2\gamma)\right) \\ &= \sqrt{1-4\gamma^2} - 2(n_0+1)\pi\gamma + 2\gamma\arccos(-2\gamma). \end{aligned} \tag{4.26}$$

Let us note that s_0 is the strict maximum point of $c'_{2,\gamma}$ on the interval $(2n_0\pi - \pi, 2n_0\pi)$. Now, let us consider the function $\gamma \mapsto c'_{2,\gamma}(s_0)$, $\gamma \in \left(\frac{2}{(n_0+1)\pi^2}, \frac{2}{n_0\pi^2}\right)$, where $n_0 \ge 2$ is fixed. We claim that the function $\gamma \mapsto c'_{2,\gamma}(s_0)$ is strictly decreasing. Indeed, using (4.26), we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\gamma} c'_{2,\gamma}(s_0) &= -2(s_0 - n_0\pi + \pi) \\ &= 2(\arccos(-2\gamma) - (n_0+1)\pi) < 0. \end{aligned}$$

Thus, the supremum of $c'_{2,\gamma}(s_0)$ over all $\gamma \in \left(\frac{2}{(n_0+1)\pi^2}, \frac{2}{n_0\pi^2}\right)$ is reached at the left endpoint $\gamma_{\mathrm{L}} := \frac{2}{(n_0+1)\pi^2}$

$$\sup_{\gamma} c'_{2,\gamma}(s_0) = \lim_{\gamma\to\gamma_{\mathrm{L}}+} c'_{2,\gamma}(s_0) = \sqrt{1-4\gamma_{\mathrm{L}}^2} - 2(n_0+1)\pi\gamma_{\mathrm{L}} + 2\gamma_{\mathrm{L}}\arccos(-2\gamma_{\mathrm{L}}).$$

Finally, we show that this supremum is negative

$$\sqrt{1-4\gamma_{\mathrm{L}}^2} - 2(n_0+1)\pi\gamma_{\mathrm{L}} + 2\gamma_{\mathrm{L}}\arccos(-2\gamma_{\mathrm{L}}) < 0. \tag{4.27}$$

Indeed, taking into account that the first term $\sqrt{1-4\gamma_{\mathrm{L}}^2}$ in (4.27) is strictly less than 1, it is enough to justify the following inequality

$$\begin{aligned} -2(n_0+1)\pi\gamma_{\mathrm{L}} + 2\gamma_{\mathrm{L}}\arccos(-2\gamma_{\mathrm{L}}) &< -1, \\ -\frac{4}{\pi} + \frac{4}{(n_0+1)\pi^2}\arccos\left(-\frac{4}{(n_0+1)\pi^2}\right) &< -1, \\ \arccos\left(-\frac{4}{(n_0+1)\pi^2}\right) &< \left(-1 + \frac{4}{\pi}\right)\frac{(n_0+1)\pi^2}{4}, \\ \arccos\left(-\frac{4}{(n_0+1)\pi^2}\right) &< \left(1 - \frac{\pi}{4}\right)(n_0+1)\pi. \end{aligned} \tag{4.28}$$

We have that $\left(1 - \frac{\pi}{4}\right) \doteq 0.215$ and thus, (4.28) holds for $n_0 \ge 4$. For $n_0 = 2$ and $n_0 = 3$, we can verify the inequality (4.28) numerically. □

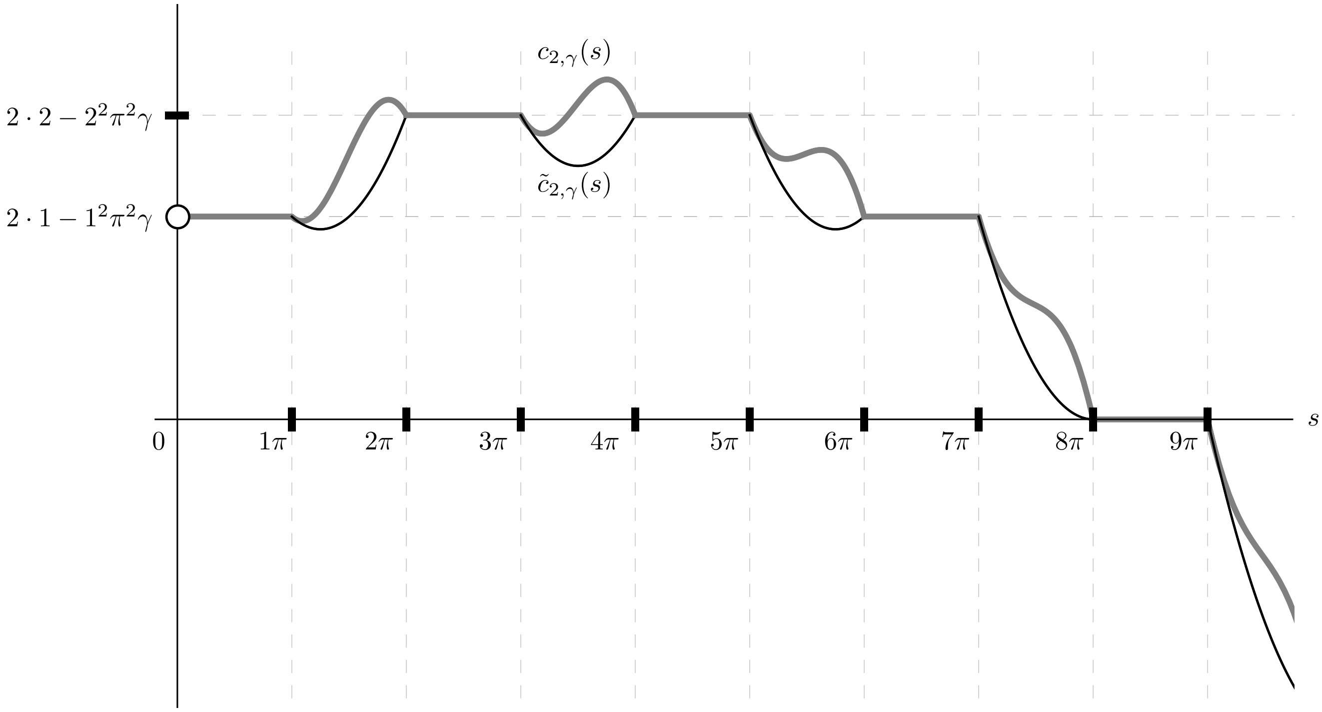


Fig. 4.10: Graph of the function $c_{2,\gamma}$ for $s > 0$ (gray curve) and graph of its lower estimate $\tilde{c}_{2,\gamma}$ for $s \in (2n\pi - \pi,\, 2n\pi)$, $n \in \mathbb{N}$ (black curves) for $\gamma = \frac{2}{5\pi^2}$.

