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# *p*-Adaptive simulations of Richards' equation with discontinuous Galerkin method

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## 1. Introduction

Richards' equation is widely used to study groundwater dynamics of saturated/unsaturated porous media, with problems ranging from oil industry and geotechnical engineering to agriculture and earth science. The numerical solution of Richards' equation can be troublesome and costly because of abrupt changes in the nonlinear hydraulic properties [1]. Typically, Richards' equation exhibits sharp wetting fronts moving dynamically in the unsaturated zone while the saturated zone remains relatively smooth. Wetting fronts may be so sharp that spurious oscillations (overshoots/undershoots) appear. Increasing mesh discretization with local adaptation (*h*-adaptation) is one well-known remedy [1]. High-order methods are known to reach accuracy with a reduced cost compared to low-order methods. In that context, the use of local space order approximation (*p*-adaptation) is a quite natural direction to be explored in order to assess the possible gains for the solution of Richards' equation. However, there have been few attempts of high-order applications to Richards' equation in the literature [2, 3, 5]. Besides, their conclusions have remained unclear about what high-order accuracy can specifically achieve for Richards' equation because these studies applied general-purpose strategies where mesh and order adaptations are used in combination (*hp*-adaptation). Firstly, this makes the benefits of high-order accuracy difficult to evaluate. Secondly, these general strategies imply many numerical and computational tools making them difficult to implement and costly to compute. This last statement is even more questionable in regards of Richards' equation which is known to be quite challenging to solve numerically and whose solvers need to be as robust and efficient as possible. Thereby, there is room to explore high-order methods applied to Richards' equation.

To this end, discontinuous Galerkin (DG) methods will be employed in this paper. DG methods are particularly suitable for high-order accuracy. Indeed, they rely on an element-wise weak formulation which can be seen as a generalization of the Finite Element or the Finite Volume frameworks. This makes DG methods flexible and attractive to design high-order schemes which can be locally adapted. In this study, the *p*-adaptive algorithm is kept simple in order to prevent computational complexity and, if needed in the future, to make the extension to *hp*-adaptation as easy as possible.

Firstly, Richards' equation and hydraulic properties are briefly introduced. Then, Richards' equation is discretized with a DG method for space and with Backward Differentiation Formula (BDF) methods for time. The adaptivity algorithm is also outlined. Finally, a test-case is presented to show the abilities of the numerical methods.

## 2. Model problem

Richards' equation is a degenerate nonlinear parabolic equation which models flows in variably-saturated porous media [1]

$$\partial_t \theta(\psi) - \nabla \cdot (\mathbb{K}(\psi) \nabla(\psi + z)) = 0, \tag{1}$$

where $\mathbb{K}$ is the hydraulic conductivity [L·T$^{-1}$] and $z$ is the elevation [L]. This equation can be rewritten in terms of hydraulic head $h = \psi + z$ [L], a more common variable in hydrology

$$\partial_t \theta(h - z) - \nabla \cdot (\mathbb{K}(h - z) \nabla h) = 0. \tag{2}$$

Solving Richards' equation (1) requires two constitutive laws: one for water content and one for hydraulic conductivity. For practical purposes, it is assumed that

$$\mathbb{K}(\psi) = \mathbb{K}_{\mathrm{s}} K_{\mathrm{r}}(\psi) \qquad \text{and} \qquad S_{\mathrm{e}}(\psi) = \frac{\theta(\psi) - \theta_{\mathrm{r}}}{\theta_{\mathrm{s}} - \theta_{\mathrm{r}}}, \tag{3}$$

where $\mathbb{K}_{\mathrm{s}}$ denotes the intrinsic or saturated hydraulic conductivity tensor [L $\cdot$ T$^{-1}$], $K_{\mathrm{r}}$ the relative hydraulic conductivity [-], $\theta_{\mathrm{s}}$ the saturated water content [-] and $\theta_{\mathrm{r}}$ the residual water content [-], corresponding to the maximal and minimal saturations, respectively. $S_{\mathrm{e}}$ and $K_{\mathrm{r}}$ are monotonic increasing functions of pressure head $\psi$ in the unsaturated zone ($\psi < 0$) and constant in the saturated zone ($\psi \geq 0$). Several relations exist to model these hydraulic properties. In this paper, the van Genuchtem-Mualem relations will be used

