Oponentský posudek na disertační práci Ing. Ivety Sobotkové

„The Fučík spectrum of the discrete Dirichlet operator"

Předložená disertační práce se zabývá studiem Fučíkova spektra diskrétního Dirichletova operátoru a opírá se o dva impaktované články, které autorka publikovala společně se svým konzultantem Ing. Petrem Nečesalem, Ph.D. Jedná se o články „The Fučík spectrum of the discrete Dirichlet operator" (2018) a „Localization of Fučík curves for the second order discrete Dirichlet operator" (2021).

Samotná práce obsahuje 6 kapitol a jsou v ní přehledně shrnuty a okomentovány klíčové výsledky s odkazem na výše uvedené články.

V první kapitole jsou zavedeny problémy P1, P2 (lineární problémy na vlastní čísla diskrétního Dirichletova operátoru – s počátečními a okrajovými podmínkami) a P3, P4 (nelineární problémy úzce související s Fučíkovým spektrem diskrétního Dirichletova operátoru – opět s počátečními i okrajovými podmínkami).

Ve druhé kapitole jsou zkoumány lineární problémy P1 a P2 a je zaveden pojem spojitého rozšíření diskrétního řešení. Dále je zde ukázána souvislost s Čebyševovými polynomy druhého druhu.

Třetí a čtvrtá kapitola se věnuje zkoumání nelineárního počátečního problému P3. Je možné zde najít i několik příkladů.

Kapitola pátá je věnována studiu nelineárního okrajového problému P4, který úzce souvisí s Fučíkovým spektrem diskrétního Dirichletova operátoru (matice A^D) a je zde možno najít řadu nových – dosud nepublikovaných – výsledků. Najdeme zde i různé popisy Fučíkova spektra, ale také odhady jednotlivých větví Fučíkova spektra. Různé typy odhadů jsou zde i vzájemně porovnávány.

V šesté kapitole pak najdeme shrnutí dosažených výsledků.

Sepsáním této práce autorka prokázala schopnost samostatně vědecky pracovat a řešit problémy. Zkoumání Fučíkova spektra diskrétních operátorů je obecně velmi obtížné. Autorka se zaměřila na studium Fučíkova spektra diskrétního Dirichletova operátoru a svými novými výsledky obohatila výzkum (nejen) na svém pracovišti.

K předložené práci nemám podstatné připomínky. Dle mého názoru byly cíle práce splněny. Proto (po úspěšné obhajobě) **doporučuji**, aby byl Ing. Ivetě Sobotkové udělen titul **Ph.D.**

V Ostravě 2.12. 2021

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doc. Mgr. Petr Vodstrčil, Ph.D.

Opponent Report on Dissertation Thesis
The Fučik spectrum of the discrete Dirichlet operator

October 20, 2021

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Supervisor: doc. Ing. Gabriela Holubová, PhD
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The PhD thesis is devoted to studying the Fučik spectrum $\sum$ for the discrete semi-linear boundary value problem with Dirichlet conditions. The author investigates the rather wide class of discrete problems: linear and semi-linear, initial and boundary value problems. Specifically, the author examines in detail the following four problems:

1. Linear initial value problem

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, \ k \in \mathbb{Z}, \\ u(0) = C_0, \ u(1) = C_1, \end{cases} \tag{1}$$

2. Linear boundary value problem

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, \ k \in \{1, ..., n\}, \\ u(0) = u(n+1) = 0, \end{cases} \tag{2}$$

3. Semi-linear initial value problem

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, \ k \in \mathbb{Z}, \\ u(0) = 0, \ u(1) = C_1, \end{cases} \tag{3}$$

4. Semi-linear boundary value problem

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, \ k \in \{1, ..., n\}, \\ u(0) = u(n+1) = 0, \end{cases} \tag{4}$$

where the second order forward difference operator is given by

$$\Delta^2 u(k-1) := u(k-1) - 2u(k) + u(k+1),$$

u^+ and u^- stand for the positive and negative parts of u.