Lemma 4.21. *For $\gamma = \frac{2}{n_0\pi^2}$, $n_0 \geq 2$, we have that*

$$\begin{aligned} c_{2,\gamma}(s) &= 0 \qquad \text{for } s \in [2n_0\pi - 2\pi,\, 2n_0\pi - \pi]\,, \\ c_{2,\gamma}(s) &< 0 \qquad \text{for } s \in (2n_0\pi - \pi,\, 2n_0\pi)\,. \end{aligned}$$

Proof. We are going to prove first, that $c_{2,\gamma}(s) = 0$ for $s \in [2n_0\pi - 2\pi,\, 2n_0\pi - \pi]$. At the point $2n_0\pi - 2\pi$, we have

$$c_{2,\gamma}(2n_0\pi - 2\pi) = 1 + \cos(2n_0\pi - 2\pi) + 2(n_0 - 1) - (2n_0\pi - 2\pi - (n_0 - 1)\pi + \pi)^2 \frac{2}{n_0\pi^2} = 0.$$

For $s \in (2n_0\pi - 2\pi,\, 2n_0\pi - \pi]$, the value of $c_{2,\gamma}(s)$ is

$$c_{2,\gamma}(s) = 2n_0 - n_0^2\pi^2 \cdot \frac{2}{n_0\pi^2} = 0.$$

To prove the negativity of the function $c_{2,\gamma}$ on the interval $(2n_0\pi - \pi,\, 2n_0\pi)$, let us recall, that

$$c_{2,\gamma}(2n_0\pi - \pi) = 0$$

and the function $c_{2,\gamma}$ is strictly decreasing on $(2n_0\pi - \pi,\, 2n_0\pi)$ according to the Lemma 4.20. □

See Figure 4.10 for the case of $\gamma = \frac{2}{n_0\pi^2}$, where the function $c_{2,\gamma}$ has the value zero on the entire interval $[2n_0\pi - 2\pi,\, 2n_0\pi - \pi]$.

A slight complication will appear for $\gamma > \frac{2}{2\pi^2}$ close to the value of $\frac{2}{2\pi^2}$, since in that case, the function $c_{2,\gamma}$ is generaly not monotone on $(2n_0\pi - 2\pi,\, 2n_0\pi)$. More careful investigation is therefore needed in order to show the uniqueness of the zero point of the function $c_{2,\gamma}$. Let us denote

$$\gamma_1 := -\frac{\cos(s_1)}{2},$$

where s_1 is the unique solution of the equation $\tan(s) = s$ on the interval $(\pi,\, 2\pi)$. Let us note that $s_1 \doteq 4.4934$ and $\gamma_1 \doteq 0.1086$.

For $s \in (\pi,\, 2\pi)$, we have that

$$\begin{aligned} c_{2,\gamma}(s) &= 3 - s^2\gamma + \cos s, \\ c'_{2,\gamma}(s) &= -2s\gamma - \sin s, \\ c''_{2,\gamma}(s) &= -2\gamma - \cos s. \end{aligned}$$

Let us denote $s_0 := 2\pi - \arccos\left(-2\gamma\right)$. Then $c''_{2,\gamma}(s_0) = 0$ and moreover, we have that

$$\begin{aligned} \max_{s\in(\pi,2\pi)} c'_{2,\gamma}(s) &= c'_{2,\gamma}(s_0) \\ &= -2\gamma(2\pi - \arccos\left(-2\gamma\right)) - \sin(2\pi - \arccos\left(-2\gamma\right)) \\ &= \sqrt{1-4\gamma^2} - 4\pi\gamma + 2\gamma\arccos(-2\gamma). \end{aligned}$$

Let us note that s_0 is the strict maximum point of $c'_{2,\gamma}$ on the interval $(\pi,\, 2\pi)$. Now, we have that the function $\gamma \mapsto c'_{2,\gamma}(s_0)$ is strictly decreasing, since

$$\tfrac{\mathrm{d}}{\mathrm{d}\gamma} c'_{2,\gamma}(s_0) = -2s_0 = -4\pi + \arccos(-2\gamma) < 0.$$

We also have that

$$c'_{2,\gamma_1}(s_0) = 4\pi\frac{\cos(s_1)}{2} - \cos(s_1)\cdot s_1 + \sin(s_1) = 0,$$

which is justified, since we have

$$\begin{aligned} 4\pi\frac{\cos(s_1)}{2} - \cos(s_1)\cdot s_1 + \sin(s_1) &= 0 \\ 2\pi + \frac{\sin(s_1)}{\cos(s_1)} &= s_1 \end{aligned}$$

and the last equation holds true due to the definition of the value s_1. Thus, for $\frac{2}{2\pi^2} < \gamma < \gamma_1$, we have $c'_{2,\gamma}(s_0) > 0$. Moreover, for $\gamma_1 < \gamma < \frac{2}{\pi^2}$, we have

$$c'_{2,\gamma}(s_0) < 0. \tag{4.29}$$

In the following lemma, we are going to introduce an estimate from below of the function cos, which is similar to the estimate in Lemma 4.2 and will become useful later.

Lemma 4.22. *For $s \in \left(\pi,\, \frac{7\pi}{4}\right)$, we have that*

$$\tfrac{8(2+\sqrt{2})}{9\pi^2}(s-\pi)^2 < 1 + \cos s.$$

Proof. Let us define $f(s) := \frac{8(2+\sqrt{2})}{9\pi^2}(s-\pi)^2 - 1 - \cos s$ for $s \in \left(\pi,\, \frac{7\pi}{4}\right)$. Our goal is to show that f is negative on $\left(\pi,\, \frac{7\pi}{4}\right)$.

We have

$$\begin{aligned} f'(s) &= \frac{16(2+\sqrt{2})}{9\pi^2}(s-\pi) + \sin s, \\ f''(s) &= \frac{16(2+\sqrt{2})}{9\pi^2} + \cos s. \end{aligned}$$

The function f'' has one zero point $s_0 = 2\pi - \arccos\left(-\frac{16(2+\sqrt{2})}{9\pi^2}\right)$ on $\left(\pi,\, \frac{7\pi}{4}\right)$, is negative on $(\pi,\, s_0)$ and positive on $\left(s_0,\, \frac{7\pi}{4}\right)$. Therefore the function f' is strictly decreasing on $(\pi,\, s_0)$ and strictly increasing on $\left(s_0,\, \frac{7\pi}{4}\right)$.

Since we have $\lim_{s\to\pi+} f'(s) = 0$ and $\lim_{s\to\frac{7\pi}{4}-} f'(s) = \frac{16(2+\sqrt{2})}{9\pi^2}\frac{3\pi}{4} - \frac{1}{\sqrt{2}} = \frac{16+8\sqrt{2}-3\pi\sqrt{2}}{6\pi} > 0$, there is exactly one zero point $s_1 \in \left(\pi,\, \frac{7\pi}{4}\right)$ of the function f' and the function f' is negative on $(\pi,\, s_1)$ and positive on $\left(s_1,\, \frac{7\pi}{4}\right)$.

Lastly, the function f has limits $\lim_{s\to\pi+} f(s) = 0 - 1 + 1 = 0$, $\lim_{s\to\frac{7\pi}{4}-} f(s) = 1 + \frac{\sqrt{2}}{2} - 1 - \frac{1}{\sqrt{2}} = 0$ and is strictly decreasing on $(\pi,\, s_1)$ and strictly increasing on $\left(s_1,\, \frac{7\pi}{4}\right)$. Therefore $f(s) < 0$ for $s \in \left(\pi,\, \frac{7\pi}{4}\right)$. □

Now we can proceed with the investigation in the case of $\frac{2}{2\pi^2} < \gamma \le \gamma_1$, where the function $c_{2,\gamma}$ is not generaly monotone on the interval $(0,\, 2\pi]$, more detailed aproach is therefore needed.

