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THE FUČÍK SPECTRUM OF THE DISCRETE DIRICHLET OPERATOR

specialization

Applied mathematics

Summary of the thesis to qualify for academic degree Doctor of Philosophy (Ph.D.)

In Plzeň, 2021

Declaration

I do hereby declare that the entire thesis is my original work and that I have used only the cited sources.

Plzeň, August 27, 2021

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Iveta Sobotková

Abstrakt

Disertační práce je zaměřena na studium Fučíkova spektra pro diskrétní operátory. Vzhledem k tomu, že obecné vyšetření Fučíkova spektra diskrétních operátorů je v dnešní době stále těžce uchopitelnou výzvou, studium v této práci je zaměřené na konkrétní operátor – Dirichletův diskrétní operátor.

Tento operátor odpovídá diferenční rovnici druhého řádu s Dirichletovými okrajovými podmínkami. V disertační práci je dopodrobna vyšetřena odpovídající semilineární úloha, zaveden pojem spojitého rozšíření diskrétního řešení úlohy a hlavně je zde uveden kompletní implicitní popis Fučíkova spektra Dirichletova diskrétního operátoru. Na závěr práce jsou popsány tři typy odhadů pro Fučíkovy větve, které umožňují lokalizovat Fučíkovy větve i pro velký rozměr odpovídající matice.

Celý text disertační práce se opírá o dva autorčiny články (v příloze práce) – [25], [31]. Samotný text disertační práce je koncipován jako shrnutí klíčových výsledků odkázaných článků a obsahuje podrobná vysvětlení jednotlivých nově zavedených konceptů pro práci s Fučíkovým spektrem pro vybraný diskrétní operátor.

Klíčová slova: Fučíkovo spektrum, diferenční operátor, Dirichletův diskrétní operátor, Chebyshevův polynom druhého druhu, asymetrické nelinearity

Abstract

The dissertation thesis is devoted to the study of Fučík spectrum for discrete operators. Considering the fact, that the problem of exploring Fučík spectrum for general discrete operators is still a significant challenge, in this thesis we focus on analyses in regards of a particular operator – Dirichlet discrete operator.

This operator corresponds to the second order difference equation with Dirichlet boundary conditions. In the thesis, we explore corresponding semi-linear problem, we define a continuous extension of a discrete solution and finally, we provide a complete implicit description of the Fučík spectrum of Dirichlet discrete operator. Last but not least, three different bounds for Fučík curves are described. This allows for a localization of Fučík curves even for large size of a corresponding matrix.

The whole text of the thesis is based on two articles of the author [25], [31]. The main goal is to summarise key results introduced in cited articles and to explain in detail new concepts of working with Fučík spectrum for the chosen discrete operator.

Key words: Fučík spectrum, difference operator, Dirichlet discrete operator, Chebyshev polynomial of the second kind, asymmetric nonlinearities

Zusammenfassung

Diese Dissertation widmet sich dem Studium des Fučík Spektrum für diskrete Operatoren. Angesichts der Tatsache, dass das Problem der Untersuchung des Fučík Spektrums für allgemeine diskrete Operatoren immer noch eine große Herausforderung darstellt, konzentrieren wir uns in dieser Arbeit auf Analysen in Bezug auf einen bestimmten Operator – den diskreten Dirichlet-Operator.

Dieser Operator entspricht der Differenzengleichung zweiter Ordnung mit Dirichlet-Randbedingungen. In der Dissertation untersuchen wir ein entsprechendes semilineares Problem, definieren eine kontinuierliche Erweiterung einer diskreten Lösung und liefern schließlich eine vollständige implizite Beschreibung des Fučík Spektrums des diskreten Dirichlet-Operatoren. Nächst werden drei Bounds von Fučík Kurven beschrieben. Diese Bounds ermöglichen eine Lokalisierung von Fučík Kurven auch bei großen Dimensionen einer entsprechenden Matrix.

Der gesamte Text der Dissertation basiert auf zwei Artikeln der Autorin: [25], [31]. Das Hauptziel besteht darin, wichtige Ergebnisse aus zitierten Artikeln zu veranschaulichen und neue Konzepte der Arbeit mit Fučík spectrum für den gewählten diskreten Operator im Detail zu erklären.

Schlüsselwörter: Fučík Spektrum, Differenzenoperator, diskrete Dirichlet-Operator, Tschebyschow-Polynome zweiter Art, asymmetrische Nichtlinearitäten.

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Contents

Chapter 1

Introduction

Svatopluk Fučík and other mathematicians studied solvability of a problem

$$-u''(x) = f(x, u(x)),$$

on some interval with various boundary conditions. Solvability of such problem with

$$f(\cdot, s) \sim \lambda s \text{ for } s \to \pm\infty$$

is dependent on the fact whether λ is (or is not) an eigenvalue of the corresponding operator. Main results are due to S. Fučík [11] and E.N. Dancer [5] who considered a different asymptotic behaviour of f, in particular

$$f(\cdot, s) \sim \mu s \text{ for } s \to +\infty, \quad f(\cdot, s) \sim \nu s \text{ for } s \to -\infty.$$

Solvability of the problem can be answered using information about all pairs $(\mu, \nu) \in \mathbb{R}^2$ such that the following problem (together with corresponding boundary conditions)

$$-u''(x) - \mu u^+(x) + \nu u^-(x) = 0$$

has a non-trivial solution. Traditionally, a set of all such pairs is called the Fučík spectrum. For more information, see [8].

Fučík spectrum for discrete operators was investigated by R. Švarc (see e.g. [38], [40]). In [40], R. Švarc considered two particular square matrices of size 4 and gave a description of their Fučík spectra. These matrices were chosen in such a way that their Fučík spectra (even for small matrices of size four) exhibit rather strange behaviour.

Authors G. Holubová and P. Nečesal [17] discussed similarities of structures in Fučík spectra for continuous and discrete operators. They also suggested an algorithm for numerical reconstruction of the Fučík spectrum for reasonably small matrices. They focused on the case of all general real square matrices of size 2 and shown all feasible structures in their Fučík spectrum. They also suggested that there are more than 300 qualitatively different patterns of the Fučík spectrum even for matrices of size 3. This illustrates that the problem of finding Fučík spectra for general matrices is a significant challenge that has not been solved yet.

Various physical phenomena are represented by continuous initial or boundary value problems. Moreover, the theory of Fučík spectrum for these problems is applied in practice for analyses of (mechanical) systems with pronounced asymmetry / asymmetric structure. One of the typical examples are suspension bridges – explored in [22, 9, 15] and the book [13] with a focus on models with asymmetric nonlinearities. Also, asymmetric nonlinearities appear in the study of competing systems of species with large interactions in biology (see [4, 6, 27]) and the Fučík spectrum of the Dirichlet Laplacian (the Laplace operator $u \mapsto -\Delta u$ with zero Dirichlet boundary conditions) is needed (see [6] for details).

Hence we contemplate that the exploration of discrete problems might be useful for practical applications. Sometimes, even though the problem is naturally discrete, researchers tend to make a simplification and look at this as a continuous problem (such examples can be found e.g. in the area of mathematical finance). On the other hand, sometimes, due to complexity of the physical phenomena, researchers tend to use a discretization of the studied continuous problem. This way, one might obtain superior analytical results or a more suitable numerical solution. Thus, we conclude that discrete problems might be relevant for both continuous and discreet natural phenomena. We note that sometimes the discrete problem can be solved in a simpler way, but quite often the discrete structure of such problems can lead to specific difficulties which pose further challenges.

We are going to make a brief comparison of the Fučík spectrum for continuous and discrete operators. We will illustrate that discrete domain brings extra challenges in finding the Fučík spectrum and we will solve several challenges for a particular problem within the thesis and in the referenced articles of the author.

Let us also mention some other articles where the structure of Fučík spectrum is studied – [1, 2, 3, 7, 10, 16, 19, 20, 21, 23, 28, 30, 34, 35, 36].

In the following paragraph, we will recall a well known result for the Fučík spectrum of the continuous second order boundary value problem.

The Fučík spectrum Σ for the continuous second order boundary value problem with Dirichlet boundary conditions, i.e.

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

is defined as the set

$$\Sigma := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \text{ the problem (1.1) has a nontrivial solution } u\right\}.$$

The description of the set Σ is well known. In fact, as shown in [11, 12], the Fučík spectrum Σ consists of two lines $\mathbf{C}_0^{\pm}$: $(\alpha - \pi^2)(\beta - \pi^2) = 0$ and countably many curves $\mathbf{C}_l^{\pm}$ given by $(j \in \mathbb{N})$

$$\mathbf{C}_{2j-1}^{\pm} : \frac{j\pi}{\sqrt{\alpha}} + \frac{j\pi}{\sqrt{\beta}} = 1, \quad \mathbf{C}_{2j}^{+} : \frac{(j+1)\pi}{\sqrt{\alpha}} + \frac{j\pi}{\sqrt{\beta}} = 1, \quad \mathbf{C}_{2j}^{-} : \frac{j\pi}{\sqrt{\alpha}} + \frac{(j+1)\pi}{\sqrt{\beta}} = 1.$$

On the other hand, investigating the Fučík spectrum for the corresponding discrete problem is a much more elaborate process to which we will devote remaining parts of the thesis.

1.1 Main definitions

In this section, we will introduce main problems of our interest and several concepts associated with the studied problems.

Studied problems:

i. linear initial value problem

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, & k \in \mathbb{Z}, \\ u(0) = C_0,\ u(1) = C_1, \end{cases} \tag{P1}$$

ii. linear boundary value problem

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, & k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases} \tag{P2}$$

iii. semi-linear initial value problem

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{Z}, \\ u(0) = 0,\ u(1) = C_1, \end{cases} \tag{P3}$$

iv. semi-linear boundary value problem

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases} \tag{P4}$$

where $n \in \mathbb{N}$, $n \geq 2$, $\mathbb{T} = \{1, \dots, n\}$, $\hat{\mathbb{T}} = \{0, \dots, n+1\}$, $u : \hat{\mathbb{T}} \to \mathbb{R}$, u^+, u^- stand for the positive and negative parts of u, i.e. $u^+(k) := \max\{+u(k), 0\}$, $u^-(k) := \max\{-u(k), 0\}$ and $\alpha, \beta, \lambda \in \mathbb{R}$. In case of problem (P1), $C_0, C_1 \in \mathbb{R}$ are constants such that $C_0^2 + C_1^2 \neq 0$. In case of problem (P3), $C_1 \in \mathbb{R} \setminus \{0\}$. The second order forward difference operator is given by $\Delta^2 u(k-1) := u(k-1) - 2u(k) + u(k+1)$.

1. **Sign property of a vector**

   Let us define a sign property of a vector $\mathbf{u} = [u_1, u_2, \dots, u_n]^T$ of size n as

$$\operatorname{sign} \mathbf{u} = [\operatorname{sign}(u_1), \operatorname{sign}(u_2), \dots, \operatorname{sign}(u_n)]^T$$

and simplify the notation. For $x \in \mathbb{R}$

$$\text{instead of } \operatorname{sign}(x) = \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0, \\ 0 & \text{for } x = 0, \end{cases} \quad \text{we denote } \operatorname{sign}(x) = \begin{cases} + & \text{for } x > 0, \\ - & \text{for } x < 0, \\ 0 & \text{for } x = 0. \end{cases}$$

2. **Positive and negative part of a vector**

   For vector $\mathbf{u}$ of size n, $n \in \mathbb{N}$, $\mathbf{u} = [u(1), \dots, u(n)]^T$, we define its positive part $\mathbf{u}^+ := [u^+(1), \dots, u^+(n)]^T$, and its negative part $\mathbf{u}^- := [u^-(1), \dots, u^-(n)]^T$ (see Figure 1.1).

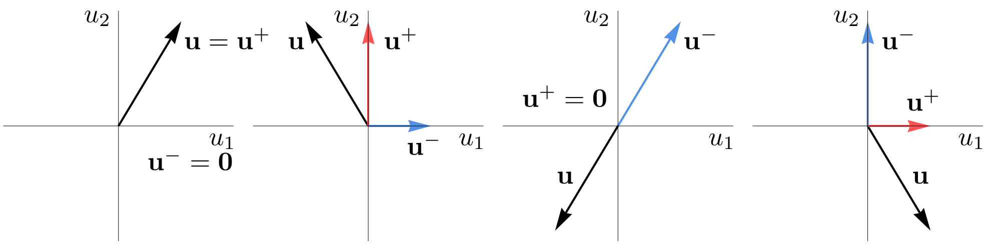

Figure 1.1: Illustration of positive $\mathbf{u}^+$ (red) and negative $\mathbf{u}^-$ (blue) part of vector $\mathbf{u} = [u_1, u_2]^T$. In this particular case, we assume $n = 2$.

3. **The Fučík spectrum of a matrix**

   The Fučík spectrum of a real square matrix $\mathbf{B}$ of size $n \times n$, $n \in \mathbb{N}$, $n \geq 2$, is the set:

$$\Sigma(\mathbf{B}) = \left\{ (\alpha, \beta) \in \mathbb{R}^2 : \text{ the problem } \mathbf{B}\mathbf{u} = \alpha \mathbf{u}^+ - \beta \mathbf{u}^- \text{ has a non-trivial solution } \mathbf{u} \right\}. \tag{1.2}$$

   The pair $(\alpha, \beta) \in \Sigma(\mathbf{B})$ is called the Fučík eigenpair and the non-trivial solution $\mathbf{u}$ is called the Fučík eigenvector for the matrix $\mathbf{B}$.