$$\text{if } \psi \geq 0, \text{ then } S_{\mathrm{e}}(\psi) = K_{\mathrm{r}}(\psi) = 1, \tag{4}$$

$$\text{if } \psi < 0, \text{ then } S_{\mathrm{e}}(\psi) = (1 + (\alpha|\psi|)^n)^{-m} \quad \text{and} \quad K_{\mathrm{r}}(\psi) = {S_{\mathrm{e}}}^{0.5} \Big(1 - \Big(1 - {S_{\mathrm{e}}}^{\frac{1}{m}}\Big)^m\Big)^2, \tag{5}$$

where $\alpha$ the parameter linked to air entry pressure inverse [L$^{-1}$], $n > 1$ the pore-size distribution [-] and $m = 1 - \dfrac{1}{n}$ the pore-size distribution [-].

## 3. Numerical methods

### *3.1 Discontinuous Galerkin discretization*

DG methods rely on an element-wise weak formulation which share properties from both the Finite Element and the Finite Volume frameworks which makes them flexible and attractive to design high-order schemes and local adaptation. Extensive introduction can be found in Rivière [4]. In this paper, Richards' equation is discretized in space by a DG method called incomplete interior penalty Galerkin (IIPG) because it is the simplest one. For time discretization, BDF methods with variable time step are chosen because they provide high-order implicit schemes in time which are needed if one wants to take advantage of high-order accuracy in space. For further developments about DG and BDF methods for Richards' equation, the reader is referred to Clément *et al.* [1].

Careful considerations should be given to the choice of the basis for the expansion of the solution when using high-order DG approximations. Indeed, modal bases which are orthogonal and hierarchical present enjoyable properties compared to other bases like nodal bases. Their hierarchical design helps the implementation along and their orthogonality provides better numerical behaviour at high-order like a low condition number for the matrices of the discrete system. In addition, integration quadrature formula should be in accordance with the maximum

order of used polynomials. In this study, the Legendre's polynomials are used because this is one of the simplest and most suitable basis for high-order. For quadrangle elements, a tensor product is performed in each direction.

*3.2 Adaptivity algorithm*

The adaptivity algorithm is made of three main steps which are sketched in Fig. 1. The first step is the evaluation of a criterion $\eta$, or better an error estimator, on each element to indicate the area of interest. In this study, the criterion is heuristically-based and aims at measuring the numerical smoothness of the solution through the volume residual, the solution jump and the flux jump, see [1]. The second step is the selection of the elements where the approximation order will be increased or decreased. To do this, the criterion is compared to user-defined threshold values $0 < \beta_{\mathrm{d}} \leq \beta_{\mathrm{i}}$. For $\beta_{\mathrm{d}} \leq \eta < \beta_{\mathrm{i}}$, the element order remains unchanged. For $\beta_{\mathrm{i}} \leq \eta$, the element order is increased by one. For $\eta < \beta_{\mathrm{d}}$, the element order is decreased by one. The last step consists in transferring the solution from the previous order approximation to the new one. Since the employed basis is hierarchical, the corresponding degrees of freedom in the solution expansion are truncated in case of order decrease, or extended by zero values in case of order increase. The adaptation process is not performed at each time step but according to a user-defined frequency.