Lemma 4.23. *For $\frac{2}{2\pi^2} < \gamma \le \gamma_1$, there exists exactly one $s^* \in (0,\, 2\pi]$ such that $c_{2,\gamma}(s^*) = 0$. Moreover, the function $c_{2,\gamma}$ is positive on $(0,\, s^*)$ and negative on $(s^*,\, 2\pi]$.*

Proof. For $s \in (0,\, \pi]$, we have that $c_{2,\gamma}(s) = 2 - \pi^2\gamma > 0$. For $s \in (\pi,\, 2\pi]$, we have $c_{2,\gamma}(s) = 3 - s^2\gamma + \cos(s)$, which is not generally monotone. However, we will show that $c_{2,\gamma}(s) > 0$ for $s \in \left(\pi,\, \frac{7\pi}{4}\right]$ and that $c_{2,\gamma}$ is strictly decreasing on $\left(\frac{7\pi}{4},\, 2\pi\right)$ and $c_{2,\gamma}(2\pi) < 0$, therefore there is exactly one zero point $s^* \in (0,\, 2\pi]$ of $c_{2,\gamma}$.

1. For $\pi < s < \frac{7\pi}{4}$, we have using Lemma 4.22 that

$$c_{2,\gamma}(s) = 1 + \cos(s) + 2 - s^2\gamma > \frac{8\left(2+\sqrt{2}\right)}{9\pi^2}(s-\pi)^2 + 2 - s^2\gamma =: g_\gamma(s).$$

   To show that the function $c_{2,\gamma}$ is positive, we investigate the minimum of the function g_γ. The function g_γ is quadratic and $g'_\gamma(s) = \frac{16(2+\sqrt{2})}{9\pi^2}(s-\pi) - 2s\gamma$, therefore the minimum is reached at $s = \frac{8(2+\sqrt{2})\pi}{16+8\sqrt{2}-9\pi^2\gamma}$ and we have

$$\min_{s\in\mathbb{R}} g_\gamma(s) = \frac{16(2+\sqrt{2}) - 2(17+4\sqrt{2})\pi^2\gamma}{8(2+\sqrt{2}) - 9\pi^2\gamma}.$$

   The minimum $\min_{s\in\mathbb{R}} g_\gamma(s)$ is positive, since both numerator and denominator are positive. At the right end point of the interval $(\pi,\, \frac{7\pi}{4})$ we also have

$$c_{2,\gamma}\left(\frac{7\pi}{4}\right) = 2 + \frac{2+\sqrt{2}}{2} - \frac{49\pi^2\gamma}{16} > 0.$$

2. Now, let us verify that $c_{2,\gamma}$ is strictly decreasing on $(\frac{7\pi}{4},\, 2\pi)$. We have

$$\frac{7\pi}{4} > \frac{3\pi}{2} > s_0 = 2\pi - \arccos(-2\gamma),$$

   where s_0 is the zero point of the function $c''_{2,\gamma}$ and also the strict maximum point of $c'_{2,\gamma}$ on $(\pi,\, 2\pi)$. Moreover the function $c'_{2,\gamma}$ is decreasing on $(s_0,\, 2\pi)$. Since

$$c'_{2,\gamma}\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \frac{7\pi\gamma}{2} < 0,$$

   we have for $s \in (\frac{7\pi}{4},\, 2\pi)$ that $c'_{2,\gamma}(s) < 0$ and $c_{2,\gamma}$ is strictly decreasing.

3. Finally, we have $c_{2,\gamma}(2\pi) = 3 - 4\pi^2\gamma + \cos(2\pi) = 4(1 - \pi^2\gamma) < 0$.

□

Now we can formulate the final lemma about the sign and zero point of the function $c_{2,\gamma}$. See Figure 4.11 for an example of such function $c_{2,\gamma}$.

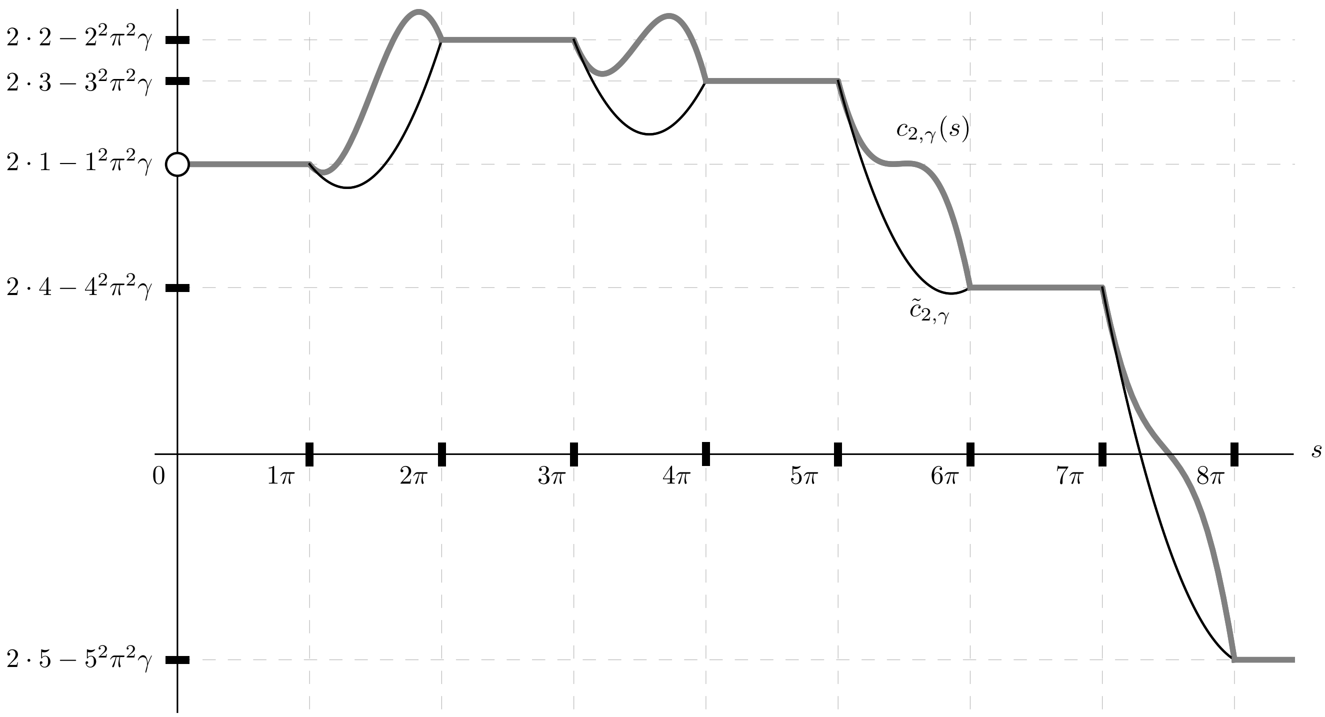


Fig. 4.11: Graph of the function $c_{2,\gamma}$ for $s > 0$ (gray curve) and graph of its lower estimate $\tilde{c}_{2,\gamma}$ for $s \in (2n\pi - \pi,\, 2n\pi)$, $n \in \mathbb{N}$ (black curves) for $\gamma = 0.045$.