4. **The Dirichlet matrix**

   Matrix $\mathbf{A}^{\mathsf{D}}$ is called the Dirichlet matrix and will be used throughout the thesis:

$$\mathbf{A}^{\mathsf{D}} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}. \tag{1.3}$$

5. **Fučík curves**

   For Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$, where Dirichlet matrix $\mathbf{A}^{\mathsf{D}}$ is of size $n \times n$ (we are going to see the relationship between matrix $\mathbf{A}^{\mathsf{D}}$ and semi-linear boundary value problem

(P4) further in the text), we define Fučík curves $\mathcal{C}_l^+$, $\mathcal{C}_l^-$, $l = 0, \dots, n-1$ as (the term of generalized zero is defined later in this text)

$$\mathcal{C}_l^+ := \{(\alpha, \beta) \in \mathbb{R}^2 : \begin{array}{l} \text{the problem (P4) has a non-trivial solution } u \\ \text{with exactly } l \text{ generalized zeros on } \mathbb{T} \text{ and } u(1) > 0\} , \end{array}$$

$$\mathcal{C}_l^- := \{(\alpha, \beta) \in \mathbb{R}^2 : \begin{array}{l} \text{the problem (P4) has a non-trivial solution } u \\ \text{with exactly } l \text{ generalized zeros on } \mathbb{T} \text{ and } u(1) < 0\} , \end{array}$$

which we jointly denote by the following simplified notation:

$$\mathcal{C}_l^\pm := \mathcal{C}_l^+ \cup \mathcal{C}_l^- .$$

1.2 Typical challenges while investigating the Fučík spectrum for matrices

Having in mind that investigating the Fučík spetrum for general matrices is at this time unsolved as far as we know, we specify a particular matrix which comes from the discretization of the continuous problem (1.1) (which has also practical applications, see [25] and [31]). We consider the following discrete problem with Dirichlet boundary conditions (P4)

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, \quad k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases}$$

where $n \in \mathbb{N}$, $n \geq 2$ and $\alpha, \beta \in \mathbb{R}$.

Equivalently, the problem (P4) can be rephrased using a matrix notation

$$\mathbf{A}^{\mathsf{D}} \mathbf{u} = \alpha \mathbf{u}^+ - \beta \mathbf{u}^-,$$

where matrix $\mathbf{A}^{\mathsf{D}}$ is the Dirichlet matrix (1.3) and and $\mathbf{u} = [u(1), \dots, u(n)]^T$, $\mathbf{u}^+ = [u^+(1), \dots, u^+(n)]^T$, $\mathbf{u}^- = [u^-(1), \dots, u^-(n)]^T$.

In particular, studying the set of all pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the problem (P4) has a non-trivial solution u, is equivalent to the investigation of the set $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$

$$\Sigma\left(\mathbf{A}^{\mathsf{D}}\right) = \left\{(\alpha, \beta) \in \mathbb{R}^2 : \text{ the problem } \mathbf{A}^{\mathsf{D}} \mathbf{u} = \alpha \mathbf{u}^+ - \beta \mathbf{u}^- \text{ has a non-trivial solution } \mathbf{u}\right\},$$

and similarly to the general notation within this thesis, $\Sigma(\mathbf{A}^{\mathsf{D}})$ is called the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$. To find the set $\Sigma(\mathbf{A}^{\mathsf{D}})$ will be the main purpose of our investigation.

Let us point out that Fučík spectrum is symmetric with respect to the line $\alpha = \beta$, i.e. $(\alpha, \beta) \in \Sigma(\mathbf{A}^{\mathsf{D}})$ with Fučík eigenvector $\mathbf{v}$ if and only if $(\beta, \alpha) \in \Sigma(\mathbf{A}^{\mathsf{D}})$ with Fučík eigenvector $-\mathbf{v}$ (see Figures 1.4 and 1.5). Before diving into particular challenges, let us recall

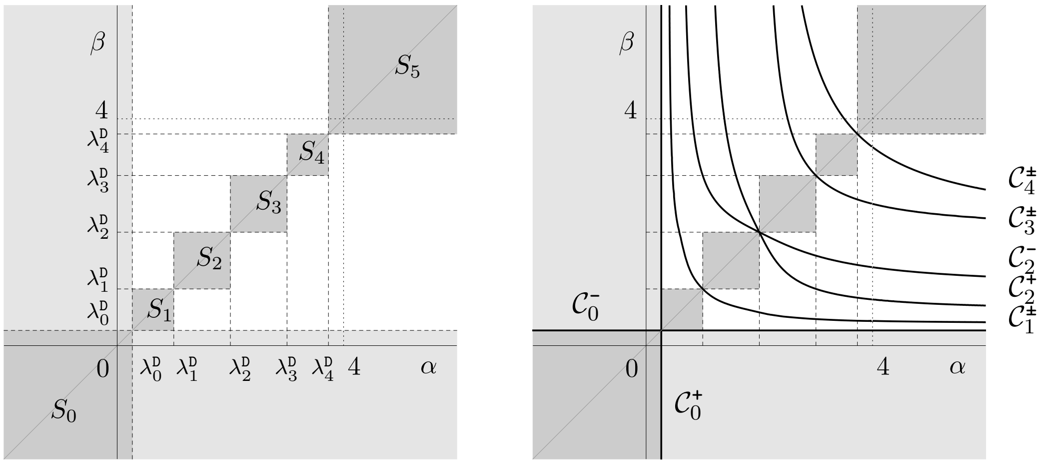

Figure 1.2: Inadmissible areas (defined in [18]) for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ (left, $n = 5$) and the particular Fučík curves $\mathcal{C}_k^\pm$ (black curves) of the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ (right, $n = 5$).

some known results about $\Sigma(\mathbf{A}^{\mathsf{D}})$ (for more details, see [25] and [31]). The eigenvalues of $\mathbf{A}^{\mathsf{D}}$ are of the form

$$\lambda_j^{\mathsf{D}} = 4\sin^2\frac{(j+1)\pi}{2(n+1)}, \qquad j = 0,\dots,n-1$$

and $\lambda_j^{\mathsf{D}} \in (0,4)$. Note that the eigenvalues λ_j^{D} of matrix $\mathbf{A}^{\mathsf{D}}$ belong to the Fučík spectrum in the sense $(\lambda_j^{\mathsf{D}}, \lambda_j^{\mathsf{D}}) \in \Sigma(\mathbf{A}^{\mathsf{D}})$, i.e. $(\lambda_j^{\mathsf{D}}, \lambda_j^{\mathsf{D}})$ is the Fučík eigenpair for matrix $\mathbf{A}^{\mathsf{D}}$. For the Fučík spectrum of $\mathbf{A}^{\mathsf{D}}$ we have

$$\Sigma\left(\mathbf{A}^{\mathsf{D}}\right) = \bigcup_{l=0}^{n-1} \mathcal{C}_l^\pm,$$

where $\mathcal{C}_l^+$ and $\mathcal{C}_l^-$ are Fučík curves (see Section 1.1 – point 5).

In [18], authors were exploring inadmissible areas of Fučík spectrum (i.e. Fučík spectrum has empty intersection with these areas in (α,β) plane – see [18] for proper definition of an inadmissible area). Since λ_0^{D} is a principal eigenvalue of $\mathbf{A}^{\mathsf{D}}$, it implies that

$$\left\{(\alpha,\beta) \in \mathbb{R}^2 : \left(\alpha - \lambda_0^{\mathsf{D}}\right)\left(\beta - \lambda_0^{\mathsf{D}}\right) < 0\right\} \cap \Sigma\left(\mathbf{A}^{\mathsf{D}}\right) = \emptyset,$$

i.e. both shifted quadrants are inadmissible areas for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$. For illustration, see Figure 1.2 where we can see inadmissible areas for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$. Thus, it is enough to investigate the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathsf{D}}\right)$ only on the set $D = ((0,4)\times(0,+\infty)) \cup ((0,+\infty)\times(0,4))$.

Also, it is enough to investigate only all Fučík curves $\mathcal{C}_l^+$ ($l = 1,\dots,n-1$), since

$$\mathcal{C}_l^- = \{(\alpha,\beta) \in D : (\beta,\alpha) \in \mathcal{C}_l^+\}.$$

Authors Ma, Xu and Gao introduced the matching-extension method for solutions of the Fučík spectrum problem for matrix $\mathbf{A}^{\mathsf{D}}$ in [26]. P. Stehlík studied the qualitative properties of the first non-trivial Fučík curve of the matrix $\mathbf{A}^{\mathsf{D}}$ in [37]. Although this topic was studied by several authors, corresponding analytic description was not introduced prior to author's articles [25] and [31] (as far as we know).

Before looking into individual results, we contemplate what possible challenges can appear while investigating the Fučík spectrum for matrix $\mathbf{A}^{\mathsf{D}}$, using illustrative examples. In Example 1 we will investigate the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ of size $n = 2$.

Example 1. Let $n = 2$, thus let us deal with the Dirichlet matrix in the form

$$\mathbf{A}^{\mathsf{D}} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Eigenvalues and corresponding eigenvectors are of the form

$$\lambda_0^{\mathsf{D}} = 1,\ \mathbf{v}_0 = [1,1]^T, \quad \lambda_1^{\mathsf{D}} = 3,\ \mathbf{v}_1 = [1,-1]^T.$$

All possible sign properties for Fučík eigenvectors are

$$[+,+]^T, [+,-]^T, [+,0]^T, [-,+]^T, [-,-]^T, [-,0]^T, [0,+]^T, [0,-]^T, [0,0]^T.$$

Similar to the case of eigenvalue problems, the sign properties $[+,+]^T$ and $[-,-]^T$ lead to the Fučík eigenvectors where we have opposite signs of the entries. The same works for couples $[+,-]^T$ and $[-,+]^T$, for $[+,0]^T$ and $[-,0]^T$ and for $[0,+]^T$ and $[0,-]^T$. Thus, it is enough to consider only $[+,+]^T$, $[+,-]^T$, $[+,0]^T$, $[0,+]^T$ and $[0,0]^T$.

1. Case $[0,0]^T$: Such case cannot happen since the Fučík eigenvector cannot be trivial.

2. Case $[0,+]^T$: The first entry of the Fučík eigenvector is zero, thus the solution of problem (P4) is zero in two consequential points (due to the zero boundary conditions). The difference equation in (P4) can be written as

   $$u(k+1) = 2u(k) - u(k-1) - \alpha u^+(k) + \beta u^-(k),$$

   thus if the solution u is zero in two consequential points, it has to be zero everywhere. That is a contradiction with the sign property $[0,+]^T$.

3. Case $[+,0]^T$: There is the same issue as in the previous case.

4. Case $[+,+]^T$: In this case the Fučík eigenvector does not change sign thus it is equivalent to the eigenvalue problem for λ_0^{D}. We have $(\alpha,\beta) \in \Sigma(\mathbf{A}^{\mathsf{D}}) : \alpha = \lambda_0^{\mathsf{D}} = 1,\ \beta \in \mathbb{R}$ with Fučík eigenvector $[1,1]^T$ and $(\alpha,\beta) \in \Sigma(\mathbf{A}^{\mathsf{D}}) : \beta = \lambda_0^{\mathsf{D}} = 1,\ \alpha \in \mathbb{R}$ with Fučík eigenvector $[-1,-1]^T$. The Fučík curves $\mathcal{C}_0^{\pm}$ are trivial ones

   $$\mathcal{C}_0^{+} = \left\{(\alpha,\beta) :\ \alpha = \lambda_0^{\mathsf{D}},\ \beta \in \mathbb{R}\right\}, \qquad \mathcal{C}_0^{-} = \left\{(\alpha,\beta) :\ \beta = \lambda_0^{\mathsf{D}},\ \alpha \in \mathbb{R}\right\}.$$

5. Case $[+,-]^T$: From the sign property of vector $\mathbf{u} = [u_1, u_2]^T$ we have $\mathbf{u}^+ = [u_1, 0]^T$ and $\mathbf{u}^- = [0, -u_2]^T$, where
$$u_1 > 0 \text{ and } u_2 < 0. \tag{1.4}$$
We can rewrite the problem $\mathbf{A}^{\mathsf{D}}\mathbf{u} = \alpha\mathbf{u}^+ - \beta\mathbf{u}^-$ as
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \alpha\begin{bmatrix} u_1 \\ 0 \end{bmatrix} - \beta\begin{bmatrix} 0 \\ -u_2 \end{bmatrix}. \tag{1.5}$$
Matrix equation in (1.5) is equivalent to
$$\begin{bmatrix} 2-\alpha & -1 \\ -1 & 2-\beta \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \;\Leftrightarrow\; \det\begin{bmatrix} 2-\alpha & -1 \\ -1 & 2-\beta \end{bmatrix} = 0. \tag{1.6}$$
The determinant in (1.6) is zero if
$$(2-\alpha)(2-\beta) - 1 = 0. \tag{1.7}$$
This leads to
$$\beta = 2 - \frac{1}{2-\alpha},\;\; \mathbf{u} = \left[-\frac{1}{2-\alpha}, -1\right]^T.$$
Let us go back to the sign property in (1.4). It is satisfied when
$$-\frac{1}{2-\alpha} > 0 \;\Leftrightarrow\; \alpha > 2.$$
If we would consider sign property $\operatorname{sign}\mathbf{u} = [-,+]^T$ we would get the same result, thus the Fučík curves $\mathcal{C}_1^{\pm}$ are
$$\mathcal{C}_1^{+} = \mathcal{C}_1^{-} = \left\{(\alpha,\beta):\; \beta = 2 - \frac{1}{2-\alpha},\; \alpha > 2\right\}.$$

While going through all possible sign properties for the Fučík eigenvectors we were able to find complete description of the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ of size $n = 2$ as
$$\Sigma(\mathbf{A}^{\mathsf{D}}) = \mathcal{C}_0^{\pm} \cup \mathcal{C}_1^{\pm},$$
where $\mathcal{C}_0^{\pm}$ and $\mathcal{C}_1^{\pm}$ are given as above. See Figure 1.3 for illustration of this example.

In the following example we will consider $n = 6$, to illustrate a dimension complexity of the problem.

