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Hamilton-connected {claw, bull}-free graphs

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Abstract
The generalized bull is the graph $B_{i,j}$ obtained by attaching endvertices of two disjoint paths of lengths i, j to two vertices of a triangle. We prove that every 3-connected $\{K_{1,3}, X\}$-free graph, where $X \in \{B_{1,6}, B_{2,5}, B_{3,4}\}$, is Hamilton-connected. The results are sharp and complete the characterization of forbidden induced bulls implying Hamilton-connectedness of a 3-connected {claw, bull}-free graph.

KEYWORDS
bull-free, claw-free, closure, forbidden subgraph, Hamilton-connected

1 | DEFINITIONS AND NOTATIONS

In this paper, by a *graph* we always mean a simple finite undirected graph; whenever we admit multiple edges, we always speak about a *multigraph*. We generally follow the most common graph-theoretical notation and terminology and for notations and concepts not defined here we refer to [4]. Specifically, multiple edges of multiplicity at least 2 (exactly 2, exactly 3) are referred to as a *multiedge* (*double edge*, *triple edge*), respectively. We use $d_G(x)$ to denote the *degree* of a vertex x in G, and for $i \geq 1$ we set $V_i(G) = \{x \in V(G) \mid d_G(x) = i\}$. If $x \in V_2(G)$ with $N_G(x) = \{y_1, y_2\}$, then the operation of replacing the path y_1xy_2 by the edge y_1y_2 is called *suppressing* the vertex x. The inverse operation is called *subdividing* the edge y_1y_2 with the vertex x. We write $F \subset H$ if F is a sub(multi)graph of H, $G_1 \simeq G_2$ if the (multi)graphs G_1, G_2 are isomorphic, and $\langle M\rangle_G$ to denote the *induced sub(multi)graph* on a set $M \subset V(G)$. We say that a vertex $x \in V(G)$ is *simplicial* if $\langle N_G(x)\rangle_G$ is a complete graph, and we use $V_{SI}(G)$ to denote the set of all simplicial vertices of G. The *circumference* of G, denoted $c(G)$, is the length of a longest cycle in G. The *line graph* of a multigraph H is the graph $G = L(H)$ with $V(G) = E(H)$, in which two vertices are adjacent if and only if the corresponding edges of H have at least one vertex in common.

By a *closed trail* in G we mean an Eulerian subgraph of G, and a connected subgraph with exactly two vertices of odd degree is called a *trail* in G. Its vertices of odd degree are its *endvertices*, and (any) its edge incident to an endvertex is a *terminal edge* (note that these definitions are equivalent with those in [4]). For $x, y \in V(G)$, a path (trail) with endvertices x, y is referred to as an *(x, y)-path* (*(x, y)-trail*), a trail with terminal edges $e, f \in E(G)$ is called an *(e, f)-trail*, and $\mathrm{Int}(T)$ denotes the set of interior vertices of a trail T. A set of vertices $M \subset V(G)$ *dominates* an edge e, if e has at least one vertex in M, and a sub(multi)graph $F \subset G$ dominates e if $V(F)$ dominates e. A closed trail T is a *dominating closed trail* (abbreviated DCT) if T dominates all edges of G, and an (e, f)-trail is an *internally dominating (e, f)-trail* (abbreviated (e, f)-IDT) if $\mathrm{Int}(T)$ dominates all edges of G. A graph is *Hamilton-connected* if, for any $u, v \in V(G)$, G has a Hamiltonian (u, v)-path, that is, a (u, v)-path P with $V(P) = V(G)$.

Finally, if $\mathcal{F}$ is a family of graphs, we say that G is *$\mathcal{F}$-free* if G does not contain an induced subgraph isomorphic to a member of $\mathcal{F}$, and the graphs in $\mathcal{F}$ are referred to in this context as *f*orbidden (induced) subgraphs. If $\mathcal{F} = \{F\}$, we simply say that G is *F-free*. Here, the *claw* is the graph $K_{1,3}$, P_i denotes the path on i vertices, and Γ_i denotes the graph obtained by joining two triangles with a path of length i (see Figure 2A). Several further graphs that will be used as forbidden subgraphs are shown in Figure 1 (specifically, the vertex of degree 2 in the triangle of the bull $B_{i,j}$ will be called its *mouth* and denoted $\mu(B_{i,j})$). Whenever we will list vertices of an $S_{i,j,k}$ in a graph, we will always write the list such that $i \le j \le k$, and we will use the notation $S_{i,j,k}(v;\, a_1a_2...a_i;\, b_1b_2...b_j;\, c_1c_2...c_k)$ (in the labeling of vertices as in Figure 1D). Similarly, when listing vertices of an induced claw $K_{1,3}$, we will always list its center as the first vertex of the list, and when listing vertices of an induced subgraph $F \simeq B_{i,j}$, we will always list first $\mu(F)$, and then vertices of the two paths, starting (if possible) with the shorter one.

We also recall two well-known graphs that will occur as exceptions in some of the results, namely, the Petersen graph Π and the Wagner graph W (see Figure 2B,C). It is a well-known fact that the Wagner graph can be obtained from the Petersen graph by removing an arbitrary edge and suppressing the two created vertices of degree 2. We will often refer to these graphs using the labeling of their vertices as indicated in Figure 2.

2 | INTRODUCTION

There are many results on forbidden induced subgraphs implying various Hamilton-type graph properties. For Hamiltonicity in 2-connected graphs (recall that 2-connectedness is the necessary connectivity level for the property), pairs of forbidden connected subgraphs are completely characterized [8]. However, for Hamilton-connectedness in 3-connected graphs (where again, 3-connectedness is the necessary connectivity level for the property), the progress

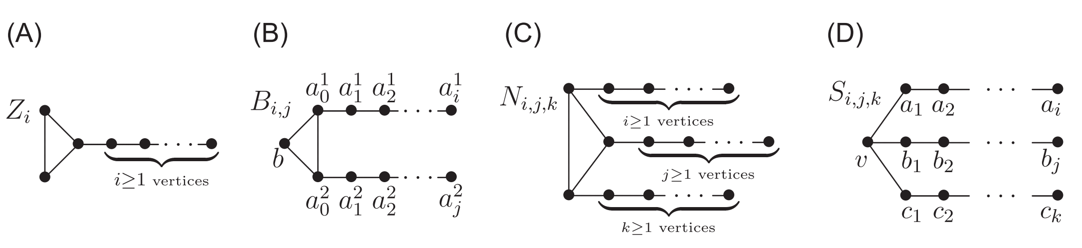

FIGURE 1 Graphs Z_i, $B_{i,j}$, $N_{i,j,k}$, and $S_{i,j,k}$

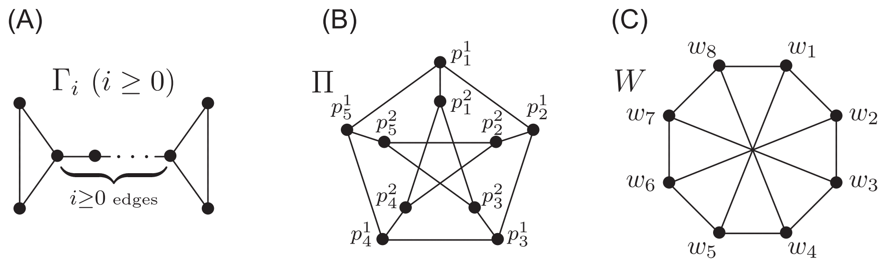

FIGURE 2 The graph Γ_i, the Petersen graph Π, and the Wagner graph W

is relatively slow. For forbidden pairs of connected graphs, there is a list of potential candidates: one of them must be the claw $K_{1,3}$, and the second one belongs to the list mentioned in Section 6. Among them, P_i and $N_{i,j,k}$ are easier to handle since if G is $\{K_{1,3}, P_i\}$-free or $\{K_{1,3}, N_{i,j,k}\}$-free, then so is its closure (more on closures in Section 3), but this is not true for $B_{i,j}$, Z_i, or Γ_i. In this paper, we introduce a technique to overcome this problem for bull-free graphs. Theorem A below lists the best-known results on pairs of forbidden subgraphs implying Hamilton-connectedness of a 3-connected graph (where, in the statement (iii), W^1 denotes the graph obtained from the Wagner graph W (see Figure 2C) by attaching exactly one pendant edge to each of its vertices).

Theorem A (Bian et al. [3], Broersma et al. [6], and Liu et al. [13–15,20]). *Let G be a 3-connected $\{K_{1,3}, X\}$-free graph, where*

(i) [6] $X = \Gamma_1$, *or*
(ii) [3] $X = P_9$, *or*
(iii) [20] $X = Z_6$, *or* $X = Z_7$ *and* $G \ncong L(W^1)$, *or*
(iv) [13–15] $X = B_{i,j}$ *for* $i + j \leq 6$, *or*
(v) [15] $X = N_{1,2,4}$, *or*
(vi) [13,14] $X \in \{N_{1,1,5}, N_{1,3,3}, N_{2,2,3}\}$.

Then G is Hamilton-connected.

Note that statement (iv) is an immediate corollary of (v) and (vi) since $B_{i,j}$ with $i + j \leq 6$ is an induced subgraph of $N_{1,1,5}$, $N_{1,2,4}$, or $N_{1,3,3}$.

Let $\mathcal{W}$ be the family of graphs obtained by attaching at least one pendant edge to each of the vertices of the Wagner graph W, and let $\mathcal{G} = \{L(H) \mid H \in \mathcal{W}\}$ be the family of their line graphs. Then any $G \in \mathcal{G}$ is 3-connected, non-Hamilton-connected (there is, e.g., no Hamiltonian $(L(w_1w_5), L(w_3w_7))$-path), P_{10}-free, $B_{i,j}$-free for $i + j = 8$, and $N_{i,j,k}$-free for $i + j + k = 8$. Thus, this example shows that parts (ii), (v), and (vi) of Theorem A are sharp, and also the next result, which is the main result of this paper, is sharp.

Theorem 1. *Let $X \in \{B_{1,6}, B_{2,5}, B_{3,4}\}$, and let G be a 3-connected $\{K_{1,3}, X\}$-free graph. Then G is Hamilton-connected.*

The proof of Theorem 1 is postponed to Section 5. In Section 3, we collect some known results and facts on line graphs and on closure operations that will be needed. In Section 3.5, we develop a method to overcome the difficulties arising from the fact that the class of $\{K_{1,3}, B_{i,j}\}$-free graphs is not stable under closure operations. In Section 4, we develop a technique that allows a significant reduction in the number of cases to be considered. Finally, in Section 6, we briefly update the discussion of remaining open cases in the characterization of forbidden pairs of connected graphs for Hamilton-connectedness from [14].

3 | PRELIMINARIES

In Sections 3.1–3.4, we summarize some known facts that will be needed in our proof of Theorem 1, and in Section 3.5, we introduce a class of graphs $\mathcal{B}_{i,j}$ such that every $\{K_{1,3}, B_{i,j}\}$-free graph is in $\mathcal{B}_{i,j}$, and for any $G \in \mathcal{B}_{i,j}$, each of its ultimate M (UM)-closures also belongs to $\mathcal{B}_{i,j}$.

3.1 | Line graphs of multigraphs and their preimages

While in line graphs of graphs, for a line graph G, the graph H such that $G = L(H)$ is uniquely determined with a single exception of $G = K_3$, in line graphs of multigraphs this is not true. Using a modification of an approach from [22], the following was proved in [18].

Theorem B (Ryjáček and Vrána [18]). *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that $G = L(H)$ and a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

The multigraph H with the properties given in Theorem B will be called the *preimage* of a line graph G and denoted $H = L^{-1}(G)$. We will also use the notation $a = L(e)$ and $e = L^{-1}(a)$ for an edge $e \in E(H)$ and the corresponding vertex $a \in V(G)$.

An edge-cut $R \subset E(H)$ of a multigraph H is *essential* if $H - R$ has at least two nontrivial components, and H is *essentially k-edge-connected* if every essential edge-cut of H is of size at least k. It is a well-known fact that a line graph G is k-connected if and only if $L^{-1}(G)$ is essentially k-edge-connected. It is also a well-known fact that if X is a line graph, then a line graph G is X-free if and only if $L^{-1}(G)$ does not contain as a sub(multi)graph (not necessarily induced) a (multi)graph F such that $L(F) = X$ (but not necessarily $F = L^{-1}(X)$). However, it is straightforward to verify that for the graph $B_{i,j}$ there is exactly one multigraph F such that $L(F) = B_{i,j}$, namely, the graph $L^{-1}(B_{i,j}) = S_{1,i+1,j+1}$ (see Figure 1D).

Harary and Nash-Williams [9] established a correspondence between a DCT in H and a Hamiltonian cycle in $L(H)$ (the result was given in [9] for line graphs of graphs, but it is easy to see that it is true also for line graphs of multigraphs). A similar result showing that $G = L(H)$ is Hamilton-connected if and only if H has an (e_1, e_2)-IDT for any pair of edges $e_1, e_2 \in E(H)$, was given in [12] (in fact, part (ii) of the following theorem is slightly stronger than the result from [12], and its easy proof is given in [13]).