Lemma 4.24. *For $0 < \gamma < \frac{2}{\pi^2}$ such that $\gamma \neq \frac{2}{n_0\pi^2}$, we have that*

$$c_{2,\gamma}(s) > 0 \qquad \text{for } s \in [2n_0\pi - 2\pi,\, 2n_0\pi - \pi]$$

and $c_{2,\gamma}$ changes sign on $(2n_0\pi - \pi,\, 2n_0\pi]$. Moreover, there exists exactly one

$$s^* \in (2n_0\pi - \pi,\, 2n_0\pi] : \, c_{2,\gamma}(s^*) = 0.$$

Proof. We are going to split the proof into four sections, in the first part we are going to prove, that $c_{2,\gamma}(s) > 0$ for $s \in [2n_0\pi - 2\pi,\, 2n_0\pi - \pi]$, then we prove that the function $c_{2,\gamma}$ changes sign on $(2n_0\pi - \pi,\, 2n_0\pi]$ for $\gamma \in \left(0,\, \frac{2}{2\pi^2}\right]$, in the third part we prove that the function $c_{2,\gamma}$ changes sign on $(2n_0\pi - \pi,\, 2n_0\pi]$ for $\gamma \in \left(\frac{2}{2\pi^2}, \gamma_1\right]$ and in the last part, we show that $c_{2,\gamma}$ changes sign on $(2n_0\pi - \pi,\, 2n_0\pi]$ for $\gamma \in \left(\gamma_1,\, \frac{2}{\pi^2}\right)$ as well.

1. Firstly, to prove that $c_{2,\gamma}(s) > 0$ for $s \in [2n_0\pi - 2\pi,\, 2n_0\pi - \pi]$, let us recall that the function $c_{2,\gamma}$ is constant on $[2n_0\pi - 2\pi,\, 2n_0\pi - \pi]$ and

$$c_{2,\gamma}(s) = 2n_0 - n_0^2\pi^2\gamma = n_0\left(2 - n_0\pi^2\gamma\right),$$

where $2 - n_0\pi^2\gamma > 0$, since it means $\gamma < \frac{2}{n_0\pi^2}$, which is true according to (4.24).

2. Let $\gamma \in \left(0,\, \frac{2}{2\pi^2}\right]$. Then we have $c_{2,\gamma}(2n_0\pi - \pi) > 0$, due to the previous part 1. We also have that

$$c_{2,\gamma}(2n_0\pi) = 2(n_0 + 1) - (n_0 + 1)^2\pi^2\gamma < 0.$$

The last inequality holds true, since it can be written as

$$\gamma > \frac{2}{(n_0 + 1)\pi^2},$$

which is true due to (4.24). Since the function $c_{2,\gamma}$ is strictly decreasing on $(2n_0\pi - \pi,\, 2n_0\pi)$ due to Lemma 4.20, we have that $c_{2,\gamma}$ changes sign and has exactly one zero point s^* on $(2n_0\pi - \pi,\, 2n_0\pi)$.

3. Let $\gamma \in \left(\frac{2}{2\pi^2}, \gamma_1\right]$. Then $n_0 = 1$ according to Definition 4.17 and the existence and uniqueness of the zero point s^* of the function $c_{2,\gamma}$ on the interval $(2n_0\pi - \pi,\, 2n_0\pi] = (\pi,\, 2\pi]$ is guaranteed by Lemma 4.23.

4. Let $\gamma \in \left(\gamma_1,\, \frac{2}{\pi^2}\right)$. In this case, the function $c_{2,\gamma}$ is strictly decreasing on $(\pi,\, 2\pi)$ due to (4.29). Now, as in the previous part 2, it is possible to justify that there is exactly one zero point $s^* \in (\pi,\, 2\pi)$ of the function $c_{2,\gamma}$.

□

Remark 4.25. *Using Lemmas 4.16, 4.18, 4.19, 4.21, 4.23 and 4.24, let us sum up known properties of the function $c_{2,\gamma}$:*

1. $c_{2,\gamma}$ *is continuous for* $\gamma \in \left(0,\, \frac{2}{\pi^2}\right)$,
2. $c_{2,\gamma}(s) > 0$ *for* $s \in (0,\, 2(n_0 - 1)\pi)$ *and* $\gamma \in \left(0,\, \frac{2}{2\pi^2}\right)$,
3. $c_{2,\gamma}(s) < 0$ *for* $s > 2n_0\pi$ *and for* $\gamma \in \left(0,\, \frac{2}{\pi^2}\right)$,
4. $c_{2,\gamma}(s) = 0$ *for* $s \in [2n_0\pi - 2\pi,\, 2n_0\pi - \pi]$ *and* $\gamma = \frac{2}{n_0\pi^2}$,
5. $c_{2,\gamma}(s) < 0$ *for* $s \in (2n_0\pi - \pi,\, 2n_0\pi)$ *and* $\gamma = \frac{2}{n_0\pi^2}$,
6. $c_{2,\gamma}(s) > 0$ *for* $s \in [2n_0\pi - 2\pi,\, 2n_0\pi - \pi]$ *and* $\gamma \in \left(0,\, \frac{2}{\pi^2}\right) \setminus \bigcup_{n_0 \in \mathbb{N}} \{\frac{2}{n_0\pi^2}\}$,
7. *For* $\gamma \in \left(0,\, \frac{2}{\pi^2}\right)$, *the function* $c_{2,\gamma}$ *has exactly one zero point* $s^* \in (2n_0\pi - \pi,\, 2n_0\pi]$.

Theorem 4.26. *For $0 < \gamma < \frac{2}{\pi^2}$ and $s > 0$, the quadratic equation (4.18) is solvable for $k > 0$ if and only if $s \in (0, s^*)$, where $s^* = 2n_0\pi - 2\pi$ for $\gamma = \frac{2}{n_0\pi^2}$ and for $\gamma < \frac{2}{n_0\pi^2}$, s^* is the unique solution of the equation*

$$1 + \cos(s) + 2n_0 = \gamma(s - n_0\pi + \pi)^2, \qquad 2n_0\pi - 2\pi < s < 2n_0\pi. \tag{4.30}$$

Moreover, for $s \in (0, s^*)$, the positive solution k of (4.18) is unique.

Proof. Let us split the proof according to the value of γ.

1. At first, let us assume that $\gamma = \frac{2}{n_0\pi^2}$. In this case, using Lemmas 4.18, 4.19 and 4.21, we have that

$$\begin{aligned} c_{2,\gamma}(s) &> 0 && \text{for } s \in (0, 2n_0\pi - 2\pi),\\ c_{2,\gamma}(s) &= 0 && \text{for } s \in [2n_0\pi - 2\pi, 2n_0\pi - \pi],\\ c_{2,\gamma}(s) &< 0 && \text{for } s \in (2n_0\pi - \pi, 2n_0\pi),\\ c_{2,\gamma}(s) &< 0 && \text{for } s \geq 2n_0\pi. \end{aligned}$$

   Now, due to Lemma 4.15, we obtain that the quadratic equation (4.18) is solvable for $k > 0$ if and only if $s \in (0, s^*)$, where $s^* = 2n_0\pi - 2\pi$.

