Real-world relationships as binary relations

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Abstract

This contribution deals with real-world relationships as examples of binary relations. It studies the properties of binary relations such as reflexivity, symmetry, transitivity, … These binary relations are defined with the usage of real-life surroundings (like family, games, public transport, etc.). First, this work will explain the basic theoretical concepts, which are needed to examination of properties of binary relations. And in the second part it will applies the theoretical concepts to specific examples from real-world such as "be a sibling" or "rock paper scissors".

Keywords: Real-world, relationships, binary relations

Introduction

Real-world relationships as binary relations – as the name says, in this contribution, we will above all deal with different real-world relationships that surround us every day. We will explore these relationships with the mathematical properties of binary relations. The simple definition of binary relation is the concrete concept investigation of a specific set.

The intention of this article was to create an original collection of examples for students of mathematical studies, in addition, at the same time support what is a relatively small amount of literature in the Czech language concerning binary relations. The second goal of this paper was to raise the fact that mathematics surrounds us ubiquitously.

The paper is divided into three parts. In the first part, we will focus on the basic mathematical concepts we need to examine for the properties of binary relations. For example, the term set, Cartesian product of binary relations. In the second part, we will directly examine specific examples of binary relations from the real world and investigate their properties. In the end, we will mention a few other thematic tasks.

Theoretical part

Now let's define the basic concepts we need to define binary relations.

Sets:

By set we mean a set of elements. The set as such can contain a different number of objects (elements of the set), which can be both finite (for example letters in the alphabet) and infinite (set of all real numbers).

Arranged pair:

Definition 1.2.1.: Thus, by an ordered pair of elements we mean the set (a, b), which we define by the relation (a, b) = {{a}, {a, b}}, where the element *a* denotes the first component and the element *b* the second folder. Subsequently, the set {a, b} indicates the components of an

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ordered pair (a, b), the set {a} then indicates which of these components we understand as the first (Chrvát**Σφάλμα! Το αρχείο προέλευσης της αναφοράς δεν βρέθηκε.**, 2009).

Theorem 1.2.2.: Suppose a pair of ordered elements (a, b) and (c, d), then:

1) $a \neq b \Rightarrow (a, b) \neq (b, a)$

2) $(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d$

The first point states that if the elements *a*, *b* are different, then their ordered pairs (a, b) and (b, a) are also different. The second point then describes that if two ordered pairs are equal, then their components (both the first and second) are equal (Vrábík, 2014).

In other words, we can create two different numbers from the numbers 5 and 6, depending on the order of the selected numbers (56 or 65). So, we can write

(5, 6) = {{5}, {5, 6}} = {{5}, {6, 5},

(6, 5) = {{6}, {6, 5}} = {{6}, {5, 6}}, and at the same time express equality

{5, 6} = {6, 5}. According to this chapter, we already know that we use curly braces {} to write a set, while we use round brackets () to write ordered pairs.

Cartesian product:

Definition 1.3.1.: If we have two arbitrary sets A, B, we can create from them A × B - cartesian product of sets A, B, which is defined by the relation:

$A \times B = \{(a, b): a \in A \wedge b \in B\}$ (Balcar & Štěpánek, 1986).

We then read this notation that the Cartesian product of the sets A, B is the set of all ordered pairs (a, b), for which it holds that *a* is an element of the set A, *b* is an element of the set B.

Example 1.3.2.: Let be set E = {1, 3, 5, 8} and F = {a, b}. determine the cartesian product E × F.

Solution: E × F = {(1, a), (1, b), (3, a), (3, b), (5, a), (5, b), (8, a), (8, b)}.