Theorem C (Harary and Nash-Williams [9] and Li et al. [12]). *Let H be a multigraph with $|E(H)| \geq 3$ and let $G = L(H)$.*

(i) [9] *The graph G is Hamiltonian if and only if H has a DCT.*

(ii) [12] *For every* $e_i \in E(H)$ *and* $a_i = L(e_i)$, $i = 1, 2$, *G has a Hamiltonian* (a_1, a_2)*-path if and only if H has an* (e_1, e_2)*-IDT.*

3.2 | SM-closure

For a graph G and $x \in V(G)$, the *local completion of G at x* is the graph $G_x^* = (V(G), E(G) \cup \{y_1y_2 | y_1, y_2 \in N_G(x)\})$ (i.e., G_x^* is obtained from G by adding all the missing edges with both vertices in $N_G(x)$). Obviously, if G is claw-free, then so is G_x^*. Note that in the special case when G is a line graph and $H = L^{-1}(G)$, G_x^* is the line graph of the (multi) graph obtained from H by contracting the edge $L^{-1}(x)$ into a vertex and replacing the created loop(s) by pendant edge(s). Also note that clearly $x \in V_{\mathrm{SI}}(G_x^*)$ for any $x \in V(G)$, and, more generally, $V_{\mathrm{SI}}(G) \subset V_{\mathrm{SI}}(G_x^*)$ for any $x \in V(G)$.

We say that a vertex $x \in V(G)$ is *eligible* if $\langle N_G(x) \rangle_G$ is a connected noncomplete graph, and we use $V_{\mathrm{EL}}(G)$ to denote the set of all eligible vertices of G. In [17], it was shown that if G is claw-free and $x \in V_{\mathrm{EL}}(G)$, then G_x^* is Hamiltonian if and only if G is Hamiltonian, and the *closure* $\mathrm{cl}(G)$ of a claw-free graph G was defined as the graph obtained from G by recursively performing the local completion operation at eligible vertices, as long as this is possible (more precisely: $\mathrm{cl}(G) = G_k$, where $G_1, \ldots, G_k$ is a sequence of graphs such that $G_1 = G$, $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{\mathrm{EL}}(G_i)$, $i = 1, \ldots, k-1$, and $V_{\mathrm{EL}}(G_k) = \emptyset$). We say that G is *closed* if $G = \mathrm{cl}(G)$. The closure $\mathrm{cl}(G)$ of a claw-free graph G is uniquely determined, is the line graph of a triangle-free graph, and is Hamiltonian if and only if so is G. However, as observed in [5], the closure operation does not preserve (non-)Hamilton-connectedness of G.

For Hamilton-connectedness, the concept of a strong M (SM)-closure G^M of a claw-free graph G was defined in [11] by the following construction.

(i) If G is Hamilton-connected, we set $G^M = \mathrm{cl}(G)$.
(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such eligible vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that
  - $G_1 = G$,
  - $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V_{\mathrm{EL}}(G_i)$, $i = 1, \ldots, k-1$,
  - G_k has no Hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
  - for any $x \in V_{\mathrm{EL}}(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and we set $G^M = G_k$.

A resulting graph G^M is called an *SM-closure* of the graph G, and a graph G equal to its SM-closure is said to be *SM-closed*. Note that for a given graph G, its SM-closure is not uniquely determined.

As shown in [18, 11], if G is SM-closed, then $G = L(H)$, where H does not contain any of the multigraphs shown in Figure 3.

The following theorem summarizes basic properties of the SM-closure operation.

Theorem D (Kužel et al. [11]). *Let G be a claw-free graph and let* G^M *be some of its SM-closures. Then* G^M *has the following properties:*

(i) $V(G) = V(G^M)$ *and* $E(G) \subset E(G^M)$,

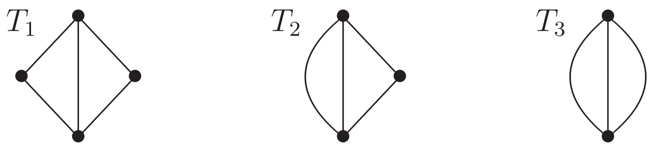

FIGURE 3 The diamond T_1, the multitriangle T_2, and the triple edge T_3

(ii) G^M *is obtained from* G *by a sequence of local completions at eligible vertices,*
(iii) G *is Hamilton-connected if and only if* G^M *is Hamilton-connected,*
(iv) *if* G *is Hamilton-connected, then* $G^M = \text{cl}(G)$,
(v) *if* G *is not Hamilton-connected, then either*
  (α) $V_{\text{EL}}(G^M) = \emptyset$ *and* $G^M = \text{cl}(G)$, *or*
  (β) $V_{\text{EL}}(G^M) \neq \emptyset$ *and* $(G^M)^*_x$ *is Hamilton-connected for any* $x \in V_{\text{EL}}(G^M)$,
(vi) $G^M = L(H)$, *where* H *contains either*
  (α) *at most two triangles and no multiedge, or*
  (β) *no triangle, at most one double edge and no other multiedge,*
(*vii*) *if* G^M *contains no Hamiltonian* (a, b)*-path for some* $a, b \in V(G^M)$ *and*
  (α) X *is a triangle in* H, *then* $E(X) \cap \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\} \neq \emptyset$,
  (β) X *is a multiedge in* H, *then* $E(X) = \{L^{-1}_{G^M}(a), L^{-1}_{G^M}(b)\}$.

We will also need the following lemma on SM-closed graphs proved in [19].

Lemma E (Ryjáček and Vrána [19]). *Let* G *be an SM-closed graph and let* $H = L^{-1}(G)$. *Then* H *does not contain a triangle with a vertex of degree* 2 *in* H.

3.3 | Core of preimage of an SM-closed graph

The definition of the core is slightly problematic for multigraphs, therefore we restrict our observations to the case that we need. Thus, let G be a 3-connected SM-closed graph and let $H = L^{-1}(G)$. The *core of* H is the multigraph $\text{co}(H)$ obtained from H by removing all pendant edges and suppressing all vertices of degree 2.

Shao [21] proved the following properties of the core of a multigraph.

Theorem F (Shao [21]). *Let* H *be an essentially* 3*-edge-connected multigraph. Then*

(i) $\text{co}(H)$ *is uniquely determined,*
(ii) $\text{co}(H)$ *is* 3*-edge-connected,*
(iii) $V(\text{co}(H))$ *dominates all edges of* H,
(iv) *if* $\text{co}(H)$ *has a spanning closed trail, then* H *has a DCT.*

3.4 | UM-closure

As shown in [13], the concept of SM-closure can be further strengthened by omitting the eligibility assumption for the application of the local completion operation (which was defined in Section 3.2 for any vertex $x \in V(G)$). Specifically, for a given claw-free graph G, we construct a graph G^U by the following construction.

(i) If G is Hamilton-connected, we set $G^U = K_{|V(G)|}$.
(ii) If G is not Hamilton-connected, we recursively perform the local completion operation at such vertices for which the resulting graph is still not Hamilton-connected, as long as this is possible. We obtain a sequence of graphs $G_1, \ldots, G_k$ such that
- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V(G_i)$, $i = 1, \ldots, k-1$,
- G_k has no Hamiltonian (a, b)-path for some $a, b \in V(G_k)$,
- for any $x \in V(G_k)$, $(G_k)^*_x$ is Hamilton-connected,

and we set $G^U = G_k$.

A graph G^U obtained by the above construction is called a *UM-closure* of the graph G, and a graph G equal to its UM-closure is said to be *UM-closed*.

Obviously, if G is UM-closed, then G is also SM-closed, implying that G is a line graph and $H = L^{-1}(G)$ has a special structure (contains no diamond, etc.—see Figure 3 and Theorem D(vi) and (vii)). The next theorem shows that for UM-closed graphs, not only H, but also $\mathrm{co}(H)$ has these strong structural properties.

Theorem G (Liu et al. [13]). *Let G be a claw-free graph and let G^U be some of its UM-closures. Then G^U has the following properties:*

(i) *$V(G) = V(G^U)$ and $E(G) \subset E(G^U)$,*
(ii) *G^U is obtained from G by a sequence of local completions at vertices,*
(iii) *G is Hamilton-connected if and only if G^U is Hamilton-connected,*
(iv) *if G is Hamilton-connected, then $G^U = K_{|V(G)|}$,*
(v) *if G is not Hamilton-connected, then $(G^U)^*_x$ is Hamilton-connected for any $x \in V(G^U)$,*
(vi) *$G^U = L(H)$, where $\mathrm{co}(H)$ contains no diamond, no multitriangle and no triple edge, and either*
 (α) *at most two triangles and no multiedge, or*
 (β) *no triangle, at most one double edge and no other multiedge, and if $\mathrm{co}(H)$ contains a double edge, then this double edge is also in H,*
(vii) *if G^U contains no Hamiltonian (a, b)-path for some $a, b \in V(G^U)$ and*
 (α) *X is a triangle in $\mathrm{co}(H)$, then $E(X) \cap \{L^{-1}_{G^U}(a), L^{-1}_{G^U}(b)\} \neq \emptyset$,*
 (β) *X is a multiedge in $\mathrm{co}(H)$, then $E(X) = \{L^{-1}_{G^U}(a), L^{-1}_{G^U}(b)\}$.*

The following lemma will be crucial in our proof of Theorem 1 (recall that W denotes the Wagner graph, see Figure 2C).

Lemma H (Liu et al. [13]). *Let G be a 3-connected non-Hamilton-connected UM-closed claw-free graph. Then G has an induced subgraph $\tilde{G}$ (possibly $\tilde{G} = G$) such that $\tilde{G}$ is 3-connected, non-Hamilton-connected and UM-closed, and, moreover, $\tilde{H}_0 = \mathrm{co}(L^{-1}(\tilde{G}))$ is 2-connected, and either $c(\tilde{H}_0) \geq 9$ and $|V(\tilde{H}_0)| \geq 10$, or $\tilde{H}_0 \simeq W$.*

3.5 | Closure operations and bull-free graphs

When applying closure techniques to {claw, bull}-free graphs, the main problem is that a closure of a $\{K_{1,3}, B_{i,j}\}$-free graph is not necessarily $\{K_{1,3}, B_{i,j}\}$-free (i.e., in the terminology of [16], the class of $\{K_{1,3}, B_{i,j}\}$-free graphs is not stable under the closure operation). Unfortunately, this is the case with all the closure operations mentioned in the previous subsections.

It turns out that this difficulty can be overcome by working in a slightly larger class of graphs which contains all the requested $\{K_{1,3}, B_{i,j}\}$-free graphs but is stable under the closure. We define the class $\mathcal{B}_{i,j}$ as follows.

For any positive integers i, j, $\mathcal{B}_{i,j}$ is the class of all claw-free graphs G such that every induced subgraph $F \subset G$, $F \simeq B_{i,j}$, satisfies $\mu(F) \in V_{\mathrm{SI}}(G)$.

Clearly, every $\{K_{1,3}, B_{i,j}\}$-free graph is in $\mathcal{B}_{i,j}$.

Theorem 2. *Let i, j be positive integers and let $G \in \mathcal{B}_{i,j}$. Then, for any $x \in V(G)$, $G_x^* \in \mathcal{B}_{i,j}$.*

Proof. Let, to the contrary, $G \in \mathcal{B}_{i,j}$ and $x \in V(G)$ be such that G_x^* contains an induced subgraph $F \simeq B_{i,j}$ with $\mu(F) \notin V_{\mathrm{SI}}(G_x^*)$. We will keep the notation of the vertices of F as in Figure 1B, and we will denote by T the triangle $\langle\{b, a_0^1, a_0^2\}\rangle_F$. Since $G \in \mathcal{B}_{i,j}$ and $b = \mu(F)$ is nonsimplicial also in G (recall that $V_{\mathrm{SI}}(G) \subset V_{\mathrm{SI}}(G_x^*)$), we have $E(F) \setminus E(G) \neq \emptyset$. The edges in $E(F) \setminus E(G)$ will be referred to as *new edges*, and we will denote $E(F) \setminus E(G) = \mathrm{new}(F)$.

Suppose first that $\mathrm{new}(F) \cap E(T) = \emptyset$. Let, say, $e = a_k^2 a_{k+1}^2$ be a new edge for some k, $0 \leq k \leq j-1$. Since $e \in E(G_x^*) \setminus E(G)$, we have $a_k^2, a_{k+1}^2 \in N_G(x)$. Since F is induced in G_x^*, the vertices a_k^2, a_{k+1}^2 are the only neighbors of x in $V(F)$ (both in G and in G_x^*). But then the graph $F' = \langle\{b, a_0^1, \dots, a_i^1, a_0^2, \dots, a_k^2, x, a_{k+1}^2, \dots, a_{j-1}^2\}\rangle_G$ is an induced $B_{i,j}$ in G with $\mu(F') \notin V_{\mathrm{SI}}(G)$, contradicting the fact that $G \in \mathcal{B}_{i,j}$.

Thus, we have $\mathrm{new}(F) \subset E(T)$. If $\mathrm{new}(F) = E(T)$, then $\langle\{x, b, a_0^1, a_0^2\}\rangle_G \simeq K_{1,3}$, a contradiction. Hence $1 \leq |\mathrm{new}(F)| \leq 2$.