2. At second, let us assume that $\gamma < \frac{2}{n_0\pi^2}$. Using Lemma 4.24, we obtain exactly one $s^* \in (2n_0\pi - \pi, 2n_0\pi)$ such that

$$c_{2,\gamma}(s) \begin{cases} > 0 & \text{for } s \in [2n_0\pi - 2\pi, s^*),\\ = 0 & \text{for } s = s^*,\\ < 0 & \text{for } s \in (s^*, 2n_0\pi]. \end{cases}$$

   Using Lemma 4.19, we get that $c_{2,\gamma}(s) < 0$ for $s > 2n_0\pi$. Moreover, in the case of $n_0 \geq 2$ (i.e. $\gamma < \frac{2}{2\pi^2}$), we obtain using Lemma 4.18 that $c_{2,\gamma}(s) > 0$ for $s \in (0, 2n_0\pi - 2\pi)$. Thus, we have exactly one $s^* > 0$ such that

$$\begin{aligned} c_{2,\gamma}(s^*) &= 0,\\ 1 + \cos(s^*) + 2n_0 &= \gamma(s^* - n_0\pi + \pi)^2, \end{aligned} \tag{4.31}$$

and that $c_{2,\gamma}(s) > 0$ for $s \in (0, s^*)$ and $c_{2,\gamma}(s) < 0$ for $s > s^*$. Finally, using Lemma 4.15, we get that the quadratic equation (4.18) is solvable for $k > 0$ if and only if $s \in (0, s^*)$.

□

For the zero point s^* of $c_{2,\gamma}$ given in Theorem 4.26, we can write

$$\Gamma(s^*) = \gamma,$$

where the function Γ with the range $\left(0, \frac{2}{\pi^2}\right)$ is defined as ($n \in \mathbb{N}$)

$$\Gamma(s) := \frac{1 + \cos(s) + 2n}{(s - n\pi + \pi)^2} \qquad \text{for } s \in (2n\pi - \pi, 2n\pi]. \tag{4.32}$$

Indeed, for $\gamma < \frac{2}{n_0\pi^2}$, we have $s^* \in (2n_0\pi - \pi, 2n_0\pi)$ and thus, we obtain $\Gamma(s^*) = \frac{1+\cos(s^*)+2n_0}{(s^*-n_0\pi+\pi)^2} = \gamma$ due to (4.31). Moreover, for $\gamma = \frac{2}{n_0\pi^2}$, we have $s^* = 2n_0\pi - 2\pi$ and

$$\Gamma(s^*) = \frac{1 + \cos(2n_0\pi - 2\pi) + 2(n_0 - 1)}{(2n_0\pi - 2\pi - (n_0 - 1)\pi + \pi)^2} = \frac{2 + 2n_0 - 2}{(2n_0\pi - \pi - n_0\pi + \pi)^2} = \frac{2n_0}{n_0^2\pi^2} = \frac{2}{n_0\pi^2} = \gamma.$$

Let us note that the function Γ is invertible and thus, we can formulate the following theorem based on Theorem 4.26.

Theorem 4.27. *For $0 < \gamma < \frac{2}{\pi^2}$ and $s > 0$, the quadratic equation (4.18) is solvable for $k > 0$ if and only if $s \in (0, s^*)$, where $s^* = \Gamma^{-1}(\gamma)$ and Γ is defined in (4.32). Moreover, for $s \in (0, s^*)$, the positive solution k of (4.18) is unique.*

Graph of the function Γ^{-1} is illustrated in Figure 4.12. The solvability of the quadratic equation (4.18) in Theorem 4.27 can be further generalized also for $\gamma \leq 0$. Without any justification, let us only reveal the extended version of the definition of Γ.

Definition 4.28. *Let us define the function $\Gamma : (0, +\infty] \to \left(-\frac{1}{2}, \frac{2}{\pi^2}\right)$ as ($n \in \mathbb{N}$)*

$$\Gamma(s) := \begin{cases} \dfrac{1 + \cos(s) - 2n}{(s - n\pi + \pi)^2} & \text{for } s \in (2n\pi - 2\pi, 2n\pi - \pi], \\ \dfrac{1 + \cos(s) + 2n}{(s - n\pi + \pi)^2} & \text{for } s \in (2n\pi - \pi, 2n\pi], \\ 0 & \text{for } s = +\infty. \end{cases} \tag{4.33}$$

Finally, the solvability result in Theorem 4.27 can be extended in the following way.

Theorem 4.29. *For $-\frac{1}{2} < \gamma < \frac{2}{\pi^2}$ and $s > 0$, the quadratic equation (4.18) is solvable for $k > 0$ if and only if $s \in (0, s^*)$, where $s^* = \Gamma^{-1}(\gamma)$ and Γ is defined in (4.33). Moreover, for $s \in (0, s^*)$, the positive solution k of (4.18) is unique.*