Let us now turn for a moment to the properties of the Cartesian product.

i) $A \times (B_1 \cup B_2) = (A \times B_1) \cup (A \times B_2)$ - distributability of the cartesian product with respect to unification

ii) $A \times (B_1 \cap B2) = (A \times B_1) \cap (A \times B_2)$ - distributability of the cartesian product with respect to intersection

iii) $A \times (B_1 - B_2) = (A \times B_1) - (A \times B_2)$ - distributability of the cartesian product due to the difference

iv) $A \subseteq A_1 \wedge B \subseteq B1 \Rightarrow A \times B \subseteq A_1 \times B_1$ - if A is a subset of the set A_1 and B is a subset of the set B_1, then their cartesian product A × B is also *a* subset of the cartesian product $A_1 \times B_1$ (Pisklák, 2004).

Relations, Binary relations:

Definition 1.4.1.: We say that by the relation R on the set M we mean any subset of the cartesian product M × M (Drábek & Honzík, 2011).

Definitions 1.4.2.: A binary relation from the set M to the set N is a set that is a subset of the cartesian product M × N. We write $R \subseteq (M \times N)$.

We denote the relation as $(x, y) \in R$ or $x \rho y$. In other words, binary relations are basically concerned with expressing the relationships between two elements, or between two sets, respectively (Drábek & Honzík, 2011).

Example 1.4.3.: The set P = {1, 3, 4, 8, 9, 10, 16} is given, where we have two-digit predicates:

T = (x, y): y = x + 1. → relation defined by the term "be one bigger".

U = (x, y): y = x^2. → i.e., "be square".

V = (x, y): y = x <y. → i.e., "be smaller".

List the resulting relations according to these predicates.

Solution: As a solution to these predicates, we finally get a solution in the form:
T = {[3, 4]; [8, 9]; [9, 10]}
U = {[1, 1]; [3, 9]; [4, 16]}
V = {[1.3]; [1, 4]; [1, 8]; [1, 9]; [1, 10]; [1, 16]; [3, 4]; [3, 8]; [3, 9], [3, 10]; [3, 16]; [4, 8]; [4, 9]; [4, 10]; [4, 16]; [8, 9]; [8, 10]; [8, 16]; [9, 10]; [9, 16]; [10, 16]}.

Properties of binary relations:

Definition reflexivity 1.5.1.: The binary relation R on the set M is reflexive just when for all elements of the set M they are in relation to themselves. We write $(\forall x \in M)$: $(x \rho x)$ (Vrábík, 2014).

Definitions areflexivity 1.5.2.: The binary relation R on the set M is areflexive if there exists at least one *x* for which the relation between *x* and *x* does not belong to the relation R. We write $(\exists x \in M)$: $[x, x] \notin R$ (Drábek & Honzík, 2004).

Definition of antireflexivity 1.5.3.: The binary relation R on the set M is antireflexive just when for all *x* it holds that the relation between *x* and *x* does not belong to the relation R. We write $(\forall x \in M)$: $[x, x] \notin R$ (Bělík, 2005).

Definition of symmetricity 1.5.4.: The binary relation R on the set M is symmetric if the element *x* is in relation to the element *y*, and vice versa the element *y* is in relation to the element *x*. Thus $(\forall x, y \in M)$: $[(x \rho y) \Rightarrow (y \rho x)]$ (Vrábík, 2014).

Definition of antisymerity 1.5.5.: The binary relation R on the set M is antisymmetric, just when *x* is in relation to *y* and at the same time *y* is in relation to *x* only if *x* and *y* are equal. We write $(\forall x, y \in M)$: $[(x \rho y) \land (y \rho x) \Rightarrow (x = y)]$ (Houb, 2021).

Definition of transitivity 1.5.6 .: The binary relation R on the set M is transitive, just when for three elements *x*, *y* and *z* it holds that if *x* is in relation to *y* and *y* then in relation to *z*, then it must naturally hold that *x* is also in relation we write $(\forall x, y, z \in M)$: $(x \rho y \land y \rho z \Rightarrow x \rho z)$ (Bělík, 2005).

Example 1.5.7.: Consider a given set of lines in a plane and the relation D = (x, y): x | | y. Hence the relation "to be parallel". Specify properties of binary relations.