The Thesis contains six chapters and two appendices. The main results are presented in Chapters 1 to 5 on 69 pages. We note right away that the study of the Fučik spectrum for discrete problems is incomparably more difficult than in the case of continuous operators.

In the Introduction, the author first defines the main definitions, the sign property of a vector, positive and negative parts of a vector, specifies a particular Dirichlet matrix A^D, which comes from the discretization of the continuous problem. It is shown that studying the set of all pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the problem (4) has a non-trivial solution u, is equivalent to the investigation of the Fučik spectrum $\sum(A^D)$ of the matrix A^D. Therefore, to find the set $\sum(A^D)$ and to investigate its properties is the main purpose of this Thesis. At first, in the Chapter 2, the author investigated in detail linear problems (1) and (2). In Lemma 3, the form of the unique solution of the problem (1) was found and the corresponding continuous extension of this solution was defined. Moreover, Lemma 7 describes the number of generalized zeros of this solution.

The author was able to establish a close relationship of problems (1) and (2) with Chebyshev polynomials of the second kind U_n. To find all values $\lambda \in \mathbb{R}$ for which (2) has a nontrivial solution is the same as to find all eigenvalues and eigevectors of the corresponding Dirichlet matrix A^D. It was shown that eigenvalues of A^D are zero points of the Chebyshev polynomial $U_n(\frac{2-\lambda}{2})$. This led to a generalized new approach suitable for a semi-linear problem (3). Lemma 22 gives an opportunity to calculate generalized zeros of a solution which leads to an implicit and recurrent description of the Fučik spectrum $\sum(A^D)$ of the matrix A^D.

In Chapter 5, the description of the Fučik spectrum for the problem (4) was obtained in Theorems 32 and 34 due to results in previous chapters. Moreover, a new modification of this description is given in Theorem 35.

As a co-author, the candidate has already published 2 papers in impacted journals. The first paper (of 45 pages) was published in the journal *Linear Algebra and Its Applications*, which belongs to Q1 in the following fields of Mathematics: *Algebra and Number Theory* and *Discrete Mathematics and Combinatorics*. The second paper (of 51 pages) was published in *Bulletin des Sciences Mathématiques*, which belongs to Q1 in *Mathematics* field. The Thesis under review is on a high mathematical level. Its main results are new and interesting, formulations are clear and elegant. The text is written very well, the proofs are presented including all the details. The level of English is very good. There are only a few typos and minor ambiguities in the Thesis.

A few questions and remarks arose when we were studying the manuscript of the Thesis. Rather, they relate to the possible continuation of these studies.

1. The left-hand side of equations (1)–(4) does not contain the first order forward difference operator. It seems that this assumption is essential for the proofs. Is it possible to extend obtained results to this case?

2. Considering that the technique used leads to an implicit and recurrent description, in principle, is it possible to give some estimates of the error in calculating the Fučik spectrum? How the number of decimal places taken into

account affects the accuracy of calculations, for example on pages 40–43?

3. It is noted that the developed theory can be applied under various boundary conditions. For example, what are these boundary conditions?

4. Were used in the calculations software packages that allow symbolic calculations?

5. A formal remark as to what might be worth the main proofs from the attached article in the appendix to include in the Thesis.

The above remarks, of course, have no influence on the positive referee's conclusion on the Thesis. Still more, it seems that the results obtained there open a source of new interesting problems.

Conslusion

To sum up, the Thesis brings original and new results which contribute significantly to the field of the discrete Fučík spectrum problems. I would like to appreciate many well-chosen examples which suitably illustrate proven theorems and make it easier for the reader to understand obtained results. Used sources are properly cited in the text and the candidates own contribution is clearly specified. I confirm that the presented Thesis meets the requirements for obtaining a PhD degree for the candidate.

Based on the above facts, **I strongly recommend the submitted Thesis for defense and warmly recommend to award to Iveta Sobotková the PhD degree.**

In Miskolc, Hungary, October 20, 2021

MISKOLCI EGYETEM Gépészmérnöki és Informatikai Kar 2. Matematikai Intézet

Prof. Miklós Rontó, DSc
Institute of Mathematics, University of Miskolc, Hungary