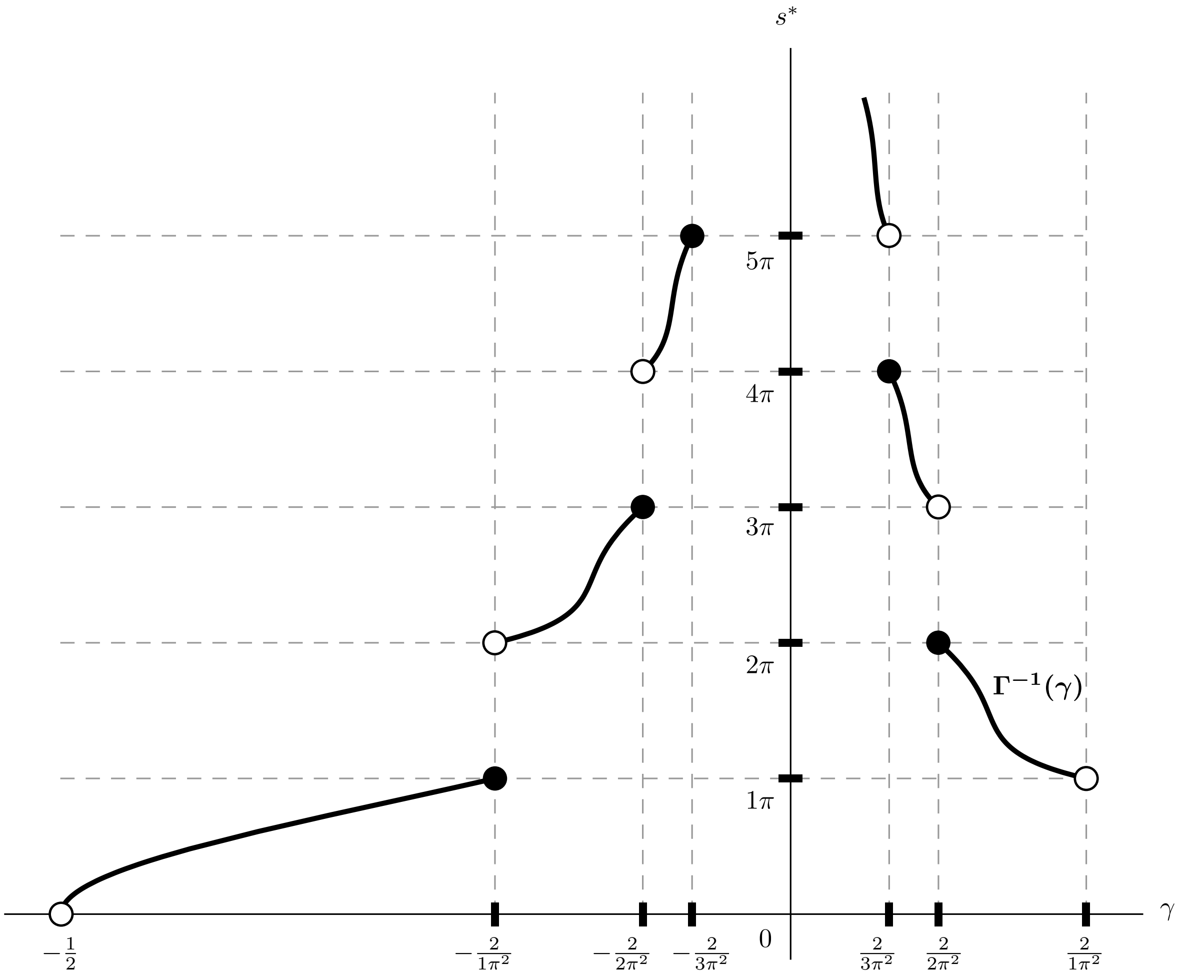


Fig. 4.12: Graph of the function $s^* = \Gamma^{-1}(\gamma)$.

Chapter 5

Parametrization of the Fučík curves

In this chapter, our goal is to find a parametrization of the set $\hat{\Sigma}^{\gamma}_{\frac{\pi}{2}}$ in the first quadrant of the $\alpha\beta$-plane, defined by the solution of the problem (1.1) for $c = \frac{\pi}{2}$, i.e. the problem

$$\begin{cases} u''(x) + \alpha u^{+}(x) - \beta u^{-}(x) = 0, & x \in (0,1) \\ u(0) = 0, \quad \int_0^1 u(x)\,\mathrm{d}x = \gamma \cdot u'(0), \end{cases} \tag{5.1}$$

where $\alpha, \beta > 0$ and $\gamma \in \mathbb{R}$.

According to the Lemma 3.1, it is sufficient to find the parametrization of the set $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$ given in (3.14).

5.1 Parametrization of the set $\mathcal{M}^{\gamma}_{c}$ for $c = \frac{\pi}{2}$

Theorem 5.1. *For $\gamma \in \left(-\frac{1}{2}, \frac{2}{\pi^2}\right)$, the set $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$ is a curve $\mu : (0, s^*) \to \mathbb{R}^2$ with the parametrization $\mu(s) := (\mu_1(s),\ \mu_2(s))$, where $s^* = \Gamma(\gamma)$, Γ is given in (4.32) and functions $\mu_1, \mu_2 : (0, s^*) \to \mathbb{R}$ are defined as*

$$\mu_1(s) = \begin{cases} s - n\pi + \pi + n\pi \cdot \dfrac{2 \cdot c_{2,\gamma}(s)}{-c_{1,\gamma}(s) + \sqrt{D(\gamma,s)}} & \text{for } 2n\pi - 2\pi < s \le 2n\pi - \pi, \\ s + \pi + 2n\pi \cdot \dfrac{2 \cdot c_{2,\gamma}(s)}{-c_{1,\gamma}(s) + \sqrt{D(\gamma,s)}} & \text{for } 2n\pi - \pi < s \le 2n\pi, \end{cases}$$

$$\mu_2(s) = \begin{cases} s - n\pi + \pi + n\pi \cdot \dfrac{-c_{1,\gamma}(s) + \sqrt{D(\gamma,s)}}{2 \cdot c_{2,\gamma}(s)} & \text{for } 2n\pi - 2\pi < s \le 2n\pi - \pi, \\ n\pi + (s - n\pi + \pi) \cdot \dfrac{-c_{1,\gamma}(s) + \sqrt{D(\gamma,s)}}{2 \cdot c_{2,\gamma}(s)} & \text{for } 2n\pi - \pi < s \le 2n\pi, \end{cases}$$

where $n \in \mathbb{N}$ and functions $c_{2,\gamma}$, $c_{1,\gamma}$, $c_{0,\gamma}$ are defined in (4.10), (4.11), (4.12) and $D(\gamma, s)$ is defined in (4.19).

Proof. Let us note, that we prove the statement only for $0 < \gamma < \frac{2}{\pi^2}$. Based on the Theorem 4.4, a pair (a, b) belongs to the set $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$ if and only if

$$c_{2,\gamma}(s) \cdot k^2 + c_{1,\gamma}(s) \cdot k + c_{0,\gamma}(s) = 0, \tag{5.2}$$

where $s > 0$ is given by (4.8) and

$$k = \tfrac{b}{a} \quad \text{and} \quad t = \tfrac{2ab}{a+b}. \tag{5.3}$$

Let us recall, that functions $c_{2,\gamma}$, $c_{1,\gamma}$ and $c_{0,\gamma}$ are defined in (4.10), (4.11) and (4.12) for $n \in \mathbb{N}$ as

$$c_{2,\gamma}(s) = \begin{cases} 2n - n^2\pi^2\gamma & \text{for } 2n\pi - 2\pi < s \le 2n\pi - \pi, \\ 1 + \cos s + 2n - (s - n\pi + \pi)^2\gamma & \text{for } 2n\pi - \pi < s \le 2n\pi, \end{cases}$$

$$c_{1,\gamma}(s) = -2n\pi(s - n\pi + \pi)\gamma \qquad \text{for } 2n\pi - 2\pi < s \le 2n\pi,$$

$$c_{0,\gamma}(s) = \begin{cases} 1 + \cos s - 2n - (s - n\pi + \pi)^2\gamma & \text{for } 2n\pi - 2\pi < s \le 2n\pi - \pi, \\ -2n - n^2\pi^2\gamma & \text{for } 2n\pi - \pi < s \le 2n\pi. \end{cases}$$

Due to Theorem 4.29, we have that the quadratic equation (5.2) for $k > 0$ is solvable if and only if $s \in (0, s^*)$. Additionaly, we can use (5.3) to find out the parameters a, b by using

$$a = \frac{(k+1)t}{2k} \qquad \text{and} \qquad b = \frac{(k+1)t}{2}. \tag{5.4}$$

To find the parametrization of the set $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$, we are going to solve the quadratic equation (5.2) for $k > 0$. Due to Lemma 4.15, we have for the quadratic polynomial in (5.2) that one root k_1 is positive and one root k_2 is negative. Thus we obtain

$$k_1 = \frac{-c_{1,\gamma}(s) + \sqrt{D(\gamma, s)}}{2 \cdot c_{2,\gamma}(s)}, \tag{5.5}$$

where $D = D(\gamma, s)$ is defined in (4.19). Now, let us split the proof according to the value of $t > 2\pi - \frac{2\pi}{1+k}$ in the same way as the proof of Lemma 4.1.