Example 2. In this example we will consider matrix $\mathbf{A}^{\mathsf{D}}$ of size $n = 6$. We will show all the possible sign properties for the Fučík eigenvectors. It is enough to investigate sign properties with positive first entries (since the Fučík spectrum is symmetric with respect to the line $\alpha = \beta$ and the Fučík eigenvectors have opposite signs). Also, for the sake of simplicity, we can investigate sign properties $\operatorname{sign}(u(k)) = 0$ and $\operatorname{sign}(u(k)) = 1$ (for some

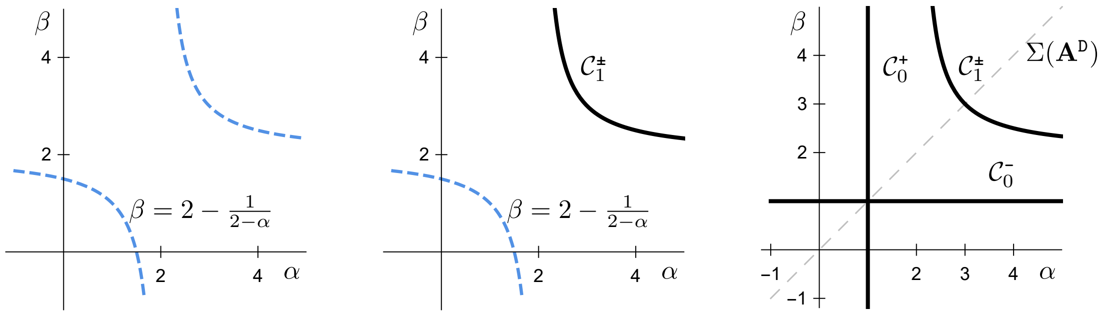

Figure 1.3: Graph of the function $\beta = 2 - \frac{1}{2-\alpha}$ (left), Fučík curves $\mathcal{C}_1^{\pm}$ (black) as part of the graph of the function $\beta = 2 - \frac{1}{2-\alpha}$ (middle) and the Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$ of matrix $\mathbf{A}^{\mathsf{D}}$ for $n = 2$.

$k \in \{1, 2, \ldots, n\}$) together. After this simplification, we need to investigate 2^{n-1} different sign properties.

All sign properties which we need to investigate are written in Table 1.1. Each column has 6 entries and represents one sign property for vector. Those sign properties which are in blue color are sign properties which at least one of the Fučík eigenvectors has (in the thesis it will be shown how to select the right ones).

To illustrate a curse of dimensionality of the studied problems, let us compare two cases of matrix $\mathbf{A}^{\mathsf{D}}$ dimension: $n_1 = 2$ and $n_2 = 6$. Within Example 1, we have shown that we need to investigate only 2 cases or more generally 2^{n_1-1} cases. However, in this example we are solving $2^{n_2-1} = 2^5 = 32$ different sign properties, each leads towards investigation of a different eigenvalue / eigenvector problem.

In particular, let us take one of the sign properties: $[+,+,+,-,+,-]^T$. For this sign property we need to solve

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \alpha \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ 0 \\ u_5 \\ 0 \end{bmatrix} - \beta \begin{bmatrix} 0 \\ 0 \\ 0 \\ -u_4 \\ 0 \\ -u_6 \end{bmatrix}.$$

This leads to the determinant equation

$$\det(\mathbf{A}^{\mathsf{D}} - \mathbf{\Lambda}) = \det\left(\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{bmatrix}\right) = 0, \tag{1.8}$$

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
+	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−

+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
−	−	−	−	−	−	−	−	−	−	−	−	−	−	−	−
+	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
+	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
+	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−

Table 1.1: Considered sign properties for $n = 6$. The blue sign properties are sign properties satisfied by some Fučík eigenvectors.

i.e.

$$\det \begin{bmatrix} 2-\alpha & -1 & 0 & 0 & 0 & 0 \\ -1 & 2-\alpha & -1 & 0 & 0 & 0 \\ 0 & -1 & 2-\alpha & -1 & 0 & 0 \\ 0 & 0 & -1 & 2-\beta & -1 & 0 \\ 0 & 0 & 0 & -1 & 2-\alpha & -1 \\ 0 & 0 & 0 & 0 & -1 & 2-\beta \end{bmatrix} = 0.$$

Since we are dealing with tridiagonal matrix, we can easily calculate its determinant and the determinant equation is a polynomial equation

$$\begin{aligned} &\alpha^4\beta^2 - 4\alpha^4\beta + 4\alpha^4 - 8\alpha^3\beta^2 + 29\alpha^3\beta - 26\alpha^3 + 22\alpha^2\beta^2 - 70\alpha^2\beta + 53\alpha^2 - 24\alpha\beta^2 \\ &+65\alpha\beta - 38\alpha + 8\beta^2 - 18\beta + 7 = 0. \end{aligned}$$

By comparing this with (1.7) (for $n = 2$), we can see that the dimension of the problem brings a lot of difficulties. We can find two values of β (dependent of the value of α as it was done in Example 1) for which we can derive that neither one of them has a corresponding eigenvector with the sign property $[+,+,+,-,+,-]^T$. That means that there does not exist Fučík eigenvector for matrix $\mathbf{A}^{\mathtt{D}}$ of size $n = 6$ with such sign property.

Since this problem depends highly on the dimension of the matrix $\mathbf{A}^{\mathtt{D}}$ (we are dealing with 2^{n-1} different eigenvalue / eigenvector problems based on the number of possible sign properties), our computational possibilities might be limiting for practical applications using the illustrated approach [1].

Let us summarize some of the challenges which appear in the investigation of the Fučík spectrum $\Sigma(\mathbf{A}^{\mathtt{D}})$ of matrix $\mathbf{A}^{\mathtt{D}}$ of size n:

[1] E.g. in a relatively reasonable time we might be able to find (numerically)Fučík spectrum up to $n = 16$.

- Number of possible sign properties is 2^{n-1} (after the simplification which was done in Example 2).
- Only some of them are sign properties satisfied by Fučík eigenvectors of $\Sigma(\mathbf{A}^\mathtt{D})$.
- We note that for a general matrix, one might struggle with computation of the matrix determinant. Whereas for the Dirichlet matrix, $\det(\mathbf{A}^\mathtt{D} - \mathbf{\Lambda})$ (see (1.8) in Example 2) can be calculated recurrently (due to having a tridiagonal symmetric matrix).
- For each sign property we need to verify which parts (if any) of the solution (curve) of $\det(\mathbf{A}^\mathtt{D} - \mathbf{\Lambda}) = 0$ are actually in the Fučík spectrum $\Sigma(\mathbf{A}^\mathtt{D})$.

On Figure 1.4 we can see the Fučík spectrum $\Sigma(\mathbf{A}^\mathtt{D})$ of size $n = 9$. In the thesis, we will introduce how to deal with the curse of dimensionality and other challenges mentioned above.

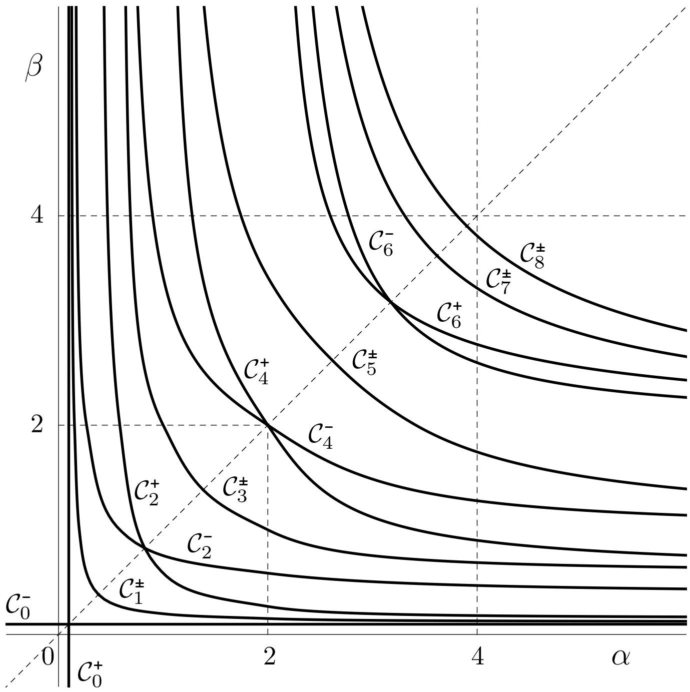

Figure 1.4: The Fučík spectrum $\Sigma\left(\mathbf{A}^\mathtt{D}\right)$ of the Dirichlet matrix $\mathbf{A}^\mathtt{D}$ of size $n = 9$ and its Fučík curves $\mathcal{C}_l^\pm$, $l = 0, 1, \ldots, 8$.

1.3 Structure of the thesis

First of all, we would like to note that the thesis is mainly based on research articles of the author: [25] and [31]. The aim of the thesis is not to provide in-depth technical details for all newly introduced concepts in [25] and [31], but rather to provide a comprehensive

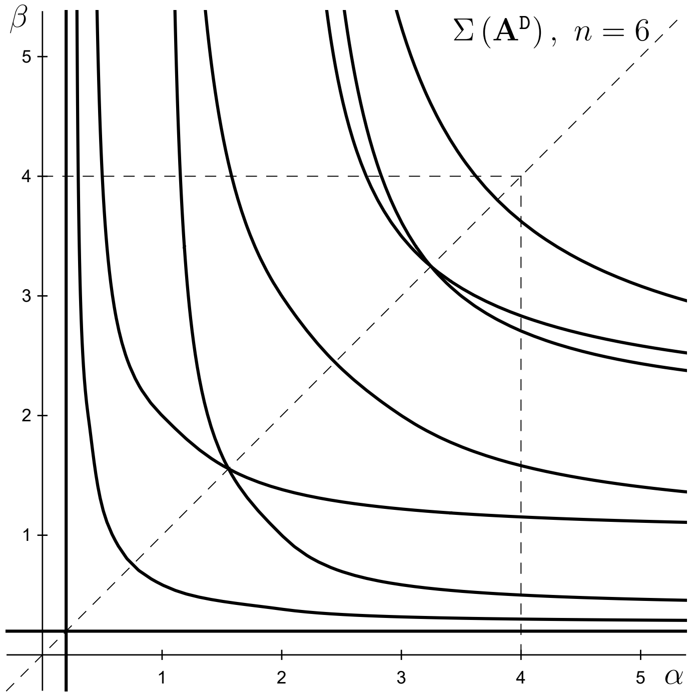

Figure 1.5: The Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$ of the Dirichlet matrix $\mathbf{A}^{\mathsf{D}}$ of size $n = 6$.

overview, illustrate the concepts on particular examples and to explain connections between individual concepts.

We note that in order to provide such comprehensive text, we also extend the results in aforementioned articles by supporting lemmas / theorems, new illustrations / examples and other new results. However, for most of the proofs of original theorems and lemmas we refer the reader to the articles which are attached to the thesis. If there is no citation (excluding citation to [25] and [31] – articles of the author) in the definitions, lemmas and theorems, then the results presented there are original and (as far as we know) not published anywhere else. We note that results in Section 5.5 of the thesis are completely new and not published anywhere yet. The thesis is organized as follows.

Chapter 1 provides an introduction to the problems and showcases possible issues which may appear while investigating discrete Fučík spectrum. In the following chapters, we are going to investigate in detail four problems, introduced in Section 1.1:

i. linear initial value problem (P1);

ii. linear boundary value problem (P2);

iii. semi-linear initial value problem (P3);

iv. semi-linear boundary value problem (P4).

Chapter 2 is devoted to the study of linear problems (P1) and (P2). We are going to define one of the most important tool-kits in the thesis – the continuous extension of respective solutions. Exploring such continuous extension will allow us to explore nodal properties of the solution. A generalization of this result will be very valuable in the analysis of semi-linear problems. Let us note that even though we are spending a substantial part of the thesis (and likewise a substantial part of research articles [25] and [31]) studying simple linear problems (P1) and (P2), the results in this chapter are new and (as far as we know) not published anywhere. We need to construct a robust theory for the linear case in order to explore semi-linear case.

In Chapters 3 and 4, we are solving and investigating semi-linear initial value problem (P3). Generalizing the theory from the linear case (such as continuous extension) will allow us to "anchor" positive and negative semi-waves. This will lead to the detailed investigation of zeros of a continuous extension of the solution. Chapter 3 leverages the main results from [25] and Chapter 4 references results from [31].

Finally, Chapter 5 is devoted to the investigation of the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ (i.e. the corresponding semi-linear boundary value problem (P4)) – which is our main goal in the thesis (and in the research articles [25] and [31]). Several descriptions of the Fučík spectrum (analytical and implicit) are introduced. As far as we know, this is the first time anyone was able to find an analytical (and implicit) description of Fučík spectrum of matrix (excluding trivial cases) for any dimension n. In Chapter 5, we also introduce bounds of the Fučík spectrum. Such bounds can be used for efficient numerical estimations as illustrated therein.

Last but not least, we provide published articles [25] and [31]. Introduction sections in both articles describe historical references related to the Fučík spectrum and also our motivation for studying this topic in detail (including more details about practical applications).

1.4 Structure of the summary text

In this summary text, we provide introduction to the problems and we showcase possible issues which may appear while investigating discrete Fučík spectrum (as it is done in Chapter 1 of the thesis). We also define all main functions and terms and provide main theorems. Chapters 2 and 3 in this summary text are devoted to the brief introduction and summary of main results of the thesis – descriptions of the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ and bounds of Fučík curves of the Fučík spectrum for matrix $\mathbf{A}^{\mathsf{D}}$.

Chapter 2

Main results – description of the Fučík spectrum of matrix $\mathbf{A}^{\mathtt{D}}$

This chapter (and also the next one) is devoted to the summary of main results. We investigate semi-linear boundary value problem (P4)

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{T}, \\ u(0) = u(n+1) = 0, \end{cases}$$

where $n \in \mathbb{N}$, $n \geq 2$, $u^{\pm}(k) = \max\{\pm u(k), 0\}$ and $\alpha, \beta \in \mathbb{R}$.

Equivalently, the problem (P4) can be rephrased using a matrix notation

$$\mathbf{A}^{\mathtt{D}}\mathbf{u} = \alpha\mathbf{u}^+ - \beta\mathbf{u}^-,$$

where matrix $\mathbf{A}^{\mathtt{D}}$ is the Dirichlet matrix defined in (1.3). Thus, our main goal is to investigate Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$. Some known results were already discussed in Section 1.2.

We are going to briefly present two main results. As a first main result we introduce (in this chapter) two different descriptions (both implicit and recurrent) of Fučík spectrum of matrix $\mathbf{A}^{\mathtt{D}}$. And as second main result, we introduce (in the next chapter) two different bounds of Fučík curves which can be used for numerical estimation of Fučík curves (they are not recurrent and they do not become more complicated when dimension n increases).

