COMPUTATIONAL MECHANICS 2022
37th conference with international participation
Srní
November 7 - 9, 2022

Construction of the Lyapunov function reflecting the physical properties of the model

J. Náprstek, C. Fischer

Institute of Theoretical and Applied Mechanics of the Czech Academy of Sciences, Prosecká 76, 190 00 Prague, Czech Republic

1. Introduction

Practical experience shows that the random excitation component can affect the system response and its dynamic stability not only negatively but also positively. For example, the presence of a certain artificially generated turbulence component can have a positive effect against the occurrence of resonance. Such mechanisms are usually developed heuristically and are often not sufficiently justified theoretically. On the other hand, the presence of random excitation can lead to dangerous interactions with deterministic processes and thus cause a reduction in the level of dynamic stability in conditions that do not seem serious at first sight (icing on cables or power lines, road roughness, etc.).

In the sense presented by Bolotin [1], the deterministic LF (as the total time derivative of a positive definite function), is replaced in the stochastic domain by the adjoint Fokker-Planck (FP) operator

$$\mathbf{L}\{\lambda(t,\mathbf{u})\} \;=\; \frac{\partial\lambda(t,\mathbf{u})}{\partial t} + \sum_{i=1}^{n}\frac{\partial\lambda(t,\mathbf{u})}{\partial u_i}\kappa_i + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\lambda(t,\mathbf{u})}{\partial u_i \partial u_j}\kappa_{ij}, \tag{1}$$

where κ_i, κ_{ij} are the drift and diffusion coefficients of the n-dimensional Markov process and m depends on the system structure

$$\kappa_i = \sum_{k=1}^{m} A_{ik}(t) f_{ik}(\mathbf{u}) + \frac{1}{2}\sum_{k,l=1}^{m}\sum_{p=1}^{n}\frac{\partial f_{ik}(\mathbf{u})}{\partial u_p} f_{ip}(\mathbf{u})\cdot s_{iklp}\,, \quad \kappa_{ij} = \sum_{k,l=1}^{m} f_{ik}(\mathbf{u}) f_{jl}(\mathbf{u})\cdot s_{ikjl}\,. \tag{2}$$

Eqs. (1) and (2) relate to the original stochastic system, the stochastic stability of which is being assessed

$$\dot{u}_i \;=\; \sum_{k-1}^{m}(A_{ik}(t) + w_{ik}(t)) f_{ik}(\mathbf{u}); \quad \mathbf{u}(t_0) \;=\; \mathbf{u}_0, \tag{3}$$

where $\lambda(t,\mathbf{u})$ is the LF candidate, $A_{ik}(t)$ are the nominal values of the system coefficients, $w_{ik}(t)$ is the Gaussian white noise of cross-intensity s_{ikjl}, and $f_{ik}(\mathbf{u})$ are the continuous non-decreasing functions.

Function $\lambda(t,\mathbf{u})$ should be a continuous positive definite. Its derivatives $\partial_t\lambda(t,\mathbf{u})$ and $\partial_{\mathbf{u},\mathbf{u}}\lambda(t,\mathbf{u})$ should be continuous as well. Let $\psi(t,\mathbf{u}) = \mathbf{L}\{\lambda(t,\mathbf{u})\} < 0$ in $\mathbf{u} \in \mathbf{\Omega}$ and $\psi(t,0) = 0$ or $\psi(t,0)$ is not defined, $\lambda(t,\mathbf{u})$ can be considered a Lyapunov function. Thus, for any $||\mathbf{u}_0|| \neq 0$ function $\lambda(t,\mathbf{u})$ decreases for $t \to \infty$ and, consequently, the trivial solution of Eq. (3) is stable in terms of probability.

It should be emphasized that an inappropriate choice of the form of the Lyapunov function can lead to inconsistent results. Therefore, it should be designed very carefully. However, it is well known that there is no universal method for constructing the Lyapunov function in either the deterministic or the stochastic case.

2. Construction of the Lyapunov function

Let us assume that the following first integrals $J_1, \ldots J_s$ satisfy the equations of motion

$$J_1(\mathbf{u}) = C_1, ..., J_s(\mathbf{u}) = C_s \,. \tag{4}$$

The Lyapunov function can be selected as a linear combination of the first integrals and their functions. The most convenient selection for practical purposes will obviously be

$$\lambda(\mathbf{u}) = \sum_{i=1}^{s} a_i[J_i(\mathbf{u}) - J_i(0)] + b_i[J_i^2(\mathbf{u}) - J_i^2(0)], \tag{5}$$

where a_i, b_i are unknown constants that must be selected so that the function (5) satisfies the conditions of the positive definiteness.

The first integrals of the type (4) are most often found in the context of cyclic coordinates. In such cases, the corresponding Lagrange equation simplifies considerably

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{u}_k} \;=\; D_k + \Gamma_k, \tag{6}$$

where D_k and Γ_k are dissipative and gyroscopic forces, respectively, and T is the kinetic energy.