Suppose first that $|\mathrm{new}(F)| = 2$. By symmetry, either $\mathrm{new}(F) = \{ba_0^1, ba_0^2\}$, or $\mathrm{new}(F) = \{ba_0^1, a_0^1 a_0^2\}$. In both cases, necessarily $N_G(x) \cap V(F) = \{b, a_0^1, a_0^2\}$ (since F is induced in G_x^*). Then, in the first case $F' = \langle\{x, a_0^1, \dots, a_i^1, a_0^2, \dots, a_j^2\}\rangle_G$, and in the second case $F' = \langle\{b, x, a_0^1, \dots, a_{i-1}^1, a_0^2, \dots, a_j^2\}\rangle_G$ is an induced $B_{i,j}$ in G with $\mu(F') \notin V_{\mathrm{SI}}(G)$, a contradiction.

Hence $|\mathrm{new}(F)| = 1$ and then, by symmetry, either $\mathrm{new}(F) = \{ba_0^1\}$, or $\mathrm{new}(F) = \{a_0^1 a_0^2\}$. However, if $\mathrm{new}(F) = \{ba_0^1\}$, then immediately $\langle\{a_0^2, a_1^2, a_0^1, b\}\rangle_G \simeq K_{1,3}$, a contradiction.

Thus, the only remaining case is $\mathrm{new}(F) = \{a_0^1 a_0^2\}$. Then $a_0^1, a_0^2 \in N_G(x)$, and $x \neq b$ (since otherwise $x = b \in V_{\mathrm{SI}}(G_x^*)$). We have $a_1^1 x, a_1^1 b \notin E(G)$ since F is induced in G_x^*, implying $bx \in E(G)$, for otherwise $\langle\{a_0^1, a_1^1, b, x\}\rangle_G \simeq K_{1,3}$. Since $b \notin V_{\mathrm{SI}}(G_x^*)$, there is a vertex $u \in N_G(b)$ such that $xu \notin E(G)$, and since $\langle\{b, u, a_0^1, a_0^2\}\rangle_G \not\simeq K_{1,3}$,

$N_G(u) \cap \{a_0^1, a_0^2\} \neq \emptyset$. By symmetry, let $ua_0^1 \in E(G)$. Since $\langle\{a_0^1, x, u, a_1^1\}\rangle_G \not\simeq K_{1,3}$, we have $ua_1^1 \in E(G)$.

We consider the graph $F'' = \langle\{x, b, u, a_0^2, ..., a_j^2\}\rangle_G$ if $i = 1$, $F'' = \langle\{x, b, u, a_1^1, a_0^2, ..., a_j^2\}\rangle_G$ if $i = 2$, or $F'' = \langle\{x, b, u, a_1^1 ..., a_{i-1}^1, a_0^2, ..., a_j^2\}\rangle_G$ if $i \geq 3$, respectively. If $F'' \simeq B_{i,j}$, then $x = \mu(F'')$, contradicting the fact that $G \in \mathcal{B}_{i,j}$ since $x \notin V_{\mathrm{SI}}(G)$. Hence $F'' \not\simeq B_{i,j}$, implying that either $ua_0^2 \in E(G)$, or, if $i \geq 3$, possibly $ua_1^2 \in E(G)$ (all other potential edges either imply a claw with center at u, or contradict the fact that F is induced in G_x^*).

Let first $ua_0^2 \in E(G)$. Since $\langle\{a_0^2, a_1^2, x, u\}\rangle_G \not\simeq K_{1,3}$, we have $ua_1^2 \in E(G)$, but then $\langle\{u, a_1^2, b, a_1^1\}\rangle_G \simeq K_{1,3}$, a contradiction.

Secondly, if $i = 1$, then $F'' = \langle\{x, b, u, a_0^2, ..., a_j^2\}\rangle_G \simeq B_{i,j}$, and if $i = 2$, then $F'' = \langle\{x, b, u, a_1^1, a_0^2, ..., a_j^2\}\rangle_G \simeq B_{i,j}$ with $\mu(F'') = x$, a contradiction again.

Hence $i \geq 3$ and $ua_1^2 \in E(G)$. But then we have $F''' = \langle\{x, b, u, a_2^1 ..., a_i^1, a_0^2, ..., a_j^2\}\rangle_G \simeq B_{i,j}$ with $\mu(F''') = x$, a contradiction. □

The following corollary is immediate.

Corollary 3. *Let G be a $\{K_{1,3}, B_{i,j}\}$-free graph for some $i, j \geq 1$, and let G^U be one of the UM-closures of G. Then $G^U \in \mathcal{B}_{i,j}$.*

4 | A SPECIAL VERSION OF THE "NINE-POINT-THEOREM"

We will use a special version of the well-known "Nine-point-theorem" by Holton et al. [10] and of its modification by Bau and Holton [2], developed in [13]. For this, we need some more terminology from [1].

Let G be a multigraph, $R \subset G$ a spanning sub(multi)graph of G, and let $\mathcal{R}$ be the set of components of R. Then G/R is the multigraph with $V(G/R) = \mathcal{R}$, in which, for each edge in $E(G)$ between two components of R, there is an edge in $E(G/R)$ joining the corresponding vertices of G/R. The (multi)graph G/R is said to be a *contraction* of G. (Roughly, in G/R, components of R are contracted to single vertices while keeping the adjacencies between them). Clearly, if R is connected, then $G/R = K_1$, and if R is edgeless, then $G/R = G$; these two contractions are called *trivial*.

The contraction operation maps $V(G)$ onto $V(G/R)$, where vertices of a component of R are mapped on a vertex of G/R. If $G/R \simeq F$, then this defines a function $\alpha : G \to F$ which is called a *contraction of G on F*.

Throughout the rest of this section, Π denotes the Petersen graph.

The following special version of the "nine-point-theorem" was proved in [13].

Theorem I (Liu et al. [13]). *Let H be a 3-edge-connected multigraph, $A \subset V(H)$, $|A| = 8$, and let $e \in E(H)$. Then either*

(i) *H contains a closed trail T such that $A \subset V(T)$ and $e \in E(T)$, or*
(ii) *there is a contraction $\alpha : H \to \Pi$ such that $\alpha(e) = xy \in E(\Pi)$ and $\alpha(A) = V(\Pi) \setminus \{x, y\}$.*

We will also need the following auxiliary result from [13].

Lemma J (Liu et al. [13]). *Let H be a graph such that* $\text{co}(H) = W$. *If there is a vertex* $x \in V(\text{co}(H))$ *such that* $N_H(x) = N_{\text{co}(H)}(x)$, *then* $L(H)$ *is Hamilton-connected.*

Theorem 4. *Let* $X \in \{B_{1,6}, B_{2,5}, B_{3,4}\}$, *and let* G *be a* 3*-connected* $\{K_{1,3}, X\}$*-free graph with a UM-closure* G^U *such that* $\text{co}(H)$, *where* $H = L^{-1}(G^U)$, *is* 2*-connected. Let* $e_1, e_2 \in E(H)$ *be such that there is no* (e_1, e_2)*-IDT in* H. *Then for every set* $A \subset V(\text{co}(H))$, $|A| = 8$, *there is an* (e_1, e_2)*-trail* T *in* H *such that* $A \subset \text{Int}(T)$.

Proof. First of all, it should be noted here that some parts of the proof of Theorem 4 are (almost) the same as the corresponding parts of the proof of Theorem 9 in [13]. Since the other parts are quite different, for the sake of completeness, we give a complete proof here, including the identical parts.

Let G be a graph satisfying the assumptions of the theorem. By Corollary 3, $G^U \in \mathcal{B}_{1,6} \cup \mathcal{B}_{2,5} \cup \mathcal{B}_{3,4}$, implying that in $H = L^{-1}(G^U)$, every subgraph (not necessarily induced) isomorphic to $S_{1,2,7}$, $S_{1,3,6}$, or $S_{1,4,5}$ has its branch of length 1 at a pendant edge (recall that a vertex in G^U is simplicial if and only if the corresponding edge in $H = L^{-1}(G^U)$ is pendant by Theorem B).

Let H' be the multigraph obtained from H by the following construction:

(i) if e_1, e_2 share a vertex of degree 2, say, $e_i = v_i v$, $i = 1, 2$ with $v \in V_2(H)$, we suppress v and set $h = v_1 v_2$,
(ii) otherwise, we subdivide either e_i if e_i is nonpendant, or some edge in $\text{co}(H)$ sharing a vertex with e_i if e_i is pendant, with a vertex v_i, $i = 1, 2$, and add a new edge $h = v_1 v_2$.

If there is no contraction $\alpha' : H' \to \Pi$ such that $\alpha'(h) = x_1 x_2 \in E(\Pi)$ and $\alpha'(A) = V(\Pi) \setminus \{x_1, x_2\}$, then, by Theorem I, there is a closed trail T' in H' such that $A \subset V(T')$ and $h \in E(T')$. Returning to H, that is, in case (i) subdividing h, or in case (ii) removing h, suppressing v_1, v_2, and extending the trail to e_i if e_i is pendant, we obtain an (e_1, e_2)-trail T in H with $A \subset \text{Int}(T)$.

Thus, we suppose that there is a contraction $\alpha' : H' \to \Pi$ such that $\alpha'(h) = x_1 x_2 \in E(\Pi)$ and $\alpha'(A) = V(\Pi) \setminus \{x_1, x_2\}$. In case (i), H can be contracted on a graph isomorphic to the Petersen graph with at least one subdivided edge which contains each of the graphs $S_{1,2,7}$, $S_{1,3,6}$, and $S_{1,4,5}$: in the labeling of vertices as in Figure 2B, if, say, the edge $p_1^1 p_1^2$ is subdivided with a vertex q, we have $S_{1,2,7}(p_1^1; q; p_2^1 p_3^1; p_5^1 p_4^1 p_4^2 p_1^2 p_3^2 p_5^2 p_2^2)$, $S_{1,3,6}(p_1^1; q; p_5^1 p_4^1 p_3^1; p_2^1 p_2^2 p_4^2 p_1^2 p_3^2 p_5^2)$, and $S_{1,4,5}(p_1^1; q; p_5^1 p_4^1 p_4^2 p_1^2; p_2^1 p_3^1 p_3^2 p_5^2 p_2^2)$ as subgraphs of H with the branch of length 1 at a nonpendant edge, a contradiction. Thus, for the rest of the proof, we suppose that H' is obtained by construction (ii).

Set $H_0 = \text{co}(H)$, and recall that H_0 is 3-edge-connected (since H is essentially 3-edge-connected). Let R' be the spanning sub(multi)graph of H' that defines α', and suppose that, say, the component $R_1 = (\alpha')^{-1}(x_1)$ of R' is nontrivial. Since $x_1 \in V(\Pi)$, R_1 is separated from the rest of H' by a 3-edge-cut containing the edge h, implying that in H_0, the sub(multi)graph $R_1 - v_1$ is separated from the rest of H_0 by a 2-edge-cut, contradicting the fact that H_0 is 3-edge-connected. Hence $(\alpha')^{-1}(x_1)$, and symmetrically also $(\alpha')^{-1}(x_2)$, are trivial, that is, $V((\alpha')^{-1}(x_i)) = \{v_i\}$, $i = 1, 2$. Removing from H' the edge h and suppressing v_1 and v_2, we obtain from R' the corresponding spanning sub (multi)graph R of H, and from R, in a standard way, a spanning sub(multi)graph R_0 of H_0. Note that clearly every component of R' except $\{v_1\}$ and $\{v_2\}$ corresponds to a nonempty component of R_0 since α' maps H' on a cubic graph and hence every component of R' must contain a vertex of degree more than 2. Then the components of R_0 define a contraction $\alpha : H_0 \to W$, where W is the Wagner graph (see Figure 2C; recall that W can be obtained from Π by removing an edge and suppressing the created vertices of degree 2).

Case 1: *$\alpha^{-1}(w)$ is trivial for any $w \in V(W)$.*

Then we have $H_0 \simeq W$. By Lemma J, every vertex of H_0 is incident in H to a pendant edge or to a subdivided edge.

Subcase 1.1: *No edge of H_0 is subdivided in H.*

Then, by Lemma J, each vertex of H_0 is incident in H with at least one pendant edge, and then H contains each of the subgraphs $S_{1,2,7}(w_1; w_1'; w_8w_8'; w_2w_3w_4w_5w_6w_7w_7')$, $S_{1,3,6}(w_1; w_1'; w_8w_7w_7'; w_2w_3w_4w_5w_6w_6')$, and $S_{1,4,5}(w_1; w_1'; w_8w_7w_6w_6'; w_2w_3w_4w_5w_5')$ (where w_i' is a vertex of degree 1 adjacent to w_i, $i = 1, \ldots, 8$).