2.1 Linear case

Firstly, let us consider the linear initial value problem (P1)

$$\begin{cases} \Delta^2 u(k-1) + \lambda u(k) = 0, & k \in \mathbb{Z}, \\ u(0) = C_0, u(1) = C_1, \end{cases}$$

where $\lambda \in \mathbb{R}$ and $C_0, C_1 \in \mathbb{R}$ are constants such that $C_0^2 + C_1^2 \neq 0$.

This problem is the easiest one to solve (considering all problems (P1), (P2), (P3) and (P4)). Yet, a complete understanding of how one can get the solution and what are the properties of such a solution, leads to valuable knowledge and tools for further study of more difficult problems such as linear boundary value problem (P2), semi-linear initial value problem (P3) and even semi-linear boundary value problem (P4).

The following lemma is used to find a solution for linear initial value problem (P1) which might be also utilized later on for more complex problems.

Lemma. *(Lemma 3 in the thesis)*
For given $\lambda \in \mathbb{R}$ and $C_0, C_1 \in \mathbb{R}$, the linear initial value problem (P1) *has a unique solution of the form*

$$u(k) = C_0 F^\lambda(1-k) + C_1 F^\lambda(k), \quad k \in \mathbb{Z},$$

where the function $F^\lambda : \mathbb{R} \to \mathbb{R}$ is defined as

$$F^\lambda(t) := \begin{cases} \sinh(\omega_\lambda t)/\sinh\omega_\lambda & \text{for } \lambda < 0, \\ t & \text{for } \lambda = 0, \\ \sin(\omega_\lambda t)/\sin\omega_\lambda & \text{for } \lambda \in (0,4), \\ -t\cos(\pi t) & \text{for } \lambda = 4, \\ -\cos(\pi t)\sinh(\omega_\lambda t)/\sinh\omega_\lambda & \text{for } \lambda > 4, \end{cases} \quad \omega_\lambda := \begin{cases} \operatorname{arcosh}\frac{2-\lambda}{2} & \text{for } \lambda \le 0, \\ \arccos\frac{2-\lambda}{2} & \text{for } \lambda \in (0,4), \\ \operatorname{arcosh}\frac{\lambda-2}{2} & \text{for } \lambda \ge 4. \end{cases}$$

For the solution u of the discrete problem (P1), let us define its continuous extension u^{c} on $\mathbb{R}$ as

$$u^{\mathsf{c}}(t) := C_0 F^\lambda(1-t) + C_1 F^\lambda(t), \quad t \in \mathbb{R}.$$

For illustration of the continuous extension of solution u for the case $\lambda \in (0,4)$ see Figure 2.1.

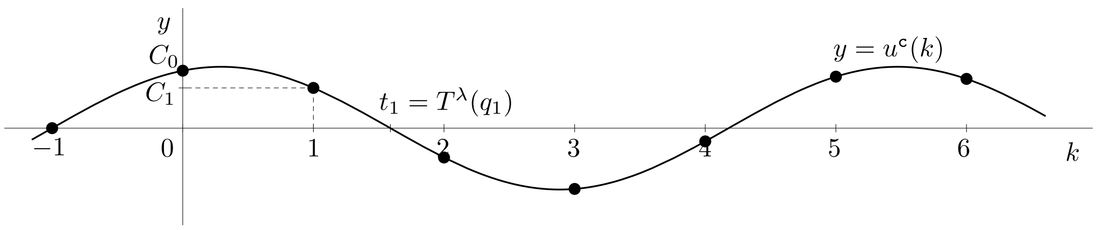

Figure 2.1: Continuous extension u^{c} of solution u of the initial value problem (P1) for $\lambda \in (0,4)$, $\lambda = 1.3$ and the first non-negative zero t_1 of u^{c}; $q_1 = \frac{C_1}{C_0}$.

Definition. (Definition 4 in the thesis)
Let us define the sequence $(q_k)_{k\in\mathbb{Z}}$ as a mapping from $\mathbb{Z}$ to $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$ (the one-point compactification of $\mathbb{R}$) as

$$q_k := \frac{u(k)}{u(k-1)}, \qquad k \in \mathbb{Z}.$$

Such sequence (q_k) is defined correctly since value of u in two consecutive integers cannot be zero. If $u(0) = C_0 = 0$, then $q_1 = \frac{C_1}{C_0} = \frac{C_1}{0} = \infty$ independent of the sign of C_1. Sequence (q_k) will be very important in the investigation of initial value problems and the properties of the solution. We are not going to focus on the values of u themselves but on these ratios (q_k). Using such approach will allow us to study the problem in detail, find zeros of solution u and describe any term of such sequence (q_k).

We can define generalized zero of a solution u (for the original definition of a generalized zero see [14]).

Definition. (Definition 5 in the thesis)
Solution u of the discrete problem (P1) has a generalized zero at $k \in \mathbb{Z}$ if

$$u(k) = 0 \quad \text{or} \quad u(k)u(k-1) < 0.$$

From the definition of (q_k) we have that u has a generalized zero at $k \in \mathbb{Z}$ if and only if $q_k \leq 0$ and $q_k \neq \infty$.

We can distinguish between three different cases dependent on the value of λ and find the number of generalized zeros of solution u of the discrete problem (P1) (distinct cases are: $\lambda \leq 0$, $\lambda \in (0,4)$ and $\lambda \geq 4$). We will (for the purpose of this summary text) focus mainly on $\lambda \in (0,4)$. For such value of λ, the solution u has infinitely many generalized zeros. In this case, $0 < \omega_\lambda < \pi$ and the continuous extension u^c is $\frac{2\pi}{\omega_\lambda}$-periodic function. The first non-negative zero of a continuous extension u^c is determined by the function T^λ (only for $\lambda \in (0,4)$; for other cases of λ, function T^λ has a different role) defined in the following definition (see again Figure 2.1 where the first non-negative zero is shown; and see Figure 2.2 for illustration of function T^λ in case $\lambda \in (0,4)$.)

Definition. (Definition 8 in the thesis)
For $\lambda \in \mathbb{R}$, let us define the function $T^\lambda : \mathbb{R}^* \to \mathbb{R}$, $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$, as

$$\begin{aligned}
\operatorname{Dom}(T^\lambda) &:= \begin{cases} \mathbb{R}^* \setminus [\mathrm{e}^{-\omega_\lambda}, \mathrm{e}^{\omega_\lambda}] & \text{for } \lambda \leq 0, \\ \mathbb{R}^* & \text{for } \lambda \in (0,4), \\ \mathbb{R}^* \setminus [-\mathrm{e}^{\omega_\lambda}, -\mathrm{e}^{-\omega_\lambda}] & \text{for } \lambda \geq 4, \end{cases} \\
T^\lambda(\infty) &:= 0 \\
T^\lambda(q) &:= \begin{cases} \frac{1}{\omega_\lambda} \operatorname{arcoth}\left(\frac{\cosh \omega_\lambda - q}{\sinh \omega_\lambda}\right) & \text{for } \lambda < 0, \\ \frac{1}{1-q} & \text{for } \lambda = 0, \\ \frac{1}{\omega_\lambda} \operatorname{arccot}\left(\frac{\cos \omega_\lambda - q}{\sin \omega_\lambda}\right) & \text{for } \lambda \in (0,4), \\ \frac{1}{1+q} & \text{for } \lambda = 4, \\ \frac{1}{\omega_\lambda} \operatorname{arcoth}\left(\frac{\cosh \omega_\lambda + q}{\sinh \omega_\lambda}\right) & \text{for } \lambda > 4. \end{cases}
\end{aligned}$$

We assume that inverse cotangent (arccotangent) has the usual principal values, thus it is defined for all real numbers and its range is interval $(0, \pi)$.

We can define function Q^λ which is the inverse function of T^λ.

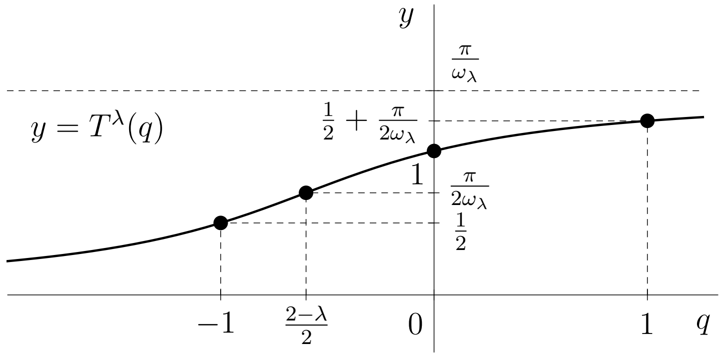

Figure 2.2: The graph of T^λ as a function of q, case $\lambda \in (0,4)$.

Definition. (Definition 9 in the thesis)
For $\lambda \in \mathbb{R}$, let us define the function $Q^\lambda : \mathbb{R} \to \mathbb{R}^*$, $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$, as

$$\begin{aligned}
\mathrm{Dom}(Q^\lambda) &:= \begin{cases} \left[0, \dfrac{\pi}{\omega_\lambda}\right) & \text{for } \lambda \in (0,4), \\ \mathbb{R} & \text{for } \lambda \in \mathbb{R} \setminus (0,4), \end{cases} \\
Q^\lambda(0) &:= \infty \\
Q^\lambda(t) &:= \begin{cases} -\dfrac{\sinh(\omega_\lambda(1-t))}{\sinh(\omega_\lambda t)} & \text{for } \lambda < 0, \\ -\dfrac{1-t}{t} & \text{for } \lambda = 0, \\ -\dfrac{\sin(\omega_\lambda(1-t))}{\sin(\omega_\lambda t)} & \text{for } \lambda \in (0,4), \\ \dfrac{1-t}{t} & \text{for } \lambda = 4, \\ \dfrac{\sinh(\omega_\lambda(1-t))}{\sinh(\omega_\lambda t)} & \text{for } \lambda > 4. \end{cases}
\end{aligned}$$

We have that the first non-negative zero t_1 can be calculated as (for $\lambda \in (0,4)$)

$$t_1 = j + T^\lambda(q_{1+j}), \qquad j = \lceil t_0 \rceil, \dots, 0, \dots, \lfloor t_1 \rfloor,$$

where t_0 is the previous zero of continuous extension u^{c} of solution u. For illustration, see Figure 2.3, where is $\lceil t_0 \rceil = -1$, $\lfloor t_1 \rfloor = 4$. For such example, there are 6 possible ways how to get t_1 using sequence $(q_k)_{k\in\mathbb{Z}}$. We have

$$t_1 = -1 + T^\lambda(q_0),\ t_1 = T^\lambda(q_1),\ t_1 = 1 + T^\lambda(q_2),$$
$$t_1 = 2 + T^\lambda(q_3),\ t_1 = 3 + T^\lambda(q_4),\ t_1 = 4 + T^\lambda(q_5).$$

There is a connection between the solution u of linear initial problem (P1) and Chebyshev polynomials of the second kind. Let us recall definition of Chebyshev polynomials of the second kind (for more details see [29]).

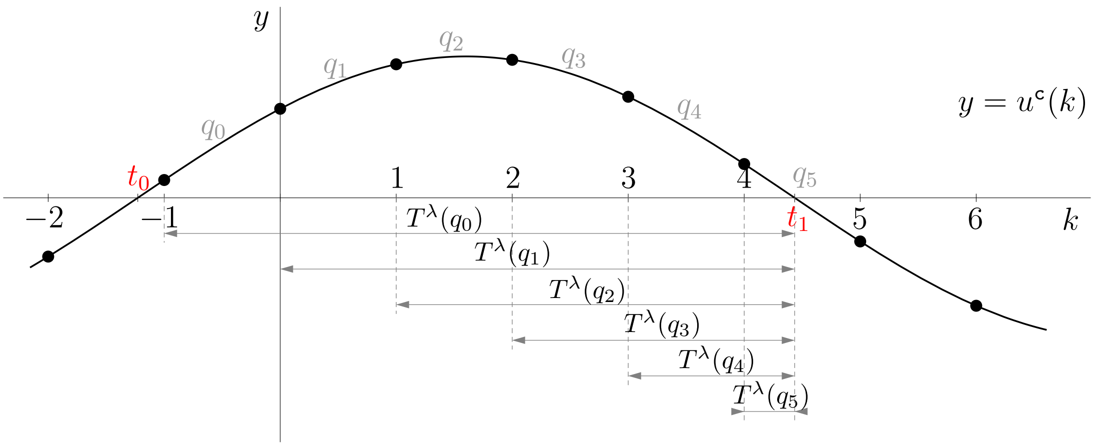

Figure 2.3: The bi-infinite sequence $(q_k)_{k\in\mathbb{Z}}$ of ratios of values of u as the solution of the initial value problem (P1) and its relation to the first non-negative zero point t_1 using function T^λ (case $\lambda \in (0,4)$).

Definition. Chebyshev polynomials U_k of the second kind of degree $k \in \mathbb{Z}$ at the point $x \in \mathbb{R}$ are defined by the recurrence formula

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x)$$

with initial conditions $U_0(x) = 1$, $U_1(x) = 2x$.

For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, let us denote

$$V_k^\lambda := U_k\left(\tfrac{2-\lambda}{2}\right).$$

For all $\lambda \in \mathbb{R}$, polynomials V_k^λ satisfy the three terms recurrence formula

$$V_{k-1}^\lambda - (2-\lambda)V_k^\lambda + V_{k+1}^\lambda = 0, \quad k \in \mathbb{Z},$$

with initial conditions $V_0^\lambda = 1$, $V_1^\lambda = 2 - \lambda$. Initial value problem (P1) has solution in recurrence form

$$u(k-1) - (2-\lambda)u(k) + u(k+1) = 0$$

with initial conditions $u(0) = C_0$ and $u(1) = C_1$. Therefore, V_k^λ is the solution of the initial value problem (P1) with $C_0 = V_0^\lambda = 1$ and $C_1 = V_1^\lambda = 2 - \lambda$.