If the system is subjected to Gaussian parametric random white noises, the system has a form

$$\mathbf{L}\{T\} = -\sum m \cdot \Theta_m, \tag{7}$$

where $\mathbf{L}\{\cdot\}$ is the adjoined FP operator and Θ_m are the homogeneous functions of phase coordinates and white noise intensities active in the system. The stability assessment procedure is then similar to the deterministic case.

3. Stability of a spherical pendulum

A spherical pendulum moves at a constant velocity around a vertical axis in the coordinates φ (angle around the vertical axis) and ξ (angle between the vertical axis and the pendulum suspension) in a horizontal circle, see, e.g., [2]. Small random perturbations of this movement can be denoted as u_1, u_3, see Fig. 1,

$$\xi = \alpha + u_1 \; ; \quad \dot{\xi} = u_2 \; ; \quad \dot{\varphi} = \omega + u_3, \tag{8}$$

where α is the angle between the suspension and the vertical in deterministic state $\omega^2 l \cdot \cos\alpha = g$, ω is the angular velocity of a circular motion and l denotes the length of the suspension. The perturbations u_1, u_2, u_3 are assumed to be small.

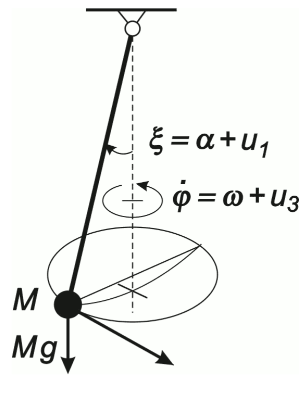

Fig. 1. Outline of a pendulum with coordinates

Two first integrals can be obtained from the general principles of dynamics—total energy $T + \Pi$ and the total momentum

$$T + \Pi = \frac{1}{2} M l^2 (\dot{\xi}^2 + \dot{\varphi}^2 \cdot \sin^2 \xi) - Mgl \cdot \cos\xi \,, \tag{9a}$$

$$\frac{\partial T}{\partial \dot{\varphi}} = M l^2 \dot{\varphi} \cdot \sin^2 \xi \,. \tag{9b}$$

Using Eqs. (9), it is possible to construct the Lyapunov function in the form of Eq. (5) and examine the stability of the pendulum movement. In the given case, it is sufficient to use the first part of Eq. (5), i.e., $b_i = 0$ $(i = 1, 2)$, which means

$$\lambda(u_1, u_2, u_3) = a_1 \left(J_1(\mathbf{u}) - J_1(0)\right) + a_2 \left(J_2(\mathbf{u}) - J_2(0)\right) , \tag{10}$$

where J_1, J_2 are written in the form originating from the substitution of Eqs. (8) into Eqs. (9)

$$\begin{aligned} J_1(\mathbf{u}) &= \frac{1}{2} M l^2 \left(u_2^2 + (\omega + u_3)^2 \sin^2(\alpha + u_1) - \frac{2g}{l} \cos(\alpha + u_1) \right) , \\ J_2(\mathbf{u}) &= \quad M l^2 \left(\, (\omega + u_3) \sin^2(\alpha + u_1) \, \right) . \end{aligned} \tag{11}$$

We shall select the parameters a_1, a_2 in the form of

$$a_1 = 2 \left(M l^2\right)^{-1} , \quad a_2 = a \left(M l^2\right)^{-1} .$$

where a should be determined from the constraint of positive definiteness of the function λ. Substitution of Eqs. (9) and (11) into (10), and the assumption that $u_i, i = 1, 2, 3$ are small, yield

$$\begin{aligned} \lambda(u_1, u_2, u_3) = & u_1^2 \omega \left((a + \omega) \cos 2\alpha + \omega \cos^2 \alpha \right) + u_2^2 + u_3^2 \sin^2 \alpha + \\ & u_1 \omega (a + 2\omega) \sin 2\alpha + u_3 (a + 2\omega) \sin^2 \alpha + u_1 u_3 (a + 2\omega) \sin 2\alpha + \ldots \end{aligned} \tag{12}$$

To make function $\lambda(u_1, u_2, u_3)$ positive definite, it is necessary to eliminate perturbations u_i in the first power. This occurs if $a = -2\omega$. Function λ as a Lyapunov function, thus, takes the form

$$\lambda(u_1, u_2, u_3) = (u_1^2 \cdot \omega^2 + u_3^2) \sin^2 \alpha + u_2^2 + \mathcal{O}(u_i^3) . \tag{13}$$