Since G is X-free for $X \in \{B_{1,6}, B_{2,5}, B_{3,4}\}$, for some vertex $w_i \in V(H_0)$, the set of edges incident to w_i corresponds in $L(H) = G^U$ to a clique obtained from a certain subgraph of G by a series of local completions. Let $G_1, \ldots, G_k$ be the sequence of graphs that yields G^U, that is, $G_1 = G$, $G_k = G^U$, and $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in V(G_i)$, $i = 1, \ldots, k-1$. Then $x_{k-1} \in V_{\mathrm{SI}}(G^U)$, thus, by Theorem B, x_{k-1} corresponds to a pendant edge in H. Choose the notation such that $L^{-1}(x_{k-1}) = w_1w_1'$. For any edge $w_iw_j \in E(W)$ set $L(w_iw_j) = v_{i,j}$, and set $L(w_iw_i') = v_i$, $i = 2, \ldots, 8$. Since $\langle\{x_{k-1}, v_{1,2}, v_{1,5}, v_{1,8}\}\rangle_{G^U}$ is a clique, x_{k-1} is adjacent in G_{k-1} to each of $v_{1,2}$, $v_{1,5}$, and $v_{1,8}$. Now, if $v_{1,2}v_{1,8} \in E(G_{k-1})$, we have $F_1 = \left\langle\{x_{k-1}, v_{1,8}, v_8, v_{1,2}, v_{2,3}, v_{3,4}, v_{4,5}, v_{5,6}, v_{6,7}, v_7\}\right\rangle_{G_{k-1}} \simeq B_{1,6}$, $F_2 = \langle\{x_{k-1}, v_{1,8}, v_{7,8}, v_7, v_{1,2}, v_{2,3}, v_{3,4}, v_{4,5}, v_{5,6}, v_6\}\rangle_{G_{k-1}} \simeq B_{2,5}$, and $F_3 = \langle\{x_{k-1}, v_{1,8}, v_{7,8}, v_{6,7}, v_6, v_{1,2}, v_{2,3}, v_{3,4}, v_{4,5}, v_5\}\rangle_{G_{k-1}} \simeq B_{3,4}$ with $\mu(F_1) = \mu(F_2) = \mu(F_3) = x_{k-1}$, contradicting the fact that $G_{k-1} \in \mathcal{B}_{1,6} \cup \mathcal{B}_{2,5} \cup \mathcal{B}_{3,4}$ (since x_{k-1} is simplicial in G_k, but not in G_{k-1}). Hence $v_{1,2}v_{1,8} \notin E(G_{k-1})$, that is, $v_{1,2}v_{1,8}$ is a new edge in $G_k = G^U$.

If both $v_{1,2}v_{1,5} \notin E(G_{k-1})$ and $v_{1,5}v_{1,8} \notin E(G_{k-1})$, we have $\langle\{x_{k-1}, v_{1,2}, v_{1,5}, v_{1,8}\}\rangle_{G_{k-1}} \simeq K_{1,3}$, a contradiction. If both $v_{1,2}v_{1,5} \in E(G_{k-1})$ and $v_{1,5}v_{1,8} \in E(G_{k-1})$, then we have $\langle\{v_{1,5}, v_{1,2}, v_{1,8}, v_{4,5}\}\rangle_{G_{k-1}} \simeq K_{1,3}$, a contradiction again. Thus, by symmetry, we can assume that $v_{1,2}v_{1,5} \in E(G_{k-1})$ and $v_{1,5}v_{1,8} \notin E(G_{k-1})$. Then $F_1 = \langle\{x_{k-1}, v_{1,2}, v_2, v_{1,5}, v_{5,6}, v_{6,7}, v_{7,8}, v_{4,8}, v_{3,4}, v_3\}\rangle_{G_{k-1}} \simeq B_{1,6}$, $F_2 = \langle\{x_{k-1}, v_{1,5}, v_{5,6}, v_6, v_{1,2}, v_{2,3}, v_{3,4}, v_{4,8}, v_{7,8}, v_7\}\rangle_{G_{k-1}} \simeq B_{2,5}$, and $F_3 = \langle\{x_{k-1}, v_{1,5}, v_{4,5}, v_{3,4}, v_3, v_{1,2}, v_{2,6}, v_{6,7}, v_{7,8}, v_8\}\rangle_{G_{k-1}} \simeq B_{3,4}$ with $\mu(F_1) = \mu(F_2) = \mu(F_3) = x_{k-1}$, a contradiction again.

Subcase 1.2: *At least one edge of H_0 is subdivided in H.*

By symmetry, we can choose the notation such that w_1w_2 or w_1w_5 is subdivided in H with a vertex w of degree 2 in H. Then we have the following possibilities.

Subdivided edge	Subgraph $S_{i,j,k}$
w_1w_2	$S_{1,2,7}(w_1; w; w_5w_5'; w_8w_4w_3w_2w_6w_7w_7')$
	$S_{1,3,6}(w_1; w_8; w_5w_4w_4'; ww_2w_3w_7w_6w_6')$
	$S_{1,4,5}(w_1; w_5; w_8w_7w_6w_6'; ww_2w_3w_4w_4')$
w_1w_5	$S_{1,2,7}(w_1; w_2; w_8w_8'; ww_5w_4w_3w_7w_6w_6')$
	$S_{1,3,6}(w_3; w_7; w_4w_8w_8'; w_2w_1ww_5w_6w_6')$
	$S_{1,4,5}(w_1; w_2; w_8w_7w_6w_6'; ww_5w_4w_3w_3')$

where w_i' is a neighbor of w_i in $H - H_0$ which exists by Lemma J (note that w_i' can be a vertex of degree 2, subdividing some of the edges incident to w_i, in which case the last two vertices of a branch can occur in reverse order).

Since in each of the cases the branch of length 1 is a nonpendant edge, we have a contradiction with the fact that $G^U \in \mathcal{B}_{1,6} \cup \mathcal{B}_{2,5} \cup \mathcal{B}_{3,4}$.

Case 2: *$\alpha^{-1}(w)$ is nontrivial for some $w \in V(W)$.*

Let $R_1^0, ..., R_8^0$ be the components of the (multi)graph R_0 that defines α, and choose the notation such that $R_i^0 = \alpha^{-1}(w_i)$, $i = 1, ..., 8$, and such that $R_1^0 = \alpha^{-1}(w_1)$ is nontrivial. Recall that $\cup_{i=1}^8(V(R_i^0)) = V(R_0) = V(H_0)$.

We observe that $e_1, e_2 \in E(H_0)\backslash E(R_0)$ since, by the construction of H', $\alpha^{-1}(x_i) = v_i$ are trivial and after deleting the edge h and suppressing the vertices v_1, v_2, each of the edges e_1, e_2 has its vertices in different components of R_0. By Theorem G(vi),(vii), this implies that each R_i^0 is a triangle-free (simple) graph. Moreover, each R_i^0 is 2-edge-connected since $R_i^0 = \alpha^{-1}(w_i)$ is separated from the rest of H_0 by a 3-edge-cut and a cut-edge in R_i^0 would create a 2-edge-cut in H_0.

We introduce the following notation. For any edge $w_iw_j \in E(W)$, we set $f_{ij} = \alpha^{-1}(w_iw_j)$ (i.e., f_{ij} joins R_i^0 and R_j^0), and we denote b_j^i its vertex in R_i^0 and b_i^j its vertex in R_j^0. Thus, we, for example, have $A_{H_0}(R_1^0) = \{b_2^1, b_5^1, b_8^1\}$, where $2 \le |\{b_2^1, b_5^1, b_8^1\}| \le 3$, and $\{f_{12}, f_{15}, f_{18}\}$ is the 3-edge-cut that separates R_1^0 from the rest of H_0.

Claim 1. Let R_i^0 be a component of R_0, $1 \le i \le 8$, and let $A_{H_0}(R_i^0) = \{b_{j_1}^i, b_{j_2}^i, b_{j_3}^i\}$. Then there is a vertex $d^i \in V(R_i^0)$ and three internally vertex-disjoint (possibly trivial) $(d^i, b_{j_k}^i)$-paths $P_{j_k}^i$, $k = 1, 2, 3$.

Proof. Let P be an arbitrary (possibly trivial) $(b_{j_1}^i, b_{j_2}^i)$-path in R_i^0, and let $P_{j_3}^i$ be a shortest path between $b_{j_3}^i$ and a vertex of P, which will be referred to as d^i. Then the vertex d^i and the paths $P_{j_1}^i = d^iPb_{j_1}^i$, $P_{j_2}^i = d^iPb_{j_2}^i$, and $P_{j_3}^i$ have the required properties. □

Claim 2. The component R_1^0 contains a cycle C of length at least 4, vertices $c_2, c_5, c_8 \in V(C)$ and paths Q_2^1, Q_5^1, Q_8^1 (possibly trivial) such that

(i) $2 \le |\{c_2, c_5, c_8\}| \le 3$,

(ii) Q_2^1 is a (c_2, b_2^1)-path, Q_5^1 is a (c_5, b_5^1)-path and Q_8^1 is a (c_8, b_8^1)-path,
(iii) the paths Q_2^1, Q_5^1, Q_8^1 are internally vertex-disjoint.

Proof. Let d^1 and P_2^1, P_5^1, P_8^1 be the vertex and paths in R_1^0 given by Claim 1. Since R_1^0 is nontrivial, at least one of P_2^1, P_5^1, P_8^1 is nontrivial. Suppose that, say, P_5^1 is nontrivial. We consider a (b_2^1, b_8^1)-path P and choose two edge-disjoint paths P_5', P_5'' such that

- P_5' is a (b_5^1, c_2)-path and P_5'' is a (b_5^1, c_8)-path for some $c_2, c_8 \in V(P)$,
- if $c_2 \neq c_8$, then c_2 is on P between c_8 and b_2^1, and
- c_2, c_8, P_5', and P_5'' are chosen such that $|E(P_5')| + |E(P_5'')|$ is smallest possible.

If $c_2 \neq c_8$, we choose c_5 as the last common vertex of P_5' and P_5'', and we set $C = c_2Pc_8P_5''c_5P_5'c_2$, $Q_2^1 = c_2Pb_2^1$, $Q_8^1 = c_8Pb_8^1$, and, say, $Q_5^1 = c_5P_5'b_5^1$. If $c_2 = c_8$, we choose c_5 as the last common vertex of P_5' and P_5'' distinct from the vertex $c_2 = c_8$ (possibly $c_5 = b_5^1$), and set $C = c_2P_5'c_5P_5''c_2$, $Q_2^1 = c_2Pb_2^1$, $Q_8^1 = c_8Pb_8^1$, and, say, $Q_5^1 = c_5P_5'b_5^1$ (recall that each R_i^0 is a triangle-free [simple] graph, hence in each case, C is of length at least 4).

If P_2^1 or P_8^1 is nontrivial, we get C, Q_2^1, Q_5^1, and Q_8^1 in the same way with the only difference that possibly $c_5 = c_8$ or $c_2 = c_5$.

By Claim 2, we have, up to a symmetry, the following possibilities (note that W has two types of symmetries—rotations and reflections, but is not edge-transitive): $|\{c_2, c_5, c_8\}| = 3$; $|\{c_2, c_5, c_8\}| = 2$ and $c_2 = c_8$; $|\{c_2, c_5, c_8\}| = 2$ and $c_2 = c_5$. For each of the requested graphs $S_{1,2,7}$, $S_{1,3,6}$, and $S_{1,4,5}$, we describe a sub(multi)graph of H_0 in which it is contained, in all three possible cases. Here, for integers i_0, j_0, k_0, $1 \leq i_0 \leq j_0 \leq k_0$, we use $S_{\geq i_0, \geq j_0, \geq k_0}$ to denote a graph containing an S_{i_0, j_0, k_0} as a subgraph. If a component R_i^0 contains the vertex of degree 3 of the $S_{\geq i_0, \geq j_0, \geq k_0}$, then it is located in the vertex d^i and uses the paths $P_{j_k}^i$, $k = 1, 2, 3$, given by Claim 1, and for any other component R_i^0, $2 \leq i \leq 8$, and $b_j^i, b_k^i \in A_{H_0}(R_i^0)$, we use $Q_{j,k}^i$ to denote an arbitrarily chosen (b_j^i, b_k^i)-path in R_i^0 (of course, if R_i^0 is trivial, all these paths collapse to a single vertex).