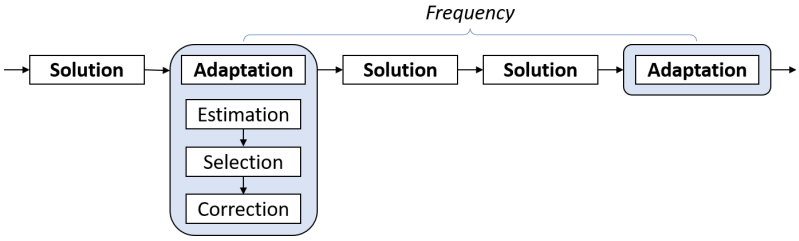


Fig. 1. Adaptivity algorithm for the present *p*-adaptive strategy

## 4. Numerical results

Numerical results are evaluated with a 1D vertical downward infiltration problem called Polmann's test-case which is described in [1]. The computational domain is a rectangle $(0, 20) \times (0, 100)$ cm. The test-case is solved for pressure head $\psi$ during $T = 48$ h with a constant time step $\tau = 120$ s. The initial condition is $\psi_0 = -1000$ cm. The implicit Euler scheme is used (1-step BDF method). The parameters of Van Genuchten-Mualem relations are $K_s = 9.22 \times 10^{-3}$ cm·s$^{-1}$, $\theta_{\mathrm{s}} = 0.368$, $\theta_{\mathrm{r}} = 0.102$, $\alpha = 3.35 \times 10^{-2}$ cm$^{-1}$ and $n = 2$. Three computations are carried out: one for a mesh M100 of 100 elements of order one, one for mesh M400 of 400 elements of order one, and one for *p*-adaptive mesh of 100 elements. In the latter case, order varies from 1 to 3, $\beta_{\mathrm{d}} = \beta_{\mathrm{i}} = 0.05$ and adaptation is done every 5 time steps.

Results are presented in Fig. 2. The left and middle figures show the abilities of the *p*-adaptive computation to follow the wetting front in time. The right figure compares the different computations. For the mesh M100, the solution holds an undershoot ahead of the wetting front. For the mesh M400 and *p*-adaptation, the oscillations in the solution vanish.

## 5. Conclusions

The paper shows that Richards' equation solution can benefit from high-order DG methods. One finding is that high-order approximation can suppress oscillations at the wetting front.

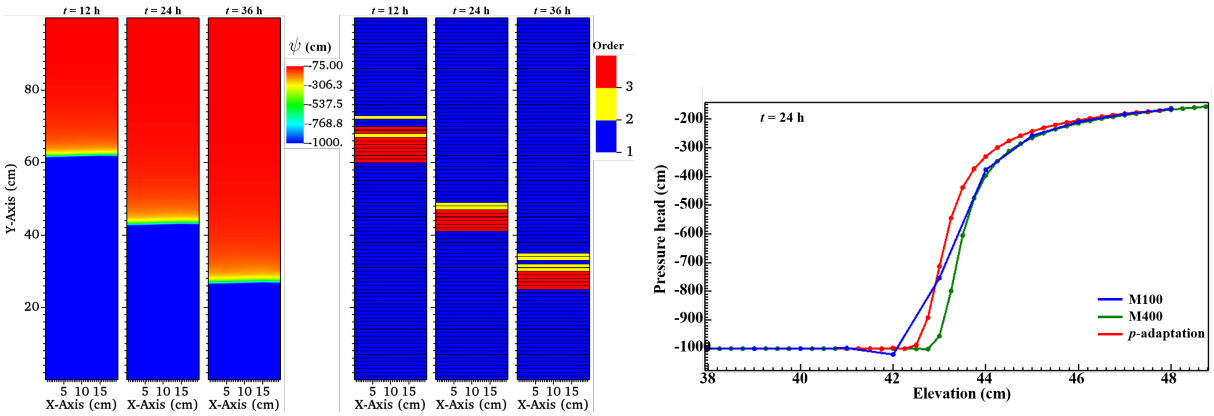


Fig. 2. Wetting front evolution (*left*), element order adaptation (*middle*) and comparison of pressure head profiles (*right*)

This is quite unexpected because wetting fronts are sharp features lacking smoothness. Since high-order approximation everywhere is costly, *p*-adaptation is necessary. A simple adaptivity algorithm was designed to follow dynamically the wetting fronts of Richards' equation where the approximation order is locally increased to remove the oscillations.

In order to take the most of high-order methods, especially in case of adaptation, lots of numerical and computational tools must be implemented. This is a challenging task and the computational cost may be increased. The choice of DG methods with orthogonal and hierarchical modal bases allows to alleviate some efforts. High-order time accuracy is needed if one wants to use high-order space accuracy. That is why BDF methods with variable time step are employed.

This study raises an interesting question to know whether *p*-adaptation is worthy in case of Richards' equation because previous studies in the literature [2, 3, 5] do not use *p*-adaptation at the wetting front but in smooth regions. In addition, *p*-adaptation is often used in combination with *h*-adaptation to make *hp*-adaptation. Toward this direction, further research is needed to assess *p*-adaptive simulations of Richards' equation.

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