Moreover, for all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$ we have

$$F^\lambda(k) = V_{k-1}^\lambda.$$

Such property allows us to get the solution u of (P1) as

$$u(k) = -C_0V_{k-2}^\lambda + C_1V_{k-1}^\lambda.$$

It is convenient to use Chebyshev polynomials V_k^λ for the definition of function W_k^λ. Function W_k^λ determines the value of k-th element q_k by the value of q_0.

Definition. (Definition 13 in the thesis)
For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, let us define the function $W_k^\lambda : \mathbb{R}^* \to \mathbb{R}^*$ in the following way

$$W_k^\lambda(q) := \begin{cases} \dfrac{q \cdot V_k^\lambda - V_{k-1}^\lambda}{q \cdot V_{k-1}^\lambda - V_{k-2}^\lambda} & \text{for } q \in \mathbb{R}, \\[2ex] \dfrac{V_k^\lambda}{V_{k-1}^\lambda} & \text{for } q = \infty. \end{cases}$$

Lemma. *(Lemma 15 in the thesis)*
For all $\lambda \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have

$$q_k = W_k^\lambda(q_0).$$

Let us assume that we have some element of bi-infinite sequence (q_k) (for example $q_1 = {}^{C_1}/_{C_0}$ given by the initial conditions). If we want to get any other element of such sequence or the first non-negative zero t_1 of u^{c}, we can use the following formulas.

1. For $\lambda \in \mathbb{R}$ and $i, j, k \in \mathbb{Z}$ such that $i + j = k$, we have that

$$q_k = W_j^\lambda(q_i).$$

   This can be used for calculation of any term in the sequence $(q_k)_{k\in\mathbb{Z}}$ from the initial condition. Let our initial condition be $C_0 = 0$, $C_1 \in \mathbb{R} \setminus \{0\}$. Then we have $q_1 = \frac{u(1)}{u(0)} = \frac{C_1}{0} = \infty$. And for any $k \in \mathbb{Z}$, we have $q_k = W_{k-1}^\lambda(q_1) = W_{k-1}^\lambda(\infty)$.

2. For $\lambda \in (0, 4)$, we have for the first non-negative zero t_1 of u^{c} that

$$t_1 = j + T^\lambda(W_j^\lambda(q_1)), \qquad j = \lceil t_0 \rceil, \dots, 0, \dots, \lfloor t_1 \rfloor.$$

2.2 Descriptions of the Fučík spectrum of matrix $\mathbf{A}^{\mathsf{D}}$ – part I

We can use theory from the linear case in order to extend the theory also for the semi-linear initial value problem (P3)

$$\begin{cases} \Delta^2 u(k-1) + \alpha u^+(k) - \beta u^-(k) = 0, & k \in \mathbb{Z}, \\ u(0) = 0, u(1) = C_1, \end{cases}$$

where $u^\pm(k) = \max\{\pm u(k), 0\}$, $C_1 \in \mathbb{R}$, $C_1 \neq 0$ and $(\alpha, \beta) \in D$,

$$D := ((0, 4) \times (0, +\infty)) \cup ((0, +\infty) \times (0, 4)).$$

We can define a continuous extension $u_{i,j}^{\mathsf{c}}$ of u on the interval $[i-1, j+1]$, where $i \in \mathbb{Z}$ is a generalized zero (similarly as we have defined a generalized zero for problem (P1), we can define it for different problems such as (P3)) and $j \in \mathbb{Z} : j > i$ is such that for all $k = i, \dots, j$, $u(k)$ is non-negative (or non-positive) and (see Figure 2.4 (left))

$$u(j)u(j+1) < 0 \quad \text{or} \quad u(j) = 0.$$

This means that i and $(j+1)$ are two consecutive generalized zeros of u if $u(j) \neq 0$. In the case of $u(j) = 0$, i and j are two consecutive generalized zeros of u.

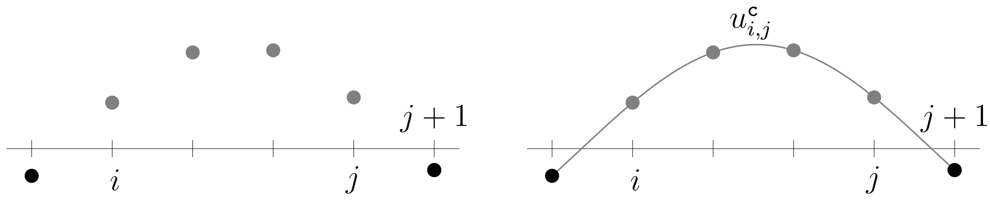

Figure 2.4: Consecutive generalized zeros i, $j+1$ and the continuous extension $u^{\mathsf{c}}_{i,j}$ of u.

On such interval, we construct a continuous extension. We define the continuous extension $u^{\mathsf{c}}_{i,j}$ of u (see Figure 2.4 (right)) on the interval $[i-1, j+1]$ as

$$u^{\mathsf{c}}_{i,j}(t) := \begin{cases} u(i-1)F^{\alpha}(1-(t-i+1)) + u(i)F^{\alpha}(t-i+1) & \text{for } u(i-1) < 0, \\ u(i-1)F^{\beta}(1-(t-i+1)) + u(i)F^{\beta}(t-i+1) & \text{for } u(i-1) > 0, \end{cases}$$

where functions F^{α} and F^{β} are given by F^{λ} for $\lambda = \alpha$ and $\lambda = \beta$, respectively.

Positive semi-wave is a continuous extension $u^{\mathsf{c}}_{i,j}$ of u such that $u(k)$ is non-negative for all $k = i, \dots, j$. Negative semi-wave is continuous extension $u^{\mathsf{c}}_{i,j}$ of u such that $u(k)$ is non-positive for all $k = i, \dots, j$. See Figure 2.5 where positive semi-waves are in orange color and negative semi-waves are in blue color.

In the following definition, we define (recurrently given) sequences $(p_k)_{k\in\mathbb{Z}}$, $(P_k)_{k\in\mathbb{Z}}$, $(\vartheta_k)_{k\in\mathbb{Z}}$, $(\mathcal{W}^{+}_k)_{k\in\mathbb{Z}}$ and $(\mathcal{W}^{-}_k)_{k\in\mathbb{Z}}$. In the text following this definition, we will explain (for the simplest case $0 < \alpha, \beta < 4$) what these sequences represent.

Definition. (Definition 20 in the thesis)
For all $j \in \mathbb{Z}$, let us denote

$$\phi_j := \begin{cases} \alpha & \text{for } j \text{ odd}, \\ \beta & \text{for } j \text{ even}. \end{cases}$$

On the domain $D = ((0,4) \times (0,+\infty)) \cup ((0,+\infty) \times (0,4))$, let us define sequences of functions (p_i) and (ϑ_i), which are given recurrently for $i \in \mathbb{N}$ in the following way

$$\vartheta_0(\alpha,\beta) := \infty,$$

$$p_i(\alpha,\beta) := \begin{cases} \left\lfloor T^{\phi_i}(\vartheta_{i-1}(\alpha,\beta)) + \frac{\pi}{\omega_{\phi_i}} \right\rfloor & \text{for } \phi_i < 4, \\ \left\lfloor T^{\phi_{i+1}}(\vartheta_{i-1}(\alpha,\beta)) + T^{\phi_{i+1}}(2-\phi_i) + 1 \right\rfloor & \text{for } \phi_i \geq 4, \end{cases}$$

$$\vartheta_i(\alpha,\beta) := W^{\phi_i}_{p_i(\alpha,\beta)}(\vartheta_{i-1}(\alpha,\beta)).$$

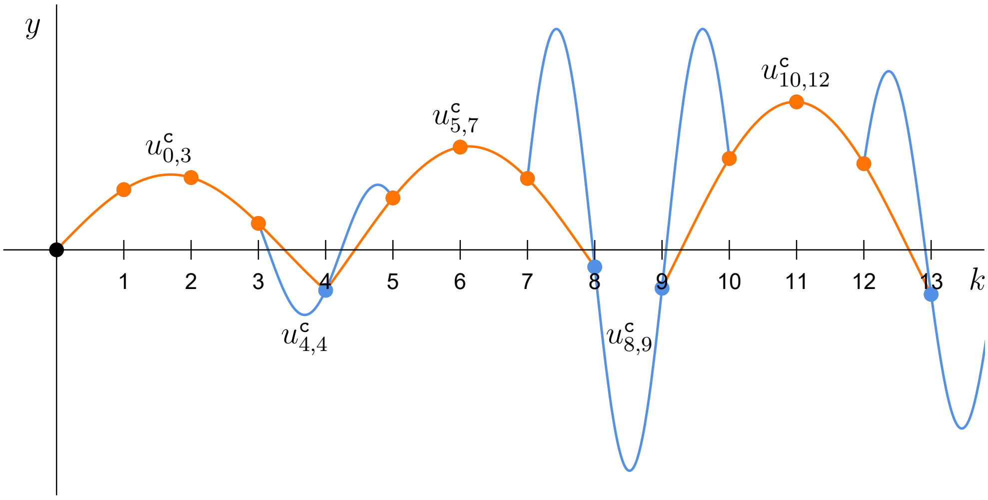

Figure 2.5: Continuous extension of solution u of semi-linear initial value problem (P3) for $\alpha = 0.8$, $\beta = 3.94$, $C_1 = 1$.

Moreover, for all $k \in \mathbb{N}$, let us define function $P_k : D \to \mathbb{N}$ and composite functions $\mathcal{W}^{\pm}_k : \mathbb{R}^* \to \mathbb{R}^*$ as

$$P_k(\alpha,\beta) := \sum_{i=1}^{k} p_i(\alpha,\beta), \qquad \begin{aligned} \mathcal{W}^{+}_k &:= W^{\phi_k}_{p_k(\alpha,\beta)} \circ \cdots \circ W^{\phi_2}_{p_2(\alpha,\beta)} \circ W^{\phi_1}_{p_1(\alpha,\beta)}, \\ \mathcal{W}^{-}_k &:= W^{\phi_{k+1}}_{p_k(\beta,\alpha)} \circ \cdots \circ W^{\phi_3}_{p_2(\beta,\alpha)} \circ W^{\phi_2}_{p_1(\beta,\alpha)}. \end{aligned}$$

We are going to illustrate what sequences in the previous definition mean for special case $0 < \alpha, \beta < 4$ and $C_1 > 0$. Let u be a solution of the semi-linear initial value problem (P3). Since we are looking for description of the sign properties of the solution u, we are interested in all positive generalized zeros of u.

For our restriction ($0 < \alpha, \beta < 4$), first few terms in the sequences $(p_j)_{j\in\mathbb{Z}}$ and $(\vartheta_j)_{j\in\mathbb{Z}}$ are:

$$\begin{aligned} p_1(\alpha,\beta) &:= \left\lfloor \tfrac{\pi}{\omega_\alpha} \right\rfloor, & \vartheta_1(\alpha,\beta) &:= W^{\alpha}_{p_1(\alpha,\beta)}(\infty), \\ p_2(\alpha,\beta) &:= \left\lfloor T^{\beta}(\vartheta_1(\alpha,\beta)) + \tfrac{\pi}{\omega_\beta} \right\rfloor, & \vartheta_2(\alpha,\beta) &:= W^{\beta}_{p_2(\alpha,\beta)}(\vartheta_1(\alpha,\beta)), \\ p_3(\alpha,\beta) &:= \left\lfloor T^{\alpha}(\vartheta_2(\alpha,\beta)) + \tfrac{\pi}{\omega_\alpha} \right\rfloor, & \vartheta_3(\alpha,\beta) &:= W^{\alpha}_{p_3(\alpha,\beta)}(\vartheta_2(\alpha,\beta)), \\ p_4(\alpha,\beta) &:= \left\lfloor T^{\beta}(\vartheta_3(\alpha,\beta)) + \tfrac{\pi}{\omega_\beta} \right\rfloor, & \vartheta_4(\alpha,\beta) &:= W^{\beta}_{p_4(\alpha,\beta)}(\vartheta_3(\alpha,\beta)), \\ &\ \vdots & &\ \vdots \end{aligned}$$

In this part of the text, for simplification, we are going to write p_1 instead of $p_1(\alpha,\beta)$ and similarly for other terms of all sequences from previous definition. For easier understanding of the following text, see Figure 2.6.

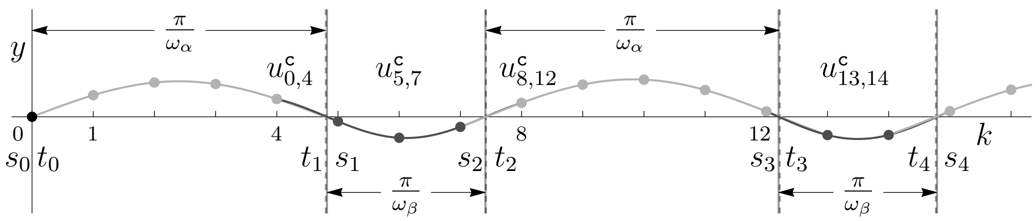

Figure 2.6: Positive and negative semi-waves of a solution of the semi-linear initial value problem (P3) for $0 < \alpha, \beta < 4$ and $C_1 > 0$ ($0 = s_0 = t_0 < t_1 < s_1 < s_2 < t_2 < s_3 < t_3 < t_4 < s_4$).

(a) First positive semi-wave: The first positive semi-wave of u (we have $C_1 > 0$) is u^{c}_{0,p_1}, thus p_1 represents the length which we need to add to $t = 0$ in order to find interval where positive semi-wave is anchored with negative semi-wave.

Positive semi-wave u^{c}_{0,p_1} is defined on $[-1, p_1 + 1]$ and has two zeros $t_0 = 0$ and $t_1 = \frac{\pi}{\omega_\alpha}$. For zero t_1 we have (remember, that function $T^\alpha(q_{p_1+1})$ returns position of zero of positive semi-wave calculated from p_1, since $q_{p_1+1} = \vartheta_1$ is the ratio $\frac{u(p_1+1)}{u(p_1)}$)

$$t_1 = p_1 + T^\alpha(q_{p_1+1}) = p_1 + T^\alpha(\vartheta_1).$$

The first positive generalized zero of u is $z_1 = p_1 + 1$ if $\vartheta_1 < 0$ or $z_1 = p_1 = t_1$ if $\vartheta_1 = \infty$.