If we relabel the vertices of the cycle C given by Claim 2 such that $C = u_1u_2 \ldots u_{|V(C)|}$ with $u_1 = c_5$ (and also $u_1 = c_5 = c_2$ in the third case), then the requested subgraphs, containing $S_{1,2,7}$ and $S_{1,4,5}$, can be (in all three cases) described as $S_{\geq 1, \geq 2, \geq 7}(d^3; P_2^3b_3^2; P_4^3Q_{3,8}^4b_4^8; P_7^3Q_{3,6}^7Q_{7,5}^6Q_{6,1}^5Q_5^1u_1u_2u_3u_4)$ and $S_{\geq 1, \geq 4, \geq 5}(d^4; P_8^4b_4^8; P_3^4Q_{4,2}^3Q_{3,6}^2Q_{2,7}^6b_6^7; P_5^4Q_{4,1}^5Q_5^1u_1u_2u_3u_4)$; finally, if we relabel the vertices of C such that $C = u_1u_2 \ldots u_{|V(C)|}$ with $u_1 = c_8$ (and also $u_1 = c_2 = c_8$ in the second case), then the subgraph, containing $S_{1,3,6}$, can be (in all three cases) described as $S_{\geq 1, \geq 3, \geq 6}(d^6; P_2^6b_6^2; P_5^6Q_{6,4}^5Q_{5,3}^4b_4^3; P_7^6Q_{6,8}^7Q_{7,1}^8Q_8^1u_1u_2u_3u_4)$. In all cases, we have obtained a subgraph $S_{1,2,7}$, $S_{1,3,6}$, and $S_{1,4,5}$ such that its branch of length 1 is nonpendant, contradicting the fact that $G^U \in \mathcal{B}_{1,6} \cup \mathcal{B}_{2,5} \cup \mathcal{B}_{3,4}$. □

5 | PROOF OF THEOREM 1

Let G be a 3-connected $\{K_{1,3}, X\}$-free graph, where $X \in \{B_{1,6}, B_{2,5}, B_{3,4}\}$, and suppose, to the contrary, that G is not Hamilton-connected. By Theorem G and by Corollary 3, we can suppose that G is UM-closed and $G \in \mathcal{B}_{1,6} \cup \mathcal{B}_{2,5} \cup \mathcal{B}_{3,4}$. Let thus $H = L^{-1}(G)$, and set $H_0 = \mathrm{co}(H)$. By Theorem F(ii), H_0 is 3-edge-connected. By Lemma H, we can assume that H_0 is 2-connected with $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$, unless $H_0 \simeq W$. However, if $H_0 \simeq W$, then, by Theorem 4 and since $|V(H_0)| = 8$, H has an (e_1, e_2)-IDT for any $e_1, e_2 \in E(H_0)$ and hence also for any $e_1, e_2 \in E(H)$, implying that $G = L(H)$ is Hamilton-connected, a contradiction. Thus, we have $c(H_0) \geq 9$ and $|V(H_0)| \geq 10$. We consider the possible cases separately and, for each of the subgraphs $B_{i,j}$, we distinguish cases according to the length of a longest cycle in H_0, and we attempt to identify a subgraph of type $S_{i,j,k}$.

Throughout the proof, in each of the cases, C always denotes a cycle such that

(i) C is a longest cycle in H_0,
(ii) subject to (i), C dominates in H maximum number of edges.

We further denote $C = x_1x_2...x_{c(H_0)}$, $R = V(H)\setminus V(C)$, $N = \{y \in V(H_0)|\, N_R(y) = \emptyset\}$, $R_0 = R \cap V(H_0)$, and if $R_0 \neq \emptyset$, we set $R_0 = \{y_1, ..., y_{|R_0|}\}$ and we choose the notation such that $y_1x_1 \in E(H_0)$. An edge $x_ix_j \in E(H_0)\setminus E(C)$ with $x_i, x_j \in V(C)$, $1 \leq i, j \leq |V(C)|$, will be called a *chord* of C, and we say that x_ix_j is a *k-chord* if the shorter one of the two subpaths of C determined by x_i and x_j has k interior vertices.

There are several general comments to some situations in the proof.

- We will often list vertices of a subgraph $S_{i,j,k}$, and then the following is possible.
  - When some edge $e = uv$ of the $S_{i,j,k}$ is in $E(H_0)$, it can always happen that e is subdivided in H, that is, formally, $e \notin E(H)$. However, it is immediate to see that if this happens, then the corresponding submultigraph of H, which instead of $e = uv$ contains a path uzv with $z \in V_2(H)$, also contains $S_{i,j,k}$ as a subgraph.
  - When a vertex $v \in V(C)$ has a (potential) neighbor $z \in R$ and the vertex z occurs as the last vertex of a branch of the $S_{i,j,k}$, then such a vertex z can be an endvertex of a pendant edge attached to v, or can be $z \in V_2(H)$ and z subdivides some of the edges incident to v. It should be noted that in the second case, the vertices v and z can occur in reverse order in the list (i.e., v being the last vertex of the branch).
- In many subcases, the cycle C will be dominating, and we will consider its potential chords, using the fact that $\delta(H_0) \geq 3$. In such situations, it is always implicitly understood that none of the edges of C can be a double edge, since if, for example, x_1x_2 is a double edge with $V(e_1) = V(e_2) = \{x_1, x_2\}$, then $T = e_1x_2x_3...x_{c(H_0)}x_1e_2$ is an (e_1, e_2)-IDT in H, contradicting Theorem G(vii)(β).

These facts will be always implicitly understood throughout the proof.

Case 1: $G \in \mathcal{B}_{1,6}$. Then H does not contain as a subgraph the graph $S_{1,2,7}$ such that its branch of length 1 is in a nonpendant edge.

Subcase 1.1: $c(H_0) = 9$ *and* $|V(H_0)| \geq 10$.
First observe that $E(\langle R\rangle_H) = \emptyset$, since if, for example, $y_1z \in E(H)$ for some $z \in R$, then H

contains the subgraph $S_{1,2,7}(x_1; x_2; y_1z; x_9x_8x_7x_6x_5x_4x_3)$ with a branch of length 1 at nonpendant edge x_1x_2, a contradiction. Hence $N_R(y_1) = \emptyset$.

Next observe that $x_2 \in N$ since otherwise, for some $z \in N_R(x_2)$, H contains the subgraph $S_{1,2,7}(x_1; y_1; x_2z; x_9x_8x_7x_6x_5x_4x_3)$ (note that $x_2y_1 \notin E(H)$ since C is longest). Similarly, we have $N_R(x_4) \subset \{y_1\}$, since otherwise, for a vertex $z \in N_R(x_4)\setminus\{y_1\}$, we have $S_{1,2,7}(x_1; y_1; x_2x_3; x_9x_8x_7x_6x_5x_4z)$ in H (note that $y_1 \in V(H_0)$, implying that the edge x_1y_1 is nonpendant in H). Symmetrically, $x_9 \in N$ and $N_R(x_7) \subset \{y_1\}$.

Now, if $x_2x_4 \notin E(H)$, then the set $A = \{x_1, x_3, x_5, x_6, x_7, x_8, x_9, y_1\}$ with $|A| = 8$ dominates all edges in H, and, by Theorem 4, $G = L(H)$ is Hamilton-connected, a contradiction. Hence $x_2x_4 \in E(H_0)$. Analogously, by Theorem 4, considering the set $A = (V(C) \cup \{y_1\})\setminus\{x_7, x_9\}$ with $|A| = 8$, we have $x_7x_9 \in E(H_0)$, and considering the set $A = (V(C) \cup \{y_1\})\setminus\{x_2, x_9\}$ with $|A| = 8$, we have $x_2x_9 \in E(H_0)$. But then the edges x_2x_4, x_7x_9, and x_2x_9 are three 1-chords in C, creating three triangles in H_0, which contradicts Theorem G(vi).

Subcase 1.2: $c(H_0) = |V(H_0)| = 10$.
Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 1.2.1: *C has a 1-chord.*
Choose the notation such that $x_1x_3 \in E(H_0)$. Then $x_2 \in N$ for otherwise, for a $z \in N_R(x_2)$, H contains $S_{1,2,7}(x_1; x_3; x_2z; x_{10}x_9x_8x_7x_6x_5x_4)$. Similarly $x_4 \in N$, for otherwise H contains $S_{1,2,7}(x_3; x_2; x_4z; x_1x_{10}x_9x_8x_7x_6x_5)$. Considering the set $A = V(C)\setminus\{x_2, x_4\}$ with $|A| = 8$, we have $x_2x_4 \in E(H_0)$ by Theorem 4. But then the two 1-chords x_1x_3 and x_2x_4 create a diamond (see Figure 3) in H_0, contradicting Theorem G(vi).

Subcase 1.2.2: *C has a 2-chord.*
Choose the notation such that $x_1x_4 \in E(H_0)$. If there is a vertex $z \in N_R(x_5) \cup N_R(x_6)$, we have $S_{1,2,7}(x_1; x_4; x_2x_3; x_{10}x_9x_8x_7x_6x_5z)$ or $S_{1,2,7}(x_4; x_5; x_3x_2; x_1x_{10}x_9x_8x_7x_6z)$ in H. Hence $\{x_5, x_6\} \subset N$, and, symmetrically, $\{x_9, x_{10}\} \subset N$. Then, using Theorem 4 and the assumption that G is not Hamilton-connected, the set $A_1 = V(C)\setminus\{x_5, x_{10}\}$ with $|A_1| = 8$ yields $x_5x_{10} \in E(H_0)$, $A_2 = V(C)\setminus\{x_6, x_9\}$ with $|A_2| = 8$ yields $x_6x_9 \in E(H_0)$, and $A_3 = V(C)\setminus\{x_5, x_9\}$ with $|A_3| = 8$ yields $x_5x_9 \in E(H_0)$. But then the chords x_5x_{10}, x_6x_9, and x_5x_9 create a diamond in H_0, contradicting Theorem G(vi).

Subcase 1.2.3: *C has a 3-chord.*
Let $x_1x_5 \in E(H_0)$. Since $\delta(H_0) \geq 3$, x_3 is in a chord, and by the previous subcases, since $|V(C)| = 10$ and by symmetry, we have $x_3x_7 \in E(H_0)$ (a 3-chord), or $x_3x_8 \in E(H_0)$ (a 4-chord).

Let first $x_3x_7 \in E(H_0)$. Then $x_2 \in N$, for otherwise, for a $z \in N_R(x_2)$, we have $S_{1,2,7}(x_3; x_4; x_2z; x_7x_6x_5x_1x_{10}x_9x_8)$ in H. Similarly, $x_4 \in N$, for otherwise, for a $z \in N_R(x_4)$, we have $S_{1,2,7}(x_3; x_2; x_4z; x_7x_6x_5x_1x_{10}x_9x_8)$ in H. Then, using the set $A = V(C)\setminus\{x_2, x_4\}$ with $|A| = 8$, we have $x_2x_4 \in E(H_0)$ by Theorem 4, and we are back in Subcase 1.2.1.

Thus, $x_3x_8 \in E(H_0)$. Then, for a $z \in N_R(x_2)$, $S_{1,2,7}(x_3; x_4; x_2z; x_8x_7x_6x_5x_1x_{10}x_9)$ is a subgraph of H, hence $x_2 \in N$. Symmetrically, $x_4 \in N$. Then Theorem 4 for the set $A = V(C)\setminus\{x_2, x_4\}$ with $|A| = 8$ implies $x_2x_4 \in E(H_0)$, and we are again back in Subcase 1.2.1.

Subcase 1.2.4: *C has a 4-chord.*

By the previous subcases, all chords in C are 4-chords. If, say, $z \in N_R(x_1)$, then H contains $S_{1,2,7}(x_5;\, x_6;\, x_4x_3;\, x_{10}x_9x_8x_7x_2x_1z)$. Hence $x_1 \in N$, and, symmetrically, $x_3 \in N$. Then, for the set $A = V(C)\setminus\{x_1, x_3\}$ with $|A| = 8$, Theorem 4 implies 1-chord $x_1x_3 \in E(H_0)$, a contradiction.

Subcase 1.3: $c(H_0) \geq 10$ *and* $|V(H_0)| > c(H_0)$.

Set $c(H_0) = t$. Then H contains $S_{1,2,7}(x_1;\, y_1;\, x_2x_3;\, x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5}x_{t-6})$ (note that $t - 6 > 3$ since $t \geq 10$, and that the edge x_1y_1 is nonpendant since $y_1 \in V(H_0)$).

Subcase 1.4: $c(H_0) = |V(H_0)| = 11$.

Since $\delta(H_0) \geq 3$, every vertex of C is in a chord. If $x_1x_3 \in E(H_0)$, H contains the subgraph $S_{1,2,7}(x_1;\, x_2;\, x_3x_4;\, x_{11}x_{10}x_9x_8x_7x_6x_5)$. Similarly, if $x_1x_4 \in E(H_0)$, H contains $S_{1,2,7}(x_1;\, x_2;\, x_4x_3;\, x_{11}x_{10}x_9x_8x_7x_6x_5)$. Hence C has only k-chords for $3 \leq k \leq 4$.

Suppose that C has a 3-chord and let $x_1x_5 \in E(H_0)$. Then x_3 has a chord, that is, by symmetry, $x_3x_7 \in E(H_0)$ or $x_3x_8 \in E(H_0)$, but in the first case H contains the subgraph $S_{1,2,7}(x_1;\, x_2;\, x_5x_6;\, x_{11}x_{10}x_9x_8x_7x_3x_4)$, and in the second case H contains the subgraph $S_{1,2,7}(x_5;\, x_4;\, x_6x_7;\, x_1x_2x_3x_8x_9x_{10}x_{11})$.

Hence the only chords in C are 4-chords. Let $x_1x_6 \in E(H_0)$. Then x_9 has a chord and, by symmetry, the only possibility is $x_3x_9 \in E(H_0)$. Then H contains the subgraph $S_{1,2,7}(x_1;\, x_2;\, x_{11}x_{10};\, x_6x_7x_8x_9x_3x_4x_5)$.

Subcase 1.5: $c(H_0) = |V(H_0)| = 12$.

If $x_1x_3 \in E(H_0)$, H contains $S_{1,2,7}(x_1;\, x_2;\, x_3x_4;\, x_{12}x_{11}x_{10}x_9x_8x_7x_6)$, and if $x_1x_k \in E(H_0)$ for $4 \leq k \leq 5$, H contains $S_{1,2,7}(x_1;\, x_k;\, x_2x_3;\, x_{12}x_{11}x_{10}x_9x_8x_7x_6)$. Hence C has only 4-chords and 5-chords.