1. Let $t \in \left(2n\pi - \frac{2\pi}{1+k}, 2n\pi\right]$, $n \in \mathbb{N}$. Then we have from (4.8) variable s, given by

$$s = \frac{1+k}{2}(t - 2n\pi) + 2n\pi - \pi.$$

By a simple manipulation, we achieve

$$t = \frac{2}{1+k}(s - 2n\pi + \pi) + 2n\pi. \tag{5.6}$$

By combining (5.5) and (5.6) with (5.4), we get

$$\begin{aligned} a &= \frac{(k_1+1)t}{2k_1} \\ &= \frac{2k_1(s - 2n\pi + \pi) + (k_1 + 1) \cdot 2n\pi}{2k_1} \\ &= (s - 2n\pi + \pi) + n\pi + \frac{n\pi}{k_1} \\ &= s - n\pi + \pi + n\pi \cdot \frac{2 \cdot c_{2,\gamma}(s)}{-c_{1,\gamma}(s) + \sqrt{D(\gamma, s)}} =: \mu_1(s) \end{aligned}$$

and

$$\begin{aligned} b &= \frac{(k_1+1)t}{2} \\ &= s - 2n\pi + \pi + n\pi(k_1 + 1) \\ &= s - n\pi + \pi + n\pi \cdot \frac{-c_{1,\gamma}(s) + \sqrt{D(\gamma, s)}}{2 \cdot c_{2,\gamma}(s)} =: \mu_2(s). \end{aligned}$$

2. Let $t \in \left(2n\pi, 2(n+1)\pi - \frac{2\pi}{1+k}\right]$, $n \in \mathbb{N}$. Then we have from (4.8) variable s, given by

$$s = \frac{1+k}{2k}(t - 2n\pi) + 2n\pi - \pi.$$

By a simple manipulation, we achieve

$$t = \frac{2k}{1+k}(s - 2n\pi + \pi) + 2n\pi. \tag{5.7}$$

By combining (5.5) and (5.7) with (5.4), we get

$$\begin{aligned} a &= \frac{(k_1+1)t}{2k_1} \\ &= (s - 2n\pi + \pi) + \frac{k_1+1}{k_1} \cdot 2n\pi \\ &= s + \pi + 2n\pi \cdot \frac{2 \cdot c_{2,\gamma}(s)}{-c_{1,\gamma}(s) + \sqrt{D(\gamma, s)}} =: \mu_1(s) \end{aligned}$$

and

$$\begin{aligned} b &= \frac{(k_1+1)t}{2} \\ &= k_1 \cdot (s - 2n\pi + \pi) + n\pi \cdot (k_1 + 1) \\ &= k_1 \cdot (s - n\pi + \pi) + n\pi \\ &= n\pi + (s - n\pi + \pi) \cdot \frac{-c_{1,\gamma}(s) + \sqrt{D(\gamma, s)}}{2 \cdot c_{2,\gamma}(s)} =: \mu_2(s). \end{aligned}$$

□

Example of the set $\mathcal{M}^{\gamma}_{\frac{\pi}{2}}$ obtained using the parametrization μ is on Figure 5.1.

Remark 5.2. *Based on the Theorem 5.1 and Lemma 3.1, we have a curve* $\nu(s) = (\nu_1(s), \nu_2(s))$ $s \in (0, s^*)$*, which belongs to the set* $\hat{\Sigma}^{\gamma}_{\frac{\pi}{2}}$*, where*

$$\begin{aligned} \nu_1(s) &:= \mu_1^2(s), \\ \nu_2(s) &:= \mu_2^2(s). \end{aligned}$$

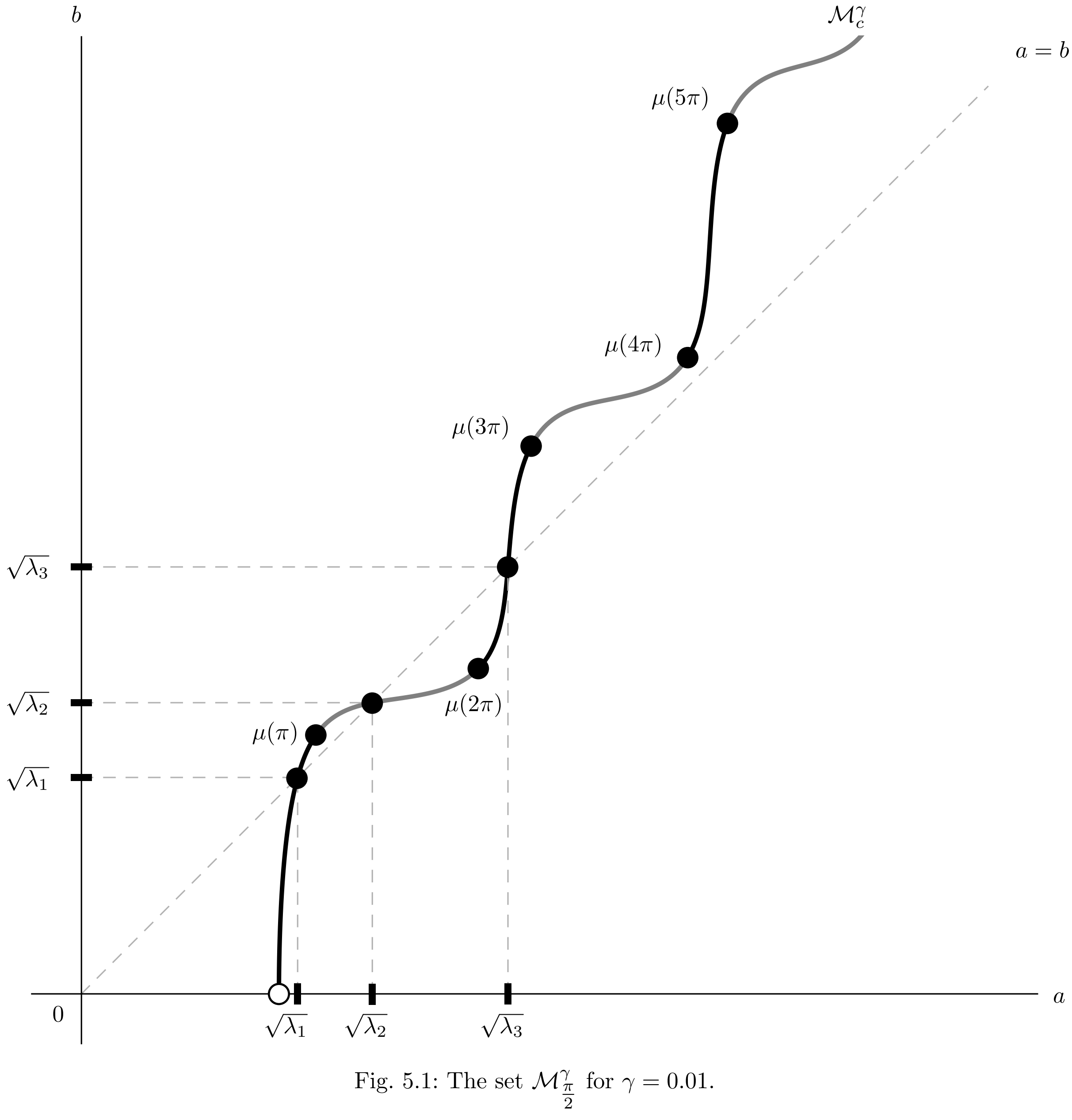


Fig. 5.1: The set $\mathcal{M}_{\frac{\pi}{2}}^{\gamma}$ for $\gamma = 0.01$.