(b) First negative semi-wave: The next semi-wave of u is negative. It has two zeros s_1 and s_2 and is defined on $[\lceil s_1 \rceil - 1, \lfloor s_2 \rfloor + 1]$. Its first zero s_1 can be calculated as

$$s_1 = \lfloor t_1 \rfloor + T^\beta\left(q_{\lfloor t_1 \rfloor + 1}\right) = p_1 + T^\beta\left(\vartheta_1\right).$$

And its second zero s_2 is

$$s_2 = s_1 + \frac{\pi}{\omega_\beta},$$

since we are just adding length of negative wave $\frac{\pi}{\omega_\beta}$ to the first zero s_1. For s_2 we have

$$\lfloor s_2 \rfloor = p_1 + p_2$$

which implies

$$q_{p_2+p_1+1} = W^\beta_{p_2}(p_1 + 1) = W^\beta_{p_2}(\vartheta_1) = W^\beta_{p_2}(W^\alpha_{p_1}(\infty)) = \vartheta_2$$

and

$$s_2 = p_1 + p_2 + T^\beta\left(\vartheta_2\right).$$

The second positive generalized zero of u is $z_2 = p_1 + p_2 + 1$ if $\vartheta_2 < 0$ or $z_2 = p_1 + p_2 = s_2$ if $\vartheta_2 = \infty$.

(c) Second positive semi-wave: The next semi-wave of u is the positive semi-wave $u^{\mathsf{c}}_{\lceil t_2\rceil,\lfloor t_3\rfloor}$, which has two zeros t_2 and t_3 and is defined on $[\lceil t_2\rceil - 1, \lfloor t_3\rfloor + 1]$. We have that $t_3 - t_2 = \frac{\pi}{\omega_\alpha}$ and

$$t_2 = \lfloor s_2\rfloor + T^\alpha\left(q_{\lfloor s_2\rfloor+1}\right) = p_1 + p_2 + T^\alpha\left(\vartheta_2\right),$$

$$\vartheta_3 = q_{p_3+p_2+p_1+1} = W^\alpha_{p_3}(W^\beta_{p_2}(W^\alpha_{p_1}(\infty))),$$

$$t_3 = \lfloor t_3\rfloor + T^\alpha\left(q_{\lfloor t_3\rfloor+1}\right) = p_1 + p_2 + p_3 + T^\alpha(\vartheta_3).$$

The third positive generalized zero of u is $z_3 = p_1 + p_2 + p_3 + 1$ if $\vartheta_3 < 0$ or $z_3 = p_1 + p_2 + p_3 = t_3$ if $\vartheta_3 = \infty$.

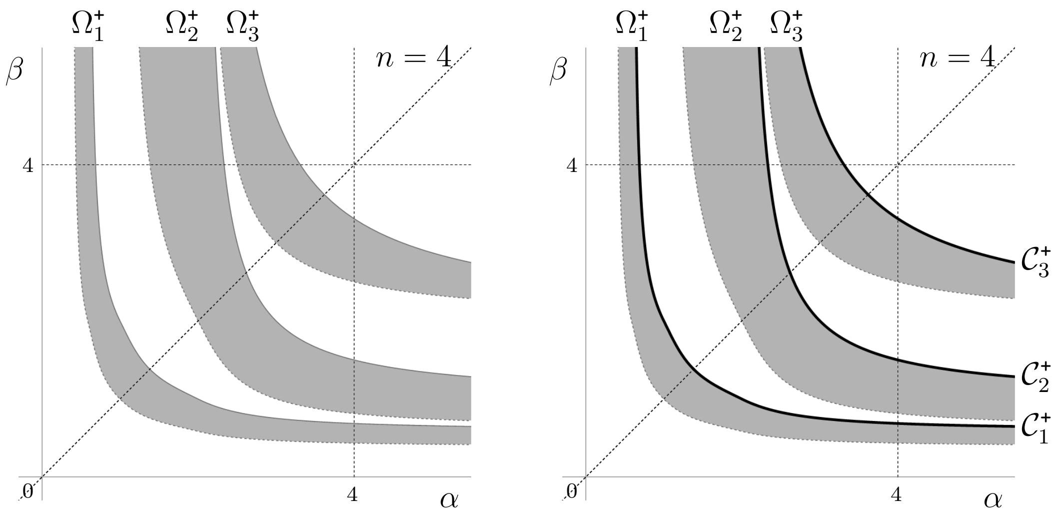

Figure 2.7: The sets Ω^+_k as grey regions for $n = 4$ (left) and the Fučík curves $\mathcal{C}^+_k$ as black curves (right).

Finally, we can formulate our main result – theorem which describes Fučík curves $\mathcal{C}^+_k$, $k = 1, \dots, n-1$ – implicit recurrent description of the Fučík spectrum $\Sigma(\mathbf{A}^{\mathsf{D}})$. In the first chapter of this summary text, we have explained that it is enough to investigate $\mathcal{C}^+_k$, $k = 1, \dots, n-1$ for the complete description of $\Sigma(\mathbf{A}^{\mathsf{D}})$.

Theorem. *(Theorem 32 in the thesis)*
For $k = 1, \dots, n-1$, we have that

$$\begin{aligned}\mathcal{C}^+_k = \ & \{(\alpha,\beta) \in (0,4)\times(0,+\infty) : \ P_{k+1}(\alpha,\beta) + T^\alpha(\vartheta_{k+1}(\alpha,\beta)) = n+1\} \ \cup \\ & \left\{(\alpha,\beta) \in (0,+\infty)\times(0,4) : \ P_{k+1}(\alpha,\beta) + T^\beta(\vartheta_{k+1}(\alpha,\beta)) = n+1\right\}.\end{aligned}$$

Moreover, if we denote

$$\Omega^+_k := \{(\alpha,\beta) \in D : \ P_{k+1}(\alpha,\beta) = n+1\}, \qquad k = 1, \dots, n-1,$$

then we have that

$$\mathcal{C}_k^+ = \left\{(\alpha,\beta) \in \Omega_k^+ : \; \mathcal{W}_{k+1}^+(\infty) = \infty\right\}.$$

An example of sets Ω_k^+ for $n = 4$ can be seen on Figure 2.7. Also, see Figure 2.8 for the complete Fučík spectrum of matrix $\mathbf{A}^{\mathtt{D}}$ for $n = 4$ and $n = 7$ (including Fučík curves $\mathcal{C}_k^-$).

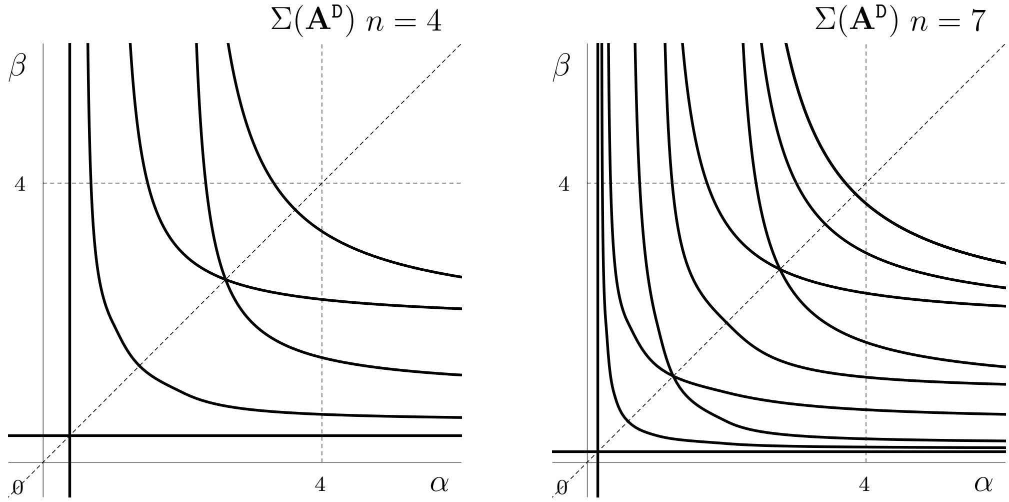

Figure 2.8: Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$ for $n = 4$ (left) and $n = 7$ (right).

2.3 Descriptions of the Fučík spectrum of matrix $\mathbf{A}^{\mathtt{D}}$ – part II

In this section, we are going to investigate problem (P3) from a different angle. A continuous extension of solution u of (P3) will be constructed in a manner considering positive semi-waves only. We will calculate the distance between every two consecutive zeros of two different (consecutive) positive semi-waves. This will allow us not only to study nodal properties of solution u of (P3) in more detail, it will also allow us to find simpler implicit description of all Fučík curves $\mathcal{C}_k^\pm$, $k = 1, \ldots, n-1$.

Continuous extension – positive semi-waves only – can be seen on the Figure 2.9. If we would have $0 < \alpha < 4$ only, then the length of all positive semi-waves is the same and is equal to $\frac{\pi}{\omega_\alpha}$. This way, localization of intervals, where positive semi-waves are anchored, can be rephrased to – "what is the distance between every two consecutive zeros of two different consecutive positive semi-waves." We denote such distance as $\rho_{\alpha,\beta}$ (we will define such function later in the text) – see Figure 2.10 for better understanding of the distance $\rho_{\alpha,\beta}$. Let us define half-strip $\mathcal{D}$ as

$$\mathcal{D} := (0,4) \times (0,+\infty).$$

In the following text, without any loss of generality, we are going to assume that $(\alpha, \beta) \in \mathcal{D}$ (it is enough to investigate $(\alpha, \beta) \in \mathcal{D}$ due to the symmetry of the Fučík spectrum). We note that it is easier to deal with zeros of positive semi-waves when $\alpha \in (0, 4)$.

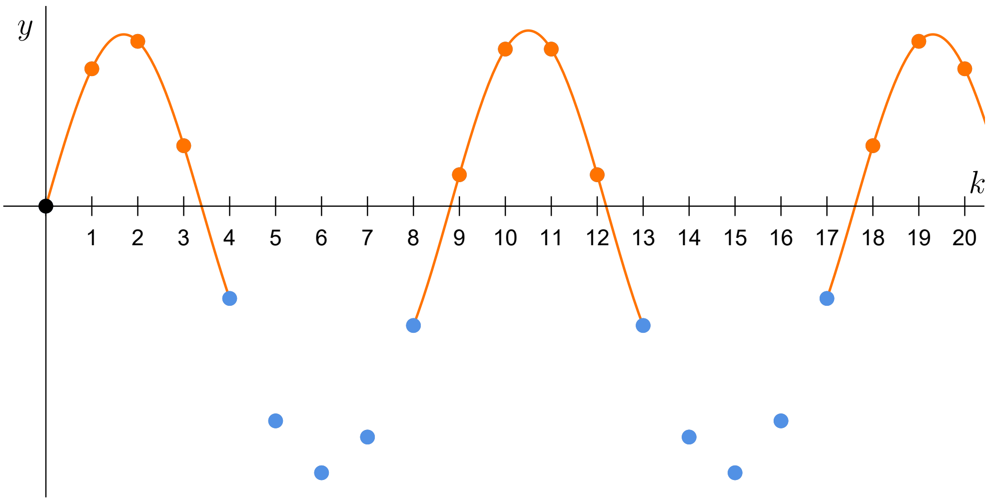

Figure 2.9: Continuous extension of only positive semi-waves for solution u of problem (P3) for $\alpha = 0.8$, $\beta = 0.33$ and $C_1 = 1 > 0$.

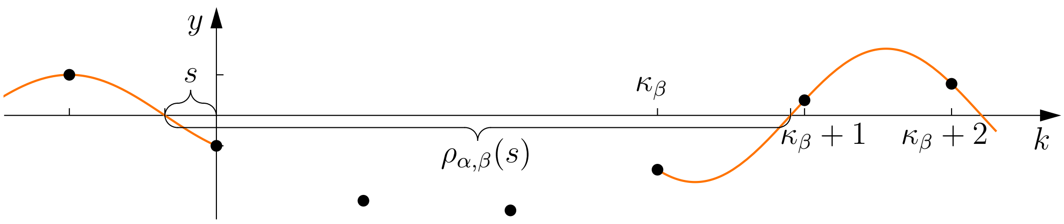

Figure 2.10: The distance $\rho_{\alpha,\beta}$ – the distance between two consecutive zeros (last and first) of two different positive semi-waves (orange color). Continuous extension for $\beta < 4$ ($\alpha = 3.5$, $\beta = 0.53$).

Let us define map $\kappa_\beta : (0, +\infty) \to \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, as

$$\kappa_\beta := \begin{cases} \left\lfloor \frac{\pi}{\omega_\beta} \right\rfloor - 1 & \text{for } 0 < \beta < 4, \\ 0 & \text{for } \beta \geq 4. \end{cases}$$

Function κ_λ allows us to determine the length of a semi-wave (as continuous extension) – see the following lemma which describes semi-wave $u^{\mathrm{c}}_{i,j}$. The semi-wave is defined on an interval $[i-1, j+1]$. Knowing i and using the ratio $q_i = \frac{u(i)}{u(i-1)}$, we can (using the value κ_λ) determine j.

Lemma. *(Lemma 23 in the thesis)*
Let $(\alpha, \beta) \in D$ and u be the solution of the initial value problem (P3). *Moreover, let $i, j \in \mathbb{Z}$ be such that $i \leq j$ and*

$$u(i-1) < 0, \quad u(k) \geq 0 \quad \textit{for } k = i, \dots, j, \quad u(j+1) < 0, \tag{2.1}$$

or

$$u(i-1) > 0, \quad u(k) \leq 0 \quad \textit{for } k = i, \dots, j, \quad u(j+1) > 0. \tag{2.2}$$

Then we have

$$j = \begin{cases} i + \kappa_\lambda & \textit{for} \quad W^{\lambda}_{\kappa_\lambda+1}(q_i) < 0, \\ i + \kappa_\lambda + 1 & \textit{for} \quad W^{\lambda}_{\kappa_\lambda+1}(q_i) \geq 0, \end{cases}$$

where we denoted $q_i := \frac{u(i)}{u(i-1)} \leq 0$ and $\lambda = \alpha$ if (2.1) *holds or $\lambda = \beta$ if* (2.2) *holds. Moreover, we have $u(k) \neq 0$ for $k \in \mathbb{Z}$ such that $i < k < j$, and $u(j) = 0$ if and only if $W^{\lambda}_{\kappa_\lambda+1}(q_i) = 0$.*

In the following definition, we will define three functions $\eta_{\alpha,\beta}$, $\tau_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$. These functions (for fixed α, β) represent important values for distance $\rho_{\alpha,\beta}$.