Let $x_1x_6 \in E(H_0)$ be a 4-chord of C. Then x_3 is in a 4-chord or in a 5-chord. There are the following possibilities.

Chord at x_3	Subgraph $S_{1,2,7}$
x_3x_8	$S_{1,2,7}(x_3;\, x_2;\, x_4x_5;\, x_8x_7x_6x_1x_{12}x_{11}x_{10})$
x_3x_9	$S_{1,2,7}(x_3;\, x_2;\, x_4x_5;\, x_9x_{10}x_{11}x_{12}x_1x_6x_7)$
x_3x_{10}	$S_{1,2,7}(x_3;\, x_2;\, x_4x_5;\, x_{10}x_{11}x_{12}x_1x_6x_7x_8)$

Thus, C has only 5-chords. Then H contains $S_{1,2,7}(x_1;\, x_{12};\, x_2x_3;\, x_7x_8x_9x_{10}x_4x_5x_6)$.

Subcase 1.6: $c(H_0) = |V(H_0)| = 13$.

If $x_1x_k \in E(H_0)$ for $3 \leq k \leq 5$, then H contains $S_{1,2,7}(x_1;\, x_2;\, x_kx_{k+1};\, x_{13}x_{12}x_{11}x_{10}x_9x_8x_7)$, and if $x_1x_6 \in E(H_0)$, then H contains $S_{1,2,7}(x_1;\, x_2;\, x_6x_5;\, x_{13}x_{12}x_{11}x_{10}x_9x_8x_7)$. Thus, the only chords in C are 5-chords. Then $x_1x_7 \in E(H_0)$ and, up to a symmetry, $x_4x_{10} \in E(H_0)$, and then H contains $S_{1,2,7}(x_1;\, x_2;\, x_{13}x_{12};\, x_7x_8x_9x_{10}x_4x_5x_6)$.

Subcase 1.7: $c(H_0) = |V(H_0)| = 14$.

If $x_1x_k \in E(H_0)$ for $3 \leq k \leq 6$, then H contains $S_{1,2,7}(x_1;\, x_2;\, x_kx_{k+1};\, x_{14}x_{13}x_{12}x_{11}x_{10}x_9x_8)$, and if $x_1x_7 \in E(H_0)$, then H contains $S_{1,2,7}(x_1;\, x_2;\, x_7x_6;\, x_{14}x_{13}x_{12}x_{11}x_{10}x_9x_8)$. Thus, the only chords in C are 6-chords, and then H contains $S_{1,2,7}(x_1;\, x_2;\, x_{14}x_{13};\, x_8x_9x_{10}x_3x_4x_5x_6)$.

Subcase 1.8: $c(H_0) = |V(H_0)| \geq 15$.

Set $c(H_0) = t$. If $x_1x_3 \in E(H_0)$, we have $S_{1,2,7}(x_1; x_2; x_3x_4; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5}x_{t-6})$ in H. Finally, if $x_1x_k \in E(H_0)$ for $4 \leq k \leq \lfloor \frac{t}{2} \rfloor + 1$, then H contains the subgraph $S_{1,2,7}(x_1; x_2; x_kx_{k-1}; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5}x_{t-6})$.

Case 2: $G \in \mathcal{B}_{2,5}$.

Then H does not contain as a subgraph the graph $S_{1,3,6}$ such that its branch of length 1 is in a nonpendant edge.

Subcase 2.1: $c(H_0) = 9$ *and* $|V(H_0)| \geq 10$.

First observe that $\langle R \rangle_H$ does not contain a path P_3 such that one of its endvertices has a neighbor on C, since if, for example, $P_3 = y_1y_2y_3 \subset \langle R \rangle_H$ is such a path with $x_1y_1 \in E(H)$, we have $S_{1,3,6}(x_1; x_9; y_1y_2y_3; x_2x_3x_4x_5x_6x_7)$ in H.

Since H is essentially 3-edge-connected, every edge in $\langle R \rangle_H$ is connected to C by at least three edges (two of them possibly being a double edge).

Subcase 2.1.1: *there is an edge* $e = y_1y_2 \in E(\langle R \rangle_H)$ *such that* $|N_C(\{y_1, y_2\})| \geq 3$. By symmetry, we assume that $y_1 \in R_0$, and either $|N_C(y_1)| \geq 3$ (with e possibly being pendant), or $|N_C(y_1)| = 2$ and $|N_C(y_2)| \geq 1$. We consider the case $|N_C(y_1)| \geq 3$, and since all our contradictions will consist in finding an $S_{1,3,6}$ with the branch of length 1 at a nonpendant edge, or in finding a cycle contradicting the choice of C, our proof remains true also in the case when $|N_C(y_1)| = 2$ and $|N_C(y_2)| \geq 1$, with only possibly reverse order of last two vertices of a branch ending at y_2 or of some branch being subdivided with y_2 in case of finding an $S_{1,3,6}$.

Thus, let $|N_C(y_1)| \geq 3$. Since C is longest, no two neighbors of y_1 are consecutive on C. Up to a symmetry, we have three possible situations: $N_C(y_1) \supset \{x_1, x_3, x_5\}$, $N_C(y_1) \supset \{x_1, x_3, x_6\}$, and $N_C(y_1) = \{x_1, x_4, x_7\}$. We consider these cases separately.

Subcase 2.1.1.1: $N_C(y_1) \supset \{x_1, x_3, x_5\}$.

If $x_2 \in N$, then the cycle $C' = x_1y_1x_3x_4x_5x_6x_7x_8x_9x_1$ dominates more edges than C, contradicting the choice of C. Hence x_2 has a neighbor $x_2' \in R$. Symmetrically, x_4 has a neighbor $x_4' \in R$, and, moreover, $x_2' \neq x_4'$, for otherwise we have $S_{1,3,6}(x_2; x_3; x_1y_1y_2; x_2'x_4x_5x_6x_7x_8)$ in H. Also, $x_2', x_4' \notin \{y_1, y_2\}$, for otherwise there is a cycle longer than C.

If $x_8y_1 \in E(H_0)$, then H contains $S_{1,3,6}(y_1; x_3; x_1x_2x_2'; x_8x_7x_6x_5x_4x_4')$, hence $x_8y_1 \notin E(H_0)$. Similarly, $x_8y_2 \notin E(H)$. Now, if there is a vertex $z \in N_R(x_8)$, H contains $S_{1,3,6}(x_1; x_2; x_9x_8z; y_1x_3x_4x_5x_6x_7)$; hence $x_8 \in N$. Since $\delta(H_0) \geq 3$, x_8 is in a chord of C. We consider all possible chords containing x_8, and for each of them we obtain an $S_{1,3,6}$ in H.

Chord at x_8	**Subgraph $S_{1,3,6}$**
x_8x_1	$S_{1,3,6}(x_8; x_9; x_1x_2x_2'; x_7x_6x_5x_4x_3y_1)$
x_8x_2	$S_{1,3,6}(y_1; x_1; x_3x_4x_4'; x_5x_6x_7x_8x_2x_2')$
x_8x_3	$S_{1,3,6}(x_8; x_9; x_3x_4x_4'; x_7x_6x_5y_1x_1x_2)$

Chord at x_8	**Subgraph $S_{1,3,6}$**
x_8x_4	$S_{1,3,6}(y_1; x_1; x_3x_2x_2'; x_5x_6x_7x_8x_4x_4')$
x_8x_5	$S_{1,3,6}(x_8; x_7; x_5x_4x_4'; x_9x_1x_2x_3y_1y_2)$
x_8x_6	$S_{1,3,6}(x_8; x_7; x_9x_1y_1; x_6x_5x_4x_3x_2x_2')$

The only remaining possibilities are that there is a double edge containing x_8. However, if x_8x_9 is a double edge, then, by symmetry, the same applies to x_7 and we have two double edges in H_0, and if x_7x_8 is a double edge, then we must have some of the above chords since otherwise $\{x_6x_7, x_8x_9\}$ is an edge-cut in H_0, a contradiction.

Subcase 2.1.1.2: $N_C(y_1) \supset \{x_1, x_3, x_6\}$.
By the choice of C, there is a vertex $x_2' \in N_R(x_2)\backslash\{y_1\}$, for otherwise the cycle $C' = x_1y_1x_3x_4x_5x_6x_7x_8x_9x_1$ dominates more edges than C. But then we have $S_{1,3,6}(x_6; y_1; x_5x_4x_3; x_7x_8x_9x_1x_2x_2')$ in H, a contradiction.

Subcase 2.1.1.3: $N_C(y_1) = \{x_1, x_4, x_7\}$.
If there is a $z \in N_R(x_2)$, H contains $S_{1,3,6}(x_4; y_1; x_3x_2z; x_5x_6x_7x_8x_9x_1)$, hence $x_2 \in N$ (note that $N_R(x_2) \cap \{y_1, y_2\} = \emptyset$ since C is a longest cycle). By symmetry, $\{x_2, x_3, x_5, x_6, x_8, x_9\} \subset N$. Since $\delta(H_0) \geq 3$, x_2 is in a chord of C, and, since the same applies to any of the vertices x_3, x_5, x_6, x_8, and x_9, by symmetry, we can assume that the chord containing x_2 is neither a 1-chord nor a double edge. Thus, by symmetry, x_2 is adjacent to x_5, x_6, or x_7.

Chord at x_2	**Contradiction**
x_2x_5	$C' = x_1y_1x_4x_3x_2x_5x_6x_7x_8x_9x_1$ longer than C
x_2x_6	$S_{1,3,6}(x_2; x_3; x_6x_5x_4; x_1x_9x_8x_7y_1y_2)$ in H
x_2x_7	$S_{1,3,6}(x_2; x_3; x_1x_9x_8; x_7x_6x_5x_4y_1y_2)$ in H

Subcase 2.1.2: For every edge $e = y_1y_2 \in E(\langle R\rangle_H)$, $|N_C(\{y_1, y_2\})| = 2$.
Let $N_C(\{y_1, y_2\}) = \{x_1, x_2\}$ with $3 \leq s \leq 8$ and $x_1y_1 \in E(H_0)$. Since H_0 is 3-edge-connected and $y_1 \in V(H_0)$, the edge e is connected to C by at least three edges.

If there is no double edge, we can choose the notation such that $x_1y_1, x_sy_1, x_1y_2 \in E(H)$. But then, if $x_2y_2 \notin E(H)$, $\langle\{x_1y_1y_2\}\rangle_H$ is a triangle in H with $d_H(y_2) = 2$, contradicting Lemma E, and if $x_2y_2 \in E(H)$, then x_1, x_s, y_1, and y_2 determine a diamond in H, contradicting Theorem G(vi). Hence, x_1y_1 is a double edge, implying that every edge in $\langle R\rangle_H$ is incident to y_1.

Now, if there is a $z \in N_R(x_3)\backslash\{y_1\}$, we have $S_{1,3,6}(x_1; y_1; x_2x_3z; x_9x_8x_7x_6x_5x_4)$ in H, and if there is a $z \in N_R(x_5)\backslash\{y_1\}$, we have $S_{1,3,6}(x_1; y_1; x_2x_3x_4; x_9x_8x_7x_6x_5z)$ in H; hence $N_R(\{x_3, x_5\}) \subset \{y_1\}$. Moreover, $x_3x_5 \notin E(H_0)$ by Theorem G(vi). Then the set $A = (V(C) \cup \{y_1\}\backslash\{x_3, x_5\}$ with $|A| = 8$ dominates all edges of H, hence $G = L(H)$ is Hamilton-connected by Theorem 4, a contradiction.

Subcase 2.1.3: $E(\langle R\rangle_H) = \emptyset$.

Choose again the notation such that $x_1y_1 \in E(H_0)$ with $y_1 \in R_0$. Note that the edge x_1y_1 is nonpendant since $y_1 \in R_0$. If there is a $z \in N_R(x_3)\setminus\{y_1\}$, we have $S_{1,3,6}(x_1; y_1; x_2x_3z; x_9x_8x_7x_6x_5x_4)$ in H; hence $N_R(x_3) \subset \{y_1\}$. Similarly, $N_R(x_5) \subset \{y_1\}$, since otherwise, for a $z \in N_R(x_5)\setminus\{y_1\}$, we have $S_{1,3,6}(x_1; y_1; x_2x_3x_4; x_9x_8x_7x_6x_5z)$ in H. Symmetrically, $N_R(x_6) \subset \{y_1\}$. Consequently, if $x_3x_5 \notin E(H_0)$, then the set $A = (V(C) \cup \{y_1\})\setminus\{x_3, x_5\}$ with $|A| = 8$ dominates all edges of H, implying that G is Hamilton-connected by Theorem 4, a contradiction. Hence $x_3x_5 \in E(H_0)$. Analogously, by Theorem 4, considering the set $A = (V(C) \cup \{y_1\})\setminus\{x_3, x_6\}$ with $|A| = 8$, we have $x_3x_6 \in E(H_0)$. But then the two chords x_3x_5 and x_3x_6 create a diamond in H_0, contradicting Theorem G(vi).

Subcase 2.2: $c(H_0) = |V(H_0)| = 10$.

Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 2.2.1: *C has a 1-chord.*

Choose the notation such that $x_1x_3 \in E(H_0)$.