The Lagrange equations of motion can be determined using the expressions for T, Π

$$\begin{gathered} \ddot{\xi} - \dot{\varphi}^2 \sin \xi \cos \xi + \frac{g}{l} \sin \xi = \mu l (\xi - \alpha) w(t) , \\ \ddot{\varphi} + 2 \dot{\varphi} \dot{\xi} \cdot \cot \xi = 0 , \end{gathered} \tag{14}$$

where parametric (white) noise $w(t)$ has been introduced into the first equation. Its effect is proportional to the deviation from the basic inclination α. In Eqs. (14) we shall substitute for $\xi, \dot{\xi}, \dot{\varphi}$ according to Eqs. (8) and modify this system into the normal form for u_1, u_2, u_3

$$\begin{aligned} \dot{u}_1 &= u_2 , \\ \dot{u}_2 &= (\omega + u_3)^2 \sin(\alpha + u_1) \cos(\alpha + u_1) - \omega^2 \cos \alpha \, \sin(\alpha + u_1) + \mu l u_1 w(t) , \\ \dot{u}_3 &= -2 u_2 (\omega + u_3) \, \cot(\alpha + u_1) . \end{aligned} \tag{15}$$

In a linearized form

$$\begin{aligned} \dot{u_1} &= u_2 , \\ \dot{u_2} &= (\omega^2 \cos 2\alpha - \frac{g}{l} \cos \alpha) u_1 + \omega \sin 2\alpha \cdot u_3 + \mu l u_1 w(t) , \\ \dot{u_3} &= -2 u_2 \cdot \omega \, \mathrm{cotg} \alpha . \end{aligned} \tag{16}$$

The general stochastic system is assumed to have a form

$$\dot{u}_i = f_i(\mathbf{u}) + \sum_{k=1}^{m} h_{ik}(\mathbf{u}) \cdot w_k(t) \quad ; \quad \mathbf{u}(t_0) = \mathbf{u}_0 , \tag{17}$$

where the diffusion coefficients κ_i, κ_{ij} are as follows:

$$\kappa_i = f_i(\mathbf{u}) \ ; \ \kappa_{ij} = \sum_{k,l=1}^{m} h_{ik}(\mathbf{u}) h_{jl}(\mathbf{u}) \cdot s_{kl} \, . \tag{18}$$

In this particular case, the respective coefficients (with $m = 1$) are

$$h_{1,1} = h_{3,1} = 0 \, , \ h_{2,1} = \mu l u_1 \, ; \qquad f_1 = \dot{u}_1 \, , \ f_2 = \dot{u}_2 - h_{2,1} w(t) \, , \ f_3 = \dot{u}_3 \, . \tag{19}$$

The first two parts of the adjoined FP operator (1) are equivalent to the total time derivative of the Lyapunov function in the deterministic domain and only the third term with the coefficients κ_{ij} represents a supplement introducing the influence of random parametric noises. Their influence on the stability of the system Eqs. (17) is determined, consequently, exclusively by the character of matrix $\mathbf{h}(\mathbf{u})$ and joint links of white noises $w_i(t)$. If, for instance, $w_i(t)$ are independent white noises $(s_{ij} = 0, i \neq j)$, the parametric noises are of destabilizing character only. However, it is also possible to construct such $\mathbf{h}(\mathbf{u})$ matrices where the random noises contribute to improve the stability of the system.

In the case of the spherical pendulum described by Eqs. (15) and LF (13), we obtain

$$\begin{aligned} \psi(\mathbf{u}) =& \, \mathbf{L}\{\lambda(t, \mathbf{u})\} \\ =& \, 2u_1 u_2 \omega^2 \sin^2\alpha + 2u_2 \left(\frac{1}{2}(\omega + u_3)^2 \sin 2(\alpha + u_1) - \omega^2 \cos\alpha \, \sin(\alpha + u_1) \right) - \\ & - 4u_2 u_3 (\omega + u_3) \sin^2\alpha \, \cot(\alpha + u_1) + u_1^2 \cdot (\mu l)^2 s_{11} \, . \end{aligned} \tag{20}$$

The destabilizing effect of the noise $w(t)$ is obvious. The stability of the system, therefore, depends on the character of other right-hand side terms of Eq. (20). If we construct the function $\psi(\mathbf{u})$ on the basis of the linearized version of the normal system Eqs. (16), the right-hand side of Eq. (20) will disappear except for the last term as a result of the character of the first integrals. This means that in the linearized state the system is not stable. It can be stabilized by inserting dissipative forces or by an adequate selection of the matrix $\mathbf{h}(\mathbf{u})$ and the characteristics of $w_i(t)$ which, naturally, are determined by the physical character of the actual system.

4. Conclusion

The Lyapunov function constructed on the basis of first integrals provides a possibility to work with the stochastic part of the problem with a much greater overview and to construct mathematical models with regard to the stabilizing or destabilizing effects of parametric random noises. Such properties are due to the fact that the structure of the actual system is fully applied to the very construction of the basic form of the function. This type of analysis is applicable to a variety of dynamic stability problems, including naturally the problem of signal and noise separation in structural health monitoring and various indirect measuring methods.

Acknowledgements

The kind support of Czech Science Foundation project No. 21-32122J and of the RVO 68378297 institutional support are gratefully acknowledged.

References

[1] Bolotin, V. V., Random vibrations of elastic systems, Springer, Netherlands, 1984.

[2] Eiserman, M. A., Classical mechanics, Nauka, Moscow, 1980. (in Russian)