Definition. (Definition 24 in the thesis)

For $0 < \alpha < 4$ and $\beta > 0$, let us define

$$\eta_{\alpha,\beta} := T^{\alpha}\left(\frac{V^{\beta}_{\kappa_\beta+1} - 1}{V^{\beta}_{\kappa_\beta}}\right), \qquad \tau_{\alpha,\beta} := T^{\alpha}\left(\frac{V^{\beta}_{\kappa_\beta+1}}{V^{\beta}_{\kappa_\beta}}\right), \qquad \mu_{\alpha,\beta} := T^{\alpha}\left(\frac{V^{\beta}_{\kappa_\beta+1}}{V^{\beta}_{\kappa_\beta} + 1}\right).$$

We will define function $\mathcal{N}_{\alpha,\beta}$ (for illustration, see Figure 2.11) which is used in the definition of distance function $\rho_{\alpha,\beta}$.

Definition. (Definition 27 in the thesis)
For $0 < \alpha < 4$ and $\beta > 0$, let us define

$$\mathrm{Dom}(\mathcal{N}_{\alpha,\beta}) := [0, 1 + \tau_{\alpha,\beta}], \qquad \mathcal{N}_{\alpha,\beta}(s) := \begin{cases} \bar{\bar{M}}_{\alpha,\beta}(s) + 1 & \text{for } s \in [0, \tau_{\alpha,\beta}], \\ \bar{M}_{\alpha,\beta}(s) & \text{for } s \in (\tau_{\alpha,\beta}, 1), \\ \bar{\bar{M}}_{\alpha,\beta}(s-1) & \text{for } s \in [1, 1 + \tau_{\alpha,\beta}], \end{cases}$$

where

$$\begin{aligned} \bar{M}_{\alpha,\beta}(s) &:= T^{\alpha}\left(W^{\beta}_{\kappa_\beta+1}\left(Q^{\alpha}(1-s)\right)\right), \quad s \in [\tau_{\alpha,\beta}, 1], \\ \bar{\bar{M}}_{\alpha,\beta}(s) &:= T^{\alpha}\left(W^{\beta}_{\kappa_\beta+2}\left(Q^{\alpha}(1-s)\right)\right), \quad s \in [0, \tau_{\alpha,\beta}]. \end{aligned}$$

And finally, we can define the distance function $\rho_{\alpha,\beta}$ – the distance between every two consecutive zeros of two different (consecutive) positive semi-waves (for illustration, see Figure 2.12).

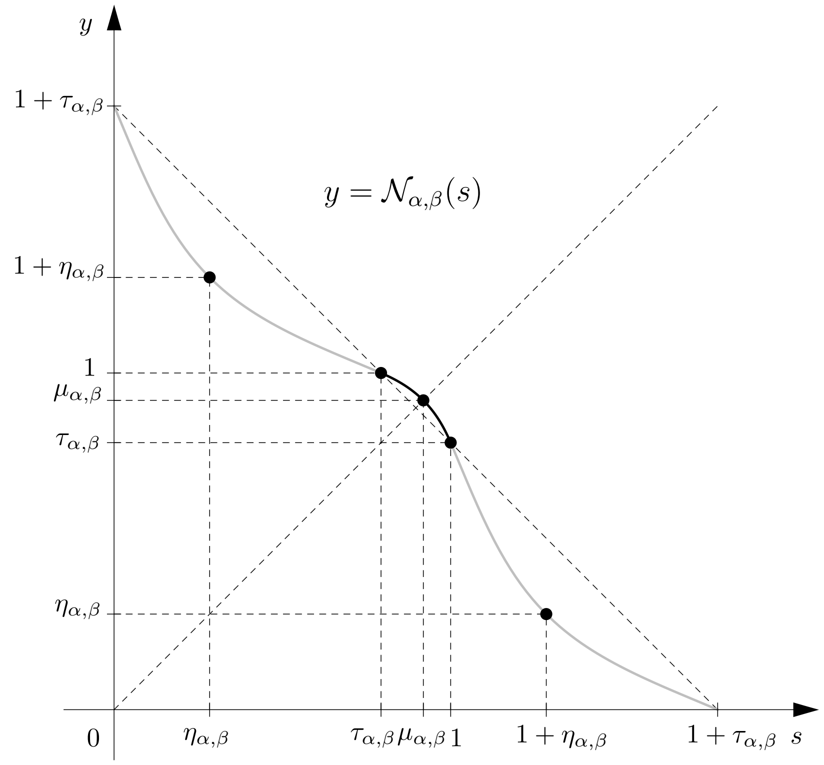

Figure 2.11: Function $\mathcal{N}_{\alpha,\beta}$ for $\alpha > \beta$ ($\alpha = 3.2$, $\beta = 1.2$).

Definition. (Definition 28 in the thesis)
Let $0 < \alpha < 4$ and $\beta > 0$. Let us define

$$\rho_{\alpha,\beta}(s) := s + \kappa_\beta + \mathcal{N}_{\alpha,\beta}(s), \qquad 0 \le s \le 1 + \tau_{\alpha,\beta}.$$

All of the theory and definitions above lead to the following theorem – another description of the Fučík spectrum $\Sigma(\mathbf{A}^{\mathrm{D}})$.

Theorem. *(Theorem 35 in the thesis)*
In the domain $\mathcal{D} = (0,4) \times (0,+\infty)$, we have the following description of Fučík curves $\mathcal{C}_l^{\pm}$, $l = 1, \dots, n-1$,

$$\begin{aligned}
\mathcal{C}_{2j-1}^{\pm} \cap \mathcal{D} &= \left\{(\alpha,\beta) \in \mathcal{D} :\ t_j^{+}(\alpha,\beta) + t_j^{-}(\alpha,\beta) = n+1\right\}, \\
\mathcal{C}_{2j}^{+} \cap \mathcal{D} &= \left\{(\alpha,\beta) \in \mathcal{D} :\ t_{j+1}^{+}(\alpha,\beta) + t_j^{+}(\alpha,\beta) = n+1\right\}, \\
\mathcal{C}_{2j}^{-} \cap \mathcal{D} &= \left\{(\alpha,\beta) \in \mathcal{D} :\ t_{j+1}^{-}(\alpha,\beta) + t_j^{-}(\alpha,\beta) = n+1\right\},
\end{aligned}$$

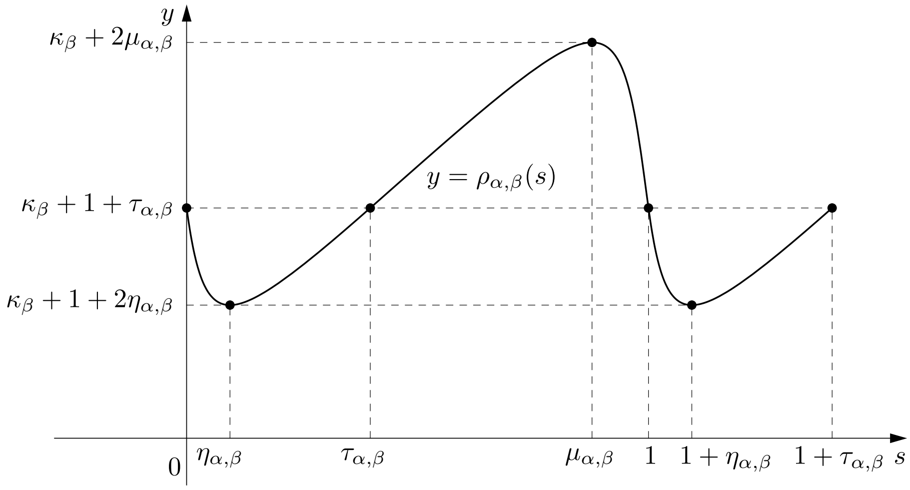

Figure 2.12: The graph of the function $\rho_{\alpha,\beta}$ for $\alpha > \beta$ ($\alpha = 3.9$, $\beta = 3.1$).

where

$$t_1^{+} := \frac{\pi}{\omega_\alpha}, \qquad t_j^{+} := \begin{cases} t_{j-1}^{+} + \rho_{\alpha,\beta}\left(\lceil t_{j-1}^{+} \rceil - t_{j-1}^{+}\right) & \text{for } j \text{ even,} \\ t_{j-1}^{+} + \frac{\pi}{\omega_\alpha} & \text{for } j \text{ odd,} \end{cases}$$

$$t_1^{-} := \rho_{\alpha,\beta}(0), \qquad t_j^{-} := \begin{cases} t_{j-1}^{-} + \frac{\pi}{\omega_\alpha} & \text{for } j \text{ even,} \\ t_{j-1}^{-} + \rho_{\alpha,\beta}\left(\lceil t_{j-1}^{-} \rceil - t_{j-1}^{-}\right) & \text{for } j \text{ odd.} \end{cases}$$

Let us note that even though our descriptions of the Fučík spectrum are for a particular matrix $\mathbf{A}^{\mathtt{D}}$ (Dirichlet matrix), theory in the thesis (and both research articles [25] and [31]) can be extended. The theory was constructed for a difference equation in (P4). In order to describe the Fučík spectrum for a problem with the same difference equation but with a different boundary conditions, one would use the same theory only changing aspects related to the boundary conditions. Thus, our results can be generalized also for different boundary conditions (one would need to explore the inadmissible areas for such matrices, since our theory does not include cases ($\alpha > 4$ and $\beta > 4$) and ($\alpha < 0$ or $\beta < 0$)).

Chapter 3

Main results – bounds of Fučík curves of matrix $\mathbf{A}^{\mathtt{D}}$

In this chapter, we introduce two different bounds of Fučík curves which can be used for numerical estimation of Fučík curves (they are not recurrent and they do not become more complicated when dimension n increases).

3.1 Delta bounds of Fučík curves for matrix $\mathbf{A}^{\mathtt{D}}$

Let us investigate in detail the “gaps” between positive and negative semi-waves – the difference between zero points of two consecutive positive and negative semi-waves. Knowing minimal and maximal such difference allows us to find regions (bounds) for Fučík curves. In the following definition we define function $\delta_{\alpha,\beta}$ which represents such difference (the length of such “gap” is then equal to the absolute value of function $\delta_{\alpha,\beta}$) – see Figure 3.1.

Definition. (Definition 40 in the thesis)
For $0 < \alpha, \beta < 4$, let us define

$$\delta_{\alpha,\beta}(q) := T^{\alpha}(q) - T^{\beta}(q), \quad q < 0.$$

We can calculate extrema of function $\delta_{\alpha,\beta}$:

Theorem. *(Theorem 42 in the thesis)*
Let $0 < \alpha, \beta < 4$. Function $\delta_{\alpha,\beta}$ has one global minimum and one global maximum (for $q < 0$):

$$\min_{q<0} \delta_{\alpha,\beta}(q) = \begin{cases} \delta_{\alpha,\beta}(q_1^{\alpha,\beta}) & \text{for } \alpha > \beta, \\ 0 & \text{for } \alpha = \beta, \\ \delta_{\alpha,\beta}(q_2^{\alpha,\beta}) & \text{for } \alpha < \beta, \end{cases} \qquad \max_{q<0} \delta_{\alpha,\beta}(q) = \begin{cases} \delta_{\alpha,\beta}(q_2^{\alpha,\beta}) & \text{for } \alpha > \beta, \\ 0 & \text{for } \alpha = \beta, \\ \delta_{\alpha,\beta}(q_1^{\alpha,\beta}) & \text{for } \alpha < \beta, \end{cases}$$

where

$$q_{1,2}^{\alpha,\beta} := J^{\alpha,\beta} \mp \sqrt{(J^{\alpha,\beta})^2 - 1}, \qquad J^{\alpha,\beta} := \frac{\omega_\alpha \sin\omega_\beta \cos\omega_\alpha - \omega_\beta \sin\omega_\alpha \cos\omega_\beta}{\omega_\alpha \sin\omega_\beta - \omega_\beta \sin\omega_\alpha}. \tag{3.1}$$

Also,

$$\max_{q<0} \delta_{\alpha,\beta}(q) = -\min_{q<0} \delta_{\alpha,\beta}(q).$$

Definition. (Definition 43 in the thesis)
For $0 < \alpha, \beta < 4$, let us define

$$\begin{aligned} \delta_{\alpha,\beta}^{\mathtt{min}} &:= -|\delta_{\alpha,\beta}(q_1^{\alpha,\beta})|, \\ \delta_{\alpha,\beta}^{\mathtt{max}} &:= +|\delta_{\alpha,\beta}(q_1^{\alpha,\beta})|, \end{aligned}$$

where $q_1^{\alpha,\beta}$ is defined in (3.1).

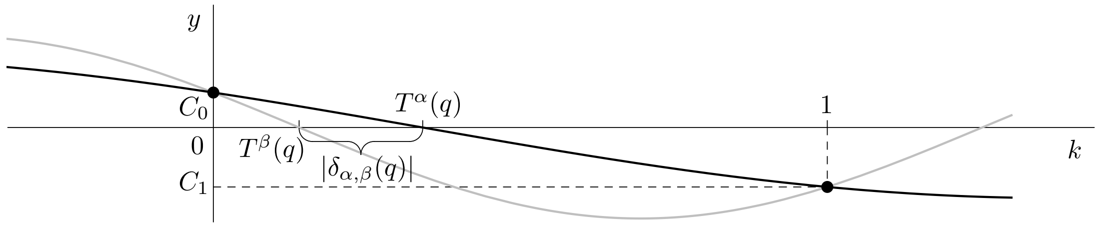

Figure 3.1: The distance between zero point of positive semi-wave and zero point of negative semi-wave – the distance $|\delta_{\alpha,\beta}|$ for $\alpha = 1.9$, $\beta = 3.9$.