If there is a $z \in N_R(x_4)$, we have $S_{1,3,6}(x_1; x_2; x_3x_4z; x_{10}x_9x_8x_7x_6x_5)$ in H, hence $x_4 \in N$. Also $x_6 \in N$, for otherwise, for $z \in N_R(x_6)$, we have $S_{1,3,6}(x_1; x_2; x_3x_4x_5; x_{10}x_9x_8x_7x_6z)$ in H. Symmetrically, $\{x_8, x_{10}\} \subset N$. Theorem 4 for $A = V(C)\setminus\{x_4, x_6\}$ with $|A| = 8$ implies $x_4x_6 \in E(H_0)$, Theorem 4 for $A = V(C)\setminus\{x_8, x_{10}\}$ implies $x_8x_{10} \in E(H_0)$, and we have three triangles in H_0, contradicting Theorem G(vi).

Subcase 2.2.2: *C has a 3-chord.*

Let $x_1x_5 \in E(H_0)$. If $z \in N_R(x_6)$, we have $S_{1,3,6}(x_1; x_5; x_2x_3x_4; x_{10}x_9x_8x_7x_6z)$ in H; hence $x_6 \in N$. Symmetrically, $x_{10} \in N$. Theorem 4 for the set $A = V(C)\setminus\{x_6, x_{10}\}$ with $|A| = 8$ implies $x_6x_{10} \in E(H_0)$, and Theorem 4 for the set $A = V(C)\setminus\{x_1, x_6\}$ implies $x_1x_6 \in E(H_0)$. The chords x_1x_5, x_6x_{10}, and x_1x_6 then determine a diamond in H_0, contradicting Theorem G(vi).

Subcase 2.2.3: *C has a 2-chord.*

Let $x_1x_4 \in E(H_0)$. Then $x_2, x_3 \in N$, since if there is a $z \in N_R(x_3)$, we have $S_{1,3,6}(x_1; x_4; x_2x_3z; x_{10}x_9x_8x_7x_6x_5)$ in H, and $x_2 \in N$ follows by symmetry. Since $\delta(H_0) \geq 3$ and by the previous subcases, x_2 is in a 2-chord or in a 4-chord of C.

If $x_2x_5 \in E(H_0)$, then, by symmetry, $x_4 \in N$, Theorem 4 for the set $A = V(C)\setminus\{x_2, x_4\}$ implies $x_2x_4 \in E(H_0)$, and we are back in Subcase 2.1.1 (since x_2x_4 is a 1-chord of C). If $x_2x_9 \in E(H_0)$, then, by symmetry, $x_1, x_{10} \in N$, and Theorem 4 for the set $A = V(C)\setminus\{x_1, x_3\}$ implies 1-chord $x_1x_3 \in E(H_0)$, a contradiction again.

Hence x_2 is in a 4-chord, that is, $x_2x_7 \in E(H_0)$. Then, for a $z \in N_R(x_6)$, we have $S_{1,3,6}(x_4; x_3; x_5x_6z; x_1x_2x_7x_8x_9x_{10})$ in H; hence $x_6 \in N$. Theorem 4 for the set $A = V(C)\setminus\{x_2, x_6\}$ then implies $x_2x_6 \in E(H_0)$, and we are back in Subcase 2.2.2.

Subcase 2.2.4: *C has only 4-chords.*

If there is a $z \in N_R(x_1)$, we have $S_{1,3,6}(x_5; x_{10}; x_6x_1z; x_4x_3x_2x_7x_8x_9)$ in H; hence $x_1 \in N$. Symmetrically, $x_3 \in N$. Theorem 4 for the set $A = V(C)\setminus\{x_1, x_3\}$ then implies the 1-chord $x_1x_3 \in E(H_0)$, and we are back in Subcase 2.2.1.

Subcase 2.3: $c(H_0) \geq 10$ and $|V(H_0)| > c(H_0)$.
Set $c(H_0) = t$. Then we have $S_{1,3,6}(x_1; y_1; x_2x_3x_4; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5})$ in H (note that $t - 5 > 4$ since $t \geq 10$, and the edge x_1y_1 is nonpendant since $y_1 \in V(H_0)$).

Subcase 2.4: $c(H_0) = |V(H_0)| = 11$. Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 2.4.1: *C has a 1-chord.*
Let $x_1x_3 \in E(H_0)$. Then H contains $S_{1,3,6}(x_1; x_2; x_3x_4x_5; x_{11}x_{10}x_9x_8x_7x_6)$.

Subcase 2.4.2: *C has a 3-chord.*
Let $x_1x_5 \in E(H_0)$. Then H contains $S_{1,3,6}(x_1; x_5; x_2x_3x_4; x_{11}x_{10}x_9x_8x_7x_6)$.

Subcase 2.4.3: *C has a 2-chord.*
Let $x_1x_4 \in E(H_0)$. If there is a $z \in N_R(x_3)$, we have $S_{1,3,6}(x_1; x_4; x_2x_3z; x_{11}x_{10}x_9x_8x_7x_6)$ in H; hence $x_3 \in N$. Similarly, if there is a $z \in N_R(x_5)$, then H contains the subgraph $S_{1,3,6}(x_1; x_2; x_4x_5z; x_{11}x_{10}x_9x_8x_7x_6)$; hence also $x_5 \in N$. Theorem 4 for the set $A = V(C)\setminus\{x_3, x_8\}$ then implies $x_3x_5 \in E(H_0)$, and we are back in Subcase 2.4.1.

Subcase 2.4.3: *C has only 4-chords.*
Since every vertex of C is in a 4-chord and $|V(C)|$ is odd, some two 4-chords have a vertex in common. Choose the notation such that $x_1x_6, x_1x_7 \in E(H_0)$. Since x_2 is in a 4-chord and the edge x_2x_7 would create a diamond, necessarily $x_2x_8 \in E(H_0)$. But then H contains $S_{1,3,6}(x_8; x_2; x_9x_{10}x_{11}; x_7x_1x_6x_5x_4x_3)$.

Subcase 2.5: $c(H_0) = |V(H_0)| = 12$.
If x_1 is in a k-chord for $1 \leq k \leq 2$, H contains $S_{1,3,6}(x_1; x_2; x_kx_{k+1}x_{k+2}; x_{12}x_{11}x_{10}x_9x_8x_7)$; if x_1 is in a k-chord for $3 \leq k \leq 4$, H contains $S_{1,3,6}(x_1; x_2; x_kx_{k-1}x_{k-2}; x_{12}x_{11}x_{10}x_9x_8x_7)$. Thus, by symmetry, every vertex of C is in a 5-chord. Then H contains the subgraph $S_{1,3,6}(x_1; x_{12}; x_7x_6x_5; x_2x_3x_4x_{10}x_9x_8)$.

Subcase 2.6: $c(H_0) = |V(H_0)| \geq 13$.
Set $c(H_0) = t$. If $x_1x_k \in E(H_0)$ for some k, $3 \leq k \leq 4$, then H contains the subgraph $S_{1,3,6}(x_1; x_2; x_kx_{k+1}x_{k+2}; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5})$, and if $x_1x_k \in E(H_0)$ for some k with $5 \leq k \leq \lfloor \frac{t}{2} \rfloor + 1$, then H contains $S_{1,3,6}(x_1; x_2; x_kx_{k-1}x_{k-2}; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4}x_{t-5})$.

Case 3: $G \in \mathcal{B}_{3,4}$. Then H does not contain as a subgraph the graph $S_{1,4,5}$ such that its branch of length 1 is in a nonpendant edge.

Subcase 3.1: $c(H_0) = 9$ and $|V(H_0)| \geq 10$.

Claim 1. The multigraph H does not contain a path P such that $\mathrm{Int}(P) \subset R$ and either

(i) $|V(P)| \geq 5$ and one of its endvertices is in $V(C)$, or
(ii) $|V(P)| \geq 4$ and both its endvertices are in $V(C)$.

Proof. (i). If $P = x_1y_1...y_k$, $k \geq 4$, is a path satisfying (i), then H contains $S_{1,4,5}(x_1; x_9; y_1y_2y_3y_4; x_2x_3x_4x_5x_6)$, a contradiction.

(ii). Let, to the contrary, $P = x_1 y_1 ... y_k x_s$, be a path satisfying (ii) for some $k \geq 2$ and $2 \leq s \leq 8$. If $s = 2$, then the cycle, obtained from C by replacing the edge $x_1 x_2$ with the path P, is longer than C, a contradiction. By symmetry, $s \in \{3, 4, 5\}$. In each of these cases we have a subgraph of H containing an $S_{1,4,5}$ with the branch of length 1 at a nonpendant edge.

Case	**Subgraph containing an $S_{1,4,5}$**
$s = 3$	$S_{1,\geq 4,5}(x_1; x_2; y_1 ... y_k x_3 x_4; x_9 x_8 x_7 x_6 x_5)$
$s = 4$	$S_{1,\geq 4,5}(x_1; x_2; y_1 ... y_k x_4 x_3; x_9 x_8 x_7 x_6 x_5)$
$s = 5$	$S_{1,4,\geq 5}(x_1; x_2; x_9 x_8 x_7 x_6; y_1 ... y_k x_5 x_4 x_3)$

□

Subcase 3.1.1: $E(\langle R \rangle_H) \neq \emptyset$.

Claim 2. Every edge in $E(\langle R \rangle_H)$ is a pendant edge of H, and one of its vertices is connected to C by at least three edges.

Proof. Let first, to the contrary, $e = y_1 y_2 \in E(\langle R \rangle_H)$ be nonpendant, and choose the notation such that $y_1 \in V(H_0)$. Since $d_H(y_1) \geq 3$, $d_H(y_2) \geq 2$, and H is essentially 3-edge-connected, e is connected to C by three edge-disjoint paths P_1, P_2, P_3, two of them, say, P_1 and P_2, starting at y_1, and P_3 starting at y_2. Let x_{i_j} be the endvertex of P_j on C, $j = 1, 2, 3$. If P_1, P_2, and P_3 can be chosen such that $|\{i_1, i_2, i_3\}| \geq 2$, then there is a path satisfying the conditions of Claim 1(ii). Hence $i_1 = i_2 = i_3$, and this vertex is a cutvertex of H, contradicting the fact that H_0 is 2-connected. Thus, e is a pendant edge of H.

By the connectivity assumption, there are three edge-disjoint paths P_1, P_2, P_3, connecting y_1 to C. Since H_0 is 2-connected, the paths P_1, P_2, P_3 can be chosen such that at least two of their endvertices are distinct. But then necessarily $\mathrm{Int}(P_i) = \emptyset$, $i = 1, 2, 3$, since otherwise we have a path satisfying the conditions of Claim 1(ii). □

Subcase 3.1.1.1: *There is an edge $e = y_1 y_2 \in E(\langle R \rangle_H)$ such that $|N_C(y_1)| \geq 3$.*
Since C is longest, no two neighbors of y_1 are consecutive on C; thus, up to a symmetry, $N_C(y_1) \supset \{x_1, x_3, x_5\}$, $N_C(y_1) \supset \{x_1, x_3, x_6\}$, or $N_C(y_1) = \{x_1, x_4, x_7\}$.

Subcase 3.1.1.1.1: $N_C(y_1) \supset \{x_1, x_3, x_5\}$. If $x_2 \in N$, then the cycle $C' = x_1 y_1 x_3 x_4 x_5 x_6 x_7 x_8 x_9 x_1$ dominates more edges than C, contradicting the choice of C. Hence there is an $x_2' \in N_R(x_2)$. We have $x_2' \neq y_1$ since C is the longest. But then H contains $S_{1,4,5}(x_5; y_1; x_4 x_3 x_2 x_2'; x_6 x_7 x_8 x_9 x_1)$.

Subcase 3.1.1.1.2: $N_C(y_1) \supset \{x_1, x_3, x_6\}$.
Then similarly there is a vertex $x_2' \in N_R(x_2) \setminus \{y_1\}$, and H contains the subgraph $S_{1,4,5}(x_6; y_1; x_7 x_8 x_9 x_1; x_5 x_4 x_3 x_2 x_2')$.

Subcase 3.1.1.1.3: $N_C(y_1) = \{x_1, x_4, x_7\}$.
If there is an $x_2' \in N_R(x_2) \setminus \{y_1\}$, we have $S_{1,4,5}(x_7; y_1; x_6 x_5 x_4 x_3; x_8 x_9 x_1 x_2 x_2')$ in H. Moreover, $N_R(x_2) \cap \{y_1, y_2\} = \emptyset$ since C is the longest. Hence $x_2 \in N$. Since $\delta(H_0) \geq 3$, there is a chord of C containing x_2. Below we consider, up to a symmetry, all possible 2-chords and 3-chords containing x_2.

Chord at x_2	Contradiction
x_2x_5	$C' = x_1y_1x_4x_3x_2x_5x_6x_7x_8x_9x_1$ longer than C
x_2x_6	$S_{1,4,5}(x_2;\ x_3;\ x_1x_9x_8x_7;\ x_6x_5x_4y_1y_2)$ in H
x_2x_7	$S_{1,4,5}(x_7;\ x_2;\ x_6x_5x_4x_3;\ x_8x_9x_1y_1y_2)$ in H

Thus, x_2 is in a 1-chord or in a double edge. However, by symmetry, the same applies to the vertices x_3, x_5, x_6, x_8, and x_9, and we have at least three triangles or double edges in H, contradicting Theorem G(vi).