Chapter 6

Conclusion

We have obtained the following main results:

1. We have described eigenvalues of the boundary value problem (2.2) (Theorem 2.4).
2. We have found the implicit description of the Fučík spectrum for the boundary value problem (3.12) (Theorem 3.9).
3. We have shown, that the problem of finding the Fučík spectrum of the problem (4.1) can be equivalently described as the problem of finding solutions of the quadratic equation (4.18) (Theorem 4.4).
4. We have found conditions under which the quadratic equation (4.18) has real solutions (Theorem 4.12) and under which has a positive solution $k > 0$ (Theorem 4.26).
5. We have found the parametrization of the Fučík spectrum of the boundary value problem (5.1) (Theorem 5.1).

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Appendix A

Example of Mathematica code

The Theorem 3.9 provides us with a way of numerically generating pairs (a, b) which belong to the set $\mathcal{M}_c^\gamma$. Here we want to provide an example of a code using this method in the software Mathematica.

```
c = Pi/4;
gamma = 0.2;
size = 50;
precision = 50;


p[a_, c_] := (-1/a) ArcCot[(1/a) Tan[c]] /; -Pi/2 < c < Pi/2;
p[a_, c_] := 0 /; c == Pi/2;
P[a_, b_, t_] := (b/a - a/b)*(t/Pi) + 1;
Gc[a_, b_, t_, c_] := (b/a)*Cos [((a + b)*t)/(2*b) - a*p[a, c]] -(b/a)*Cos[a*p[a, c]]
+ P[a, b, t] /; 0 < t <= (2*b*(Pi + a*p[a, c]))/(a + b);
Gc[a_, b_, t_, c_] := (a/b)*Cos[((a + b)*(t - 2*Pi))/(2*a) - b*p[a, c]] - (b/a)*Cos[a*p[a,
c]] + P[a, b, t - Pi] /; (2*b*(Pi + a*p[a, c]))/(a + b) < t <= 2*Pi + (2*a*b*p[a, c])/(a
+ b);
Gc[a_, b_, t_, c_] := (b/a)*Cos[(a + b) (t - 2 Pi)/(2 b) - a*p[a, c]] - (b/a)*Cos[a*p[a,
c]] + P[a, b, t - 2 Pi] /; 2*Pi + (2*a*b*p[a, c])/(a + b) <= t <= 2 Pi;
G[a_, b_, t_, c_] := Gc[a, b, Mod[t, 2*Pi], c];

ContourPlot[G[a, b, (2*a*b)/(a + b),c] == P[a, b, (2*a*b)/(a + b)] - gamma*a*b*Cos[a*p[a,
c]], {a, 0, size }, {b, 0, size }, PlotPoints -> precision]
```

Using the code above results in the Figure A.1.

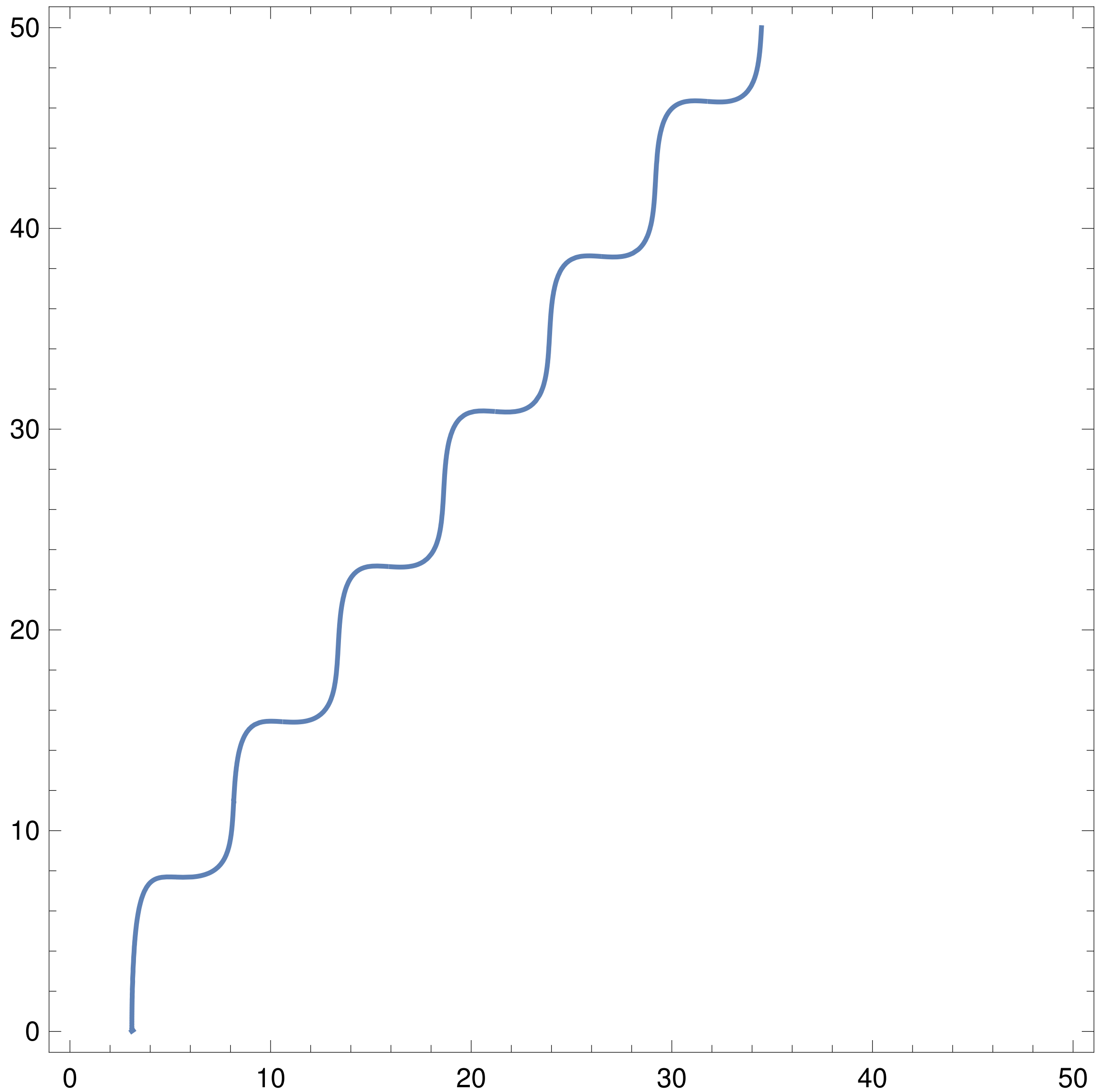

Fig. A.1: Figure of pairs (a, b) belonging to the set $\mathcal{M}_c^\gamma$ generated using the code from Appendix A in the software Mathematica.