In the following theorem, we derive "delta bounds" for Fučík curves $C_l^{\pm}$, $l = 1, \ldots, n-1$ using values $\delta_{\alpha,\beta}^{\mathtt{min}}$ and $\delta_{\alpha,\beta}$. For illustration of such bounds, see Figure 3.2.

Theorem. *(Theorem 45 in the thesis)*
In the domain $D_{0,4} = (0,4) \times (0,4)$, we have the following "delta" bounds for Fučík curves $\mathcal{C}_l^{\pm}$, $l = 1, \ldots, n-1$,

$$\begin{aligned} \left(\mathcal{C}_{2j-1}^{\pm} \cap D_{0,4}\right) &\subset \Psi_{j,j} &&=: \Psi_{2j-1}^{\pm}, \\ \left(\mathcal{C}_{2j}^{+} \cap D_{0,4}\right) &\subset \Psi_{j+1,j} &&=: \Psi_{2j}^{+}, \\ \left(\mathcal{C}_{2j}^{-} \cap D_{0,4}\right) &\subset \Psi_{j,j+1} &&=: \Psi_{2j}^{-}, \end{aligned}$$

$j \in \mathbb{N}$, where for $k, s \in \mathbb{N}$, sets $\Psi_{k,s}$ are given by

$$\Psi_{k,s} \;:=\; \{(\alpha,\beta) \in D_{0,4} : \; F_{k,s}(\alpha,\beta) \le n+1 \le G_{k,s}(\alpha,\beta)\}$$

and

$$F_{k,s}(\alpha,\beta) := k\frac{\pi}{\omega_\alpha} + s\frac{\pi}{\omega_\beta} + (k+s-1)\delta_{\alpha,\beta}^{\mathtt{min}}, \quad G_{k,s}(\alpha,\beta) := k\frac{\pi}{\omega_\alpha} + s\frac{\pi}{\omega_\beta} + (k+s-1)\delta_{\alpha,\beta}^{\mathtt{max}}.$$

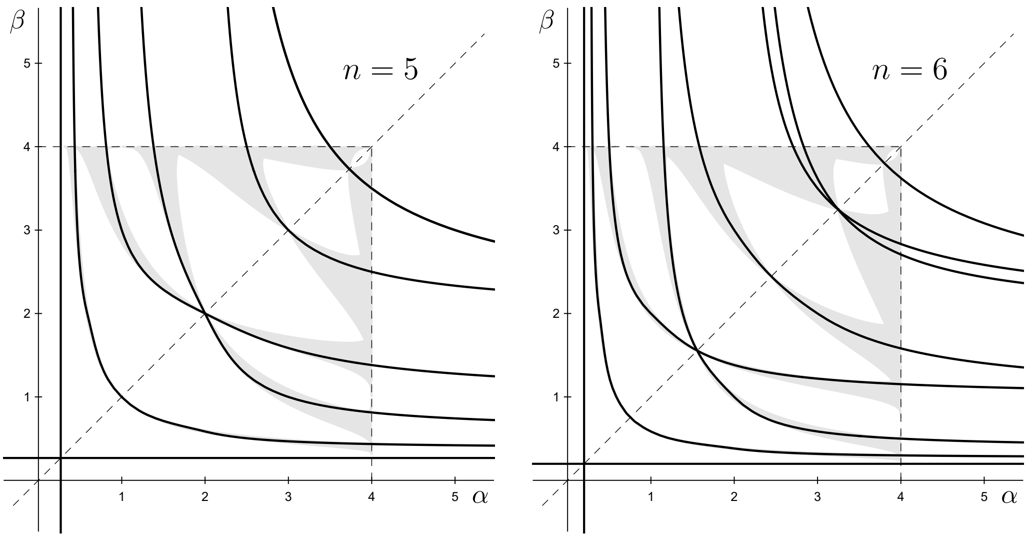

Figure 3.2: Delta bounds for $\Sigma(\mathbf{A}^{\mathtt{D}})$ of $n = 5$ (left) and $n = 6$ (right).

3.2 Rho bounds of Fučík curves for matrix $\mathbf{A}^{\mathtt{D}}$

Another bounds we are going to discuss are referred here as “rho bounds” (originally introduced as “improved bounds” in [31], see Figure 3.3 for illustration). The main idea is based on using extrema of function $\rho_{\alpha,\beta}$.

Theorem. *(Theorem 46 in the thesis)*
Let $0 < \alpha < 4$ and $\beta > 0$. Then the function $\rho_{\alpha,\beta}$ attains its global extrema at $\eta_{\alpha,\beta}$ and $\mu_{\alpha,\beta}$. More precisely, we have that

$$\min_{s\in[0,1+\tau_{\alpha,\beta}]} \rho_{\alpha,\beta}(s) = \begin{cases} \rho_{\alpha,\beta}(\mu_{\alpha,\beta}) = 2\mu_{\alpha,\beta} + \kappa_\beta & \text{for } \alpha \le \beta, \\ \rho_{\alpha,\beta}(\eta_{\alpha,\beta}) = 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \text{for } \alpha > \beta, \end{cases}$$

$$\max_{s\in[0,1+\tau_{\alpha,\beta}]} \rho_{\alpha,\beta}(s) = \begin{cases} \rho_{\alpha,\beta}(\eta_{\alpha,\beta}) = 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \text{for } \alpha \le \beta, \\ \rho_{\alpha,\beta}(\mu_{\alpha,\beta}) = 2\mu_{\alpha,\beta} + \kappa_\beta & \text{for } \alpha > \beta. \end{cases}$$

Definition. (Definition 47 in the thesis)

For $0 < \alpha < 4$ and $\beta > 0$, let us define

$$\rho^{\mathtt{min}}_{\alpha,\beta} := \begin{cases} 2\mu_{\alpha,\beta} + \kappa_\beta & \alpha \le \beta, \\ 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \alpha > \beta, \end{cases}$$

$$\rho^{\mathtt{max}}_{\alpha,\beta} := \begin{cases} 2\eta_{\alpha,\beta} + \kappa_\beta + 1 & \alpha \le \beta, \\ 2\mu_{\alpha,\beta} + \kappa_\beta & \alpha > \beta. \end{cases}$$

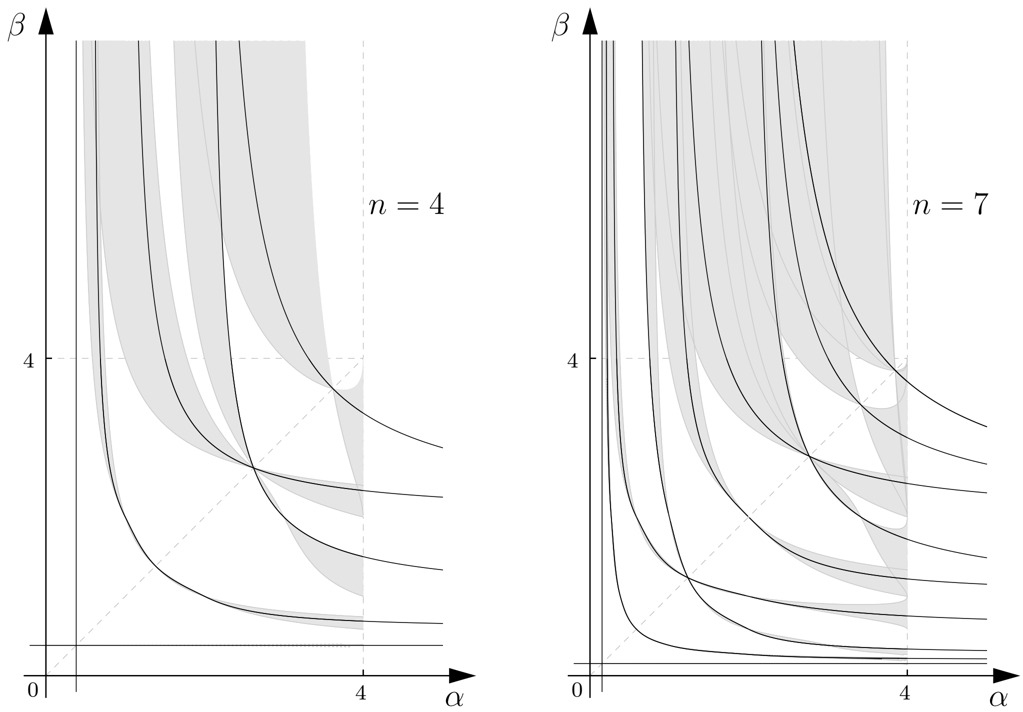

Figure 3.3: Rho bounds $\Upsilon_l^{\pm}$ $(l = 1, \ldots, n-1)$ for the Fučík spectrum $\Sigma\left(\mathbf{A}^{\mathtt{D}}\right)$ of $n = 4$ (left) and $n = 7$ (right).

Finally, rho bounds are such bounds, that we take $\rho_{\alpha,\beta}^{\mathtt{min}}$ and $\rho_{\alpha,\beta}^{\mathtt{max}}$ instead of function $\rho_{\alpha,\beta}$ in the description of Fučík spectrum $\Sigma(\mathbf{A}^{\mathtt{D}})$.

Theorem. *(Theorem 48 in the thesis)*
In the domain $\mathcal{D} = (0,4)\times(0,+\infty)$, we have the following "rho bounds" for Fučík curves $\mathcal{C}_l^{\pm}$, $l = 1, \ldots, n-1$,

$$\begin{aligned} \left(\mathcal{C}_{2j-1}^{\pm} \cap \mathcal{D}\right) &\subset \Upsilon_{j,j} &&=: \Upsilon_{2j-1}^{\pm}, \\ \left(\mathcal{C}_{2j}^{+} \cap \mathcal{D}\right) &\subset \Upsilon_{j+1,j} &&=: \Upsilon_{2j}^{+}, \\ \left(\mathcal{C}_{2j}^{-} \cap \mathcal{D}\right) &\subset \Upsilon_{j,j+1} &&=: \Upsilon_{2j}^{-}, \end{aligned}$$

$j \in \mathbb{N}$, where for $k, s \in \mathbb{N}$, sets $\Upsilon_{k,s}$ are given by

$$\Upsilon_{k,s} := \left\{(\alpha,\beta) \in \mathcal{D} : \ \rho_{\alpha,\beta}^{\mathtt{min}} \ \le \ \tfrac{1}{s}\left(n + 1 - k\tfrac{\pi}{\omega_\alpha}\right) \ \le \ \rho_{\alpha,\beta}^{\mathtt{max}}\right\}.$$

Chapter 4

Abstracts of published research articles of the author

Research articles in impacted journals:

Abstract of [25]:

I. Looseová (Sobotková), P. Nečesal, The Fučík spectrum of the discrete Dirichlet operator, Linear Algebra Appl. 553 (2018) 58–103

In this paper, we deal with the discrete Dirichlet operator of the second order and we investigate its Fučík spectrum, which consists of a finite number of algebraic curves. For each non-trivial Fučík curve, we are able to detect a finite number of its points, which are given explicitely. We provide the exact implicit description of all non-trivial Fučík curves in terms of Chebyshev polynomials of the second kind. Moreover, for each non-trivial Fučík curve, we give several different implicit descriptions, which differ in the level of depth of used nested functions. Our approach is based on the Möbius transformation and on the appropriate continuous extension of solutions of the discrete problem. Let us note that all presented descriptions of Fučík curves have the form of necessary and sufficient conditions. Finally, our approach can be also directly used in the case of difference operators of the second order with other local boundary conditions.

This article was published in Linear Algebra and Its Applications (Elsevier). For 2020, it has impact factor 1.401, cite score 2.1 and it belongs to Q1 in “Algebra and Number Theory” and “Discrete Mathematics and Combinatorics” fields of Mathematics.

Abstract of [31]:

P. Nečesal, I. Sobotková, Localization of Fučík curves for the second order discrete Dirichlet operator, Bulletin des Sciences Mathématiques 171 (2021) 103014

In this paper, we deal with the second order difference equation with asymmetric nonlinearities on the integer lattice and we investigate the distribution of zeros of continuous extensions of positive semi-waves. The distance between two consecutive zeros of two different positive semi-waves depends not only on the parameters of the problem but also on the position of one of these zeros with respect to the integer lattice. We provide an explicit formula for this distance, which allows us to obtain a new simple implicit description of all non-trivial Fučík curves for the discrete Dirichlet operator. Moreover, for fixed parameters of the problem, we show that this distance is bounded and attains its global extrema that are explicitly described in terms of Chebyshev polynomials of the second kind. Finally, for each non-trivial Fučík curve, we provide suitable bounds by two curves with a simple description similar to the description of the first non-trivial Fučík curve.

This article was published in Bulletin des Sciences Mathématiques (Elsevier). For 2020, it has impact factor 1.118, cite score 1.6 and it belongs to Q1 in "Mathematics (miscellaneous)" field.

Other activities:

Abstract of [24] in Proceedings:

I. Looseová (Sobotková), Conjecture on Fučík curve asymptotes for a particular discrete operator, in: S. Pinelas, T. Caraballo, P. Kloeden, J. R. Graef (eds.), Differential and Difference Equations with Applications, Springer International Publishing, Cham, 2018

In this paper we study properties of the Neumann discrete problem. We investigate so called polar Pareto spectrum of a specific matrix which represents the Neumann discrete operator. There is a known relation between polar Pareto spectrum of any discrete operator and its Fučík spectrum. We also state a conjecture about asymptotes of Fučík curves with respect to the matrix and we illustrate a variety of polar Pareto eigenvectors corresponding to a fixed polar Pareto eigenvalue.

Conferences:

1. Equadiff 13, Praha, 26.-30.8.2013, The asymptotes of Fučík curves for asymmetric difference operator

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3. Setkání studentů matematické analýzy a diferenciálních rovnic, Praha 2016, The Fučík spectrum of the Neumann discrete operator
4. XXX Seminar in Differential Equations, Ostrov u Tisé, 30. 5. – 3. 6. 2016, The Fučík spectrum of the second order discrete operators
5. International Conference on Differential & Difference Equations and Applications 2017, Amadora, Portugal, 5. 6. – 9. 6. 2017, The Fučík spectrum of the discrete Dirichlet operator

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