Subcase 3.1.1.2: *For every edge* $e = y_1y_2 \in E(\langle R\rangle_H)$, *we have* $|N_C(y_1)| = 2$.

Then x_1y_1 is a double edge, implying that every edge in $\langle R\rangle_H$ contains y_1. If there is an $x_4' \in N_R(x_4)\setminus\{y_1\}$, then H contains $S_{1,4,5}(x_1; y_1; x_2x_3x_4x_4'; x_9x_8x_7x_6x_5)$, hence $N_R(x_4) \subset \{y_1\}$. Similarly, if there is an $x_5' \in N_R(x_5)\setminus\{y_1\}$, then H contains $S_{1,4,5}(x_1; y_1; x_9x_8x_7x_6; x_2x_3x_4x_5x_5')$, hence $N_R(x_5) \subset \{y_1\}$. By symmetry, also $N_R(\{x_6, x_7\}) \subset \{y_1\}$. Considering the set $A_1 = (V(C) \cup \{y_1\})\setminus\{x_4, x_6\}$ with $|A_1| = 8$ and the fact that G is not Hamilton-connected, Theorem 4 implies $x_4x_6 \in E(H_0)$. But then the chord x_4x_6 creates a triangle in H_0, contradicting Theorem G(vi) since x_1y_1 is a double edge.

Subcase 3.1.2: $E(\langle R\rangle_H) = \emptyset$.

Let $y_1 \in R_0$ with $x_1y_1 \in E(H_0)$ (this is always possible by Claim 1 and since H_0 is 3-edge-connected). Similarly as in Subcase 3.1.1.2, $N_R(x_4) \subset \{y_1\}$ (otherwise, for an $x_4' \in N_R(x_4)\setminus\{y_1\}$, H contains $S_{1,4,5}(x_1; y_1; x_2x_3x_4x_4'; x_9x_8x_7x_6x_5)$), and $N_R(x_5) \subset \{y_1\}$ (otherwise, for an $x_5' \in N_R(x_4)\setminus\{y_1\}$, H contains $S_{1,4,5}(x_1; y_1; x_9x_8x_7x_6; x_2x_3x_4x_5x_5')$). By symmetry, also $N_R(\{x_6, x_7\}) \subset \{y_1\}$. Considering the sets $A_1 = (V(C) \cup \{y_1\})\setminus\{x_4, x_6\}$ and $A_2 = (V(C) \cup \{y_1\})\setminus\{x_4, x_7\}$ with $|A_1| = |A_2| = 8$, Theorem 4 implies $x_4x_6 \in E(H_0)$ and $x_4x_7 \in E(H_0)$, and then the two chords x_4x_6 and x_4x_7 create a diamond in H_0, contradicting Theorem G(vi).

Subcase 3.2: $c(H_0) = |V(H_0)| = 10$.

Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 3.2.1: *C has a 1-chord.*

Choose the notation such that $x_1x_3 \in E(H_0)$. If there is a $z \in N_R(x_5)$, then H contains $S_{1,4,5}(x_1;\ x_2;\ x_3x_4x_5z;\ x_{10}x_9x_8x_7x_6)$; hence $x_5 \in N$. Symmetrically, $x_9 \in N$. If there is a $z \in N_R(x_7)$, then H contains $S_{1,4,5}(x_1;\ x_2;\ x_3x_4x_5x_6;\ x_{10}x_9x_8x_7z)$; hence also $x_7 \in N$. Theorem 4 for $A_1 = V(C)\setminus\{x_5, x_7\}$ then implies $x_5x_7 \in E(H_0)$, Theorem 4 for $A_2 = V(C)\setminus\{x_7, x_9\}$ implies $x_7x_9 \in E(H_0)$, and the three 1-chords x_1x_3, x_5x_7, and x_7x_9 determine three triangles in H_0, contradicting Theorem G(vi).

Subcase 3.2.2: *C has a 3-chord.*

Let $x_1x_5 \in E(H_0)$. If there is a $z \in N_R(x_4)$, we have $S_{1,4,5}(x_1;\ x_5;\ x_2x_3x_4z;\ x_{10}x_9x_8x_7x_6)$ in H; hence $x_4 \in N$. Symmetrically, $x_2 \in N$. From Theorem 4 for the set $A = V(C)\setminus\{x_2, x_4\}$ we then have $x_2x_4 \in E(H_0)$; however, x_2x_4 is a 1-chord of C, and we are back in Subcase 3.2.1.

Subcase 3.2.3: *C has a 4-chord.*
Let $x_1x_6 \in E(H_0)$. Then $x_7 \in N$, since otherwise, for a $z \in N_R(x_7)$, H contains $S_{1,4,5}(x_1; x_6; x_2x_3x_4x_5; x_{10}x_9x_8x_7z)$. Symmetrically, $x_5 \in N$. Theorem 4 for the set $A = V(C)\setminus\{x_5, x_7\}$ then yields $x_5x_7 \in E(H_0)$, and we are again back in Subcase 3.2.1.

Subcase 3.2.4: *Every chord in C is a 2-chord.*
Let $x_1x_4 \in E(H_0)$. Since x_2 is in a 2-chord, we have $x_2x_9 \in E(H_0)$ or $x_2x_5 \in E(H_0)$.
Let first $x_2x_9 \in E(H_0)$. Then $x_{10} \in N$, since otherwise, for a $z \in N_R(x_{10})$, H contains $S_{1,4,5}(x_4; x_3; x_5x_6x_7x_8; x_1x_2x_9x_{10}z)$. Symmetrically, $x_5 \in N$. Theorem 4 for the set $A = V(C)\setminus\{x_5, x_{10}\}$ then implies $x_5x_{10} \in E(H_0)$, and we are back in Subcase 3.2.3. Thus, $x_2x_5 \in E(H_0)$. Since x_3 is in a 2-chord, we have, up to a symmetry, $x_3x_6 \in E(H_0)$. But then we are in a situation symmetric to the first case.

Subcase 3.3: $c(H_0) \geq 10$ *and* $|V(H_0)| > c(H_0)$.
Set $c(H_0) = t$. Then H contains the subgraph $S_{1,4,5}(x_1; y_1; x_2x_3x_4x_5; x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4})$, a contradiction.

Subcase 3.4: $c(H_0) = |V(H_0)| = 11$.
Since $\delta(H_0) \geq 3$, every vertex of C is in a chord.

Subcase 3.4.1: *C has a 1-chord.*
Let $x_1x_3 \in E(H_0)$. Then H contains $S_{1,4,5}(x_1; x_2; x_3x_4x_5x_6; x_{11}x_{10}x_9x_8x_7)$.

Subcase 3.4.2: *C has a 4-chord.*
Let $x_1x_6 \in E(H_0)$. Then H contains $S_{1,4,5}(x_1; x_6; x_2x_3x_4x_5; x_{11}x_{10}x_9x_8x_7)$.

Subcase 3.4.3: *C has a 3-chord.* Let $x_1x_5 \in E(H_0)$.
By the previous subcases, x_3 is in a 2-chord or in a 3-chord. Thus, up to a symmetry, $x_3x_6 \in E(H_0)$ or $x_3x_7 \in E(H_0)$. However, if $x_3x_6 \in E(H_0)$, H contains $S_{1,4,5}(x_1; x_2; x_{11}x_{10}x_9x_8; x_5x_4x_3x_6x_7)$, and if $x_3x_7 \in E(H_0)$, H contains $S_{1,4,5}(x_1; x_2; x_{11}x_{10}x_9x_8; x_5x_4x_3x_7x_6)$.

Subcase 3.4.4: *Every chord in C is a 2-chord.*
Let $x_1x_4 \in E(H_0)$. Then x_2 is in a 2-chord, that is, $x_2x_{10} \in E(H_0)$ or $x_2x_5 \in E(H_0)$. If $x_2x_{10} \in E(H_0)$, H contains $S_{1,4,5}(x_4; x_3; x_1x_2x_{10}x_{11}; x_5x_6x_7x_8x_9)$. Hence $x_2x_5 \in E(H_0)$, and then, for any 2-chord containing x_3 we are in a situation symmetric to the first case.

Subcase 3.5: $c(H_0) = |V(H_0)| = 12$. We show that C does not have a k-chord for $k \in \{1, 2, 4, 5\}$.

Chord in *C*	**Subgraph $S_{1,4,5}$**
1-Chord x_1x_3	$S_{1,4,5}(x_1; x_2; x_3x_4x_5x_6; x_{12}x_{11}x_{10}x_9x_8)$
2-Chord x_1x_4	$S_{1,4,5}(x_1; x_2; x_4x_5x_6x_7; x_{12}x_{11}x_{10}x_9x_8)$

Chord in C	Subgraph $S_{1,4,5}$
4-Chord x_1x_6	$S_{1,4,5}(x_1;\ x_6;\ x_2x_3x_4x_5;\ x_{12}x_{11}x_{10}x_9x_8)$
5-Chord x_1x_7	$S_{1,4,5}(x_1;\ x_7;\ x_2x_3x_4x_5;\ x_{12}x_{11}x_{10}x_9x_8)$

Hence any chord in C is a 3-chord. Let $x_1x_5 \in E(H_0)$ be a 3-chord. Up to a symmetry, $x_3x_7 \in E(H_0)$, and then H contains $S_{1,4,5}(x_1;\ x_2;\ x_5x_4x_3x_7;\ x_{12}x_{11}x_{10}x_9x_8)$.

Subcase 3.6: $c(H_0) = |V(H_0)| \geq 13$.
Set $c(H_0) = t$. If $x_1x_k \in E(H_0)$ for some k, $3 \leq k \leq 5$, then H contains the subgraph $S_{1,4,5}(x_1;\ x_2;\ x_kx_{k+1}x_{k+2}x_{k+3};\ x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4})$, and if $x_1x_k \in E(H_0)$ for some k with $5 \leq k \leq \lfloor \frac{t}{2} \rfloor + 1$, then H contains $S_{1,4,5}(x_1;\ x_2;\ x_kx_{k-1}x_{k-2}x_{k-3};\ x_tx_{t-1}x_{t-2}x_{t-3}x_{t-4})$.

6 | CONCLUDING REMARKS

1. Theorem 1 admits a slight extension as follows. For $s \geq 0$, a graph G is s-*Hamilton-connected* if the graph $G - M$ is Hamilton-connected for any set $M \subset V(G)$ with $|M| \leq s$. Obviously, an s-Hamilton-connected graph must be $(s+3)$-connected. Since an induced subgraph of a $\{K_{1,3}, B_{i,j}\}$-free graph is also $\{K_{1,3}, B_{i,j}\}$-free, we immediately have the following fact, showing that, in $\{K_{1,3}, B_{i,j}\}$-free graphs with $i + j \leq 7$, the obvious necessary condition is also sufficient.

Corollary 5. *Let s, i, j be integers such that $s \geq 0$, $i, j \geq 1$ and $i + j \leq 7$, and let G be a $\{K_{1,3}, B_{i,j}\}$-free graph. Then G is s-Hamilton-connected if and only if G is $(s+3)$-connected.*

2. We can now update the discussion of potential pairs X, Y of connected graphs that might imply Hamilton-connectedness of a 3-connected $\{X, Y\}$-free graph, as summarized in [14].

As shown in [6], up to a symmetry, necessarily $X = K_{1,3}$, and, summarizing the discussions from [3, 6, 7, 14], there are the following possibilities for Y (see Figure 1 for the graphs Z_i, $B_{i,j}$, and $N_{i,j,k}$, and Figure 2A for the graph Γ_i):

(i) $Y \in \{\Gamma_1, \Gamma_3\}$, or $Y = \Gamma_5$ for $n = |V(G)| \geq 21$,
(ii) $Y = P_i$ with $4 \leq i \leq 9$,
(iii) $Y = Z_i$ with $i \leq 6$, or $Y = Z_7$ for $n = |V(G)| \geq 21$,
(iv) $Y = B_{i,j}$ with $i + j \leq 7$,
(v) $Y = N_{i,j,k}$ with $i + j + k \leq 7$.

The best-known results in the direction of each of these subgraphs are summarized in Theorem A, and we summarize the current status of the problem in the following table.

Y	Possible	Best known	Reference	Open
Γ_i	Γ_1, Γ_3, Γ_5 for $n \geq 21$	Γ_1	[6]	Γ_3; Γ_5 for $n \geq 21$
P_i	$4 \leq i \leq 9$	P_9	[3]	–
Z_i	$i \leq 6$; Z_7 for $n \geq 21$	Z_6; Z_7 for $G \not\simeq L(W^1)$	[20]	–
$B_{i,j}$	$i + j \leq 7$	$i + j \leq 7$	This paper	–
$N_{i,j,k}$	$i + j + k \leq 7$	$i + j + k \leq 7$	[13–15]	–

Thus, the only remaining cases are the Γ_3 and the Γ_5 for $n \geq 21$. The problem here is that although we are able to construct a closure operation that turns a $\{K_{1,3}, \Gamma_i\}$-free graph into the line graph of a multigraph and preserves both Hamilton-connectedness and the property of being Γ_i-free, the structure still remains too complicated to be reasonably handled.

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