Solution: **Reflexivity:** $(x \rho x)$ - we are interested in whether the line *x* is parallel to itself. This property holds, and therefore the session is reflexive.

Note: if we know that the session is reflexive, then we do not need to examine are areflexivity and antireflexivity, as these conditions can only occur if the session is not reflexive.

Symmetry: $[(x \rho y) \Rightarrow (y \rho y)]$ - so if we know that *x* is parallel to *y*, then *y* should also be parallel to *x*. this situation is true for parallels, and therefore the relation is symmetric.

Note: if we know that the relation is symmetric, we do not have to examine the antisymmetry property, as this can only occur if the relation is not symmetric.

Transitivity: $[(x \rho y) \land (y \rho z) \Rightarrow (x \rho z)]$ - we have the assumption that *x* is parallel to *y* and at the same time *y* is parallel to *z*. We are therefore interested in whether *x* is parallel to *z*. This the assumption in the plane also applies and the relation is therefore also transitive.

Definition of equivalence 1.5.8.: as a relation of equivalence on the set M we call a relation which is at the same time reflexive, symmetric and transitive.

So, about relation in example 1.5.7. we can say that it is an equivalence relation (Vrábík, 2014).

Practical part:

In this section we will try to explain the principle of solving the properties of binary relations on specific examples.

Being a sibling (with the same parents)

For us, this relation is defined by the fact that the elements (individuals) x, y and z have the same parents A, B (see Figure 1). We will now examine the properties:

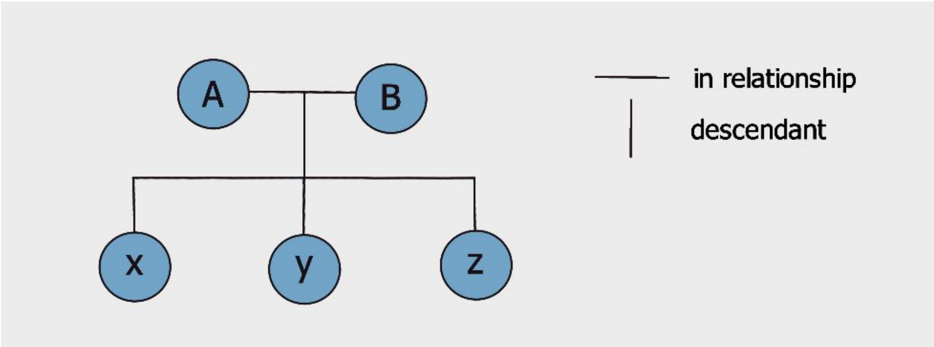

Figure 1. Family tree (source: own)

Reflexivity: (x ρ x) - in practice for us this property means "to be our own sibling". Of course, this is not the case, so the relation is not symmetric. At the same time, we can say that this assumption never applies, and therefore the relation is anti-reflective.

Symmetry: [(x ρ y) ⇒ (y ρ x)] - if individual x is a sibling of y, then then individual y is a sibling of x. This assumption holds true, which also leads us to the fact that the session is reflexive.

Transitivity: [(x ρ y) ∧ (y ρ z) ⇒ (x ρ z)] - So if x is a sibling of y and at the same time y is a sibling of z, then then it must hold that x is a sibling of z. therefore, we can say that this relation is also transitive.

So, in this case we got a relation that is antireflexive, symmetric, and transitive.

The relationship with “being a half-sibling” or “being a parent” could be solved in the same way.

Rock scissors Paper

The first game that demonstrates a binary relation is the old classic game "Rock, scissors, paper". It is a game for two or more players. The history of this game can be traced back to the 19th century in Japan. This game works on the principle of elimination. Players will show one of the symbols in sync with the countdown (stone - clenched fist, scissors - close hand with two raised fingers (index finger, middle finger), paper - open palm (see Figure 2). The rules are as follows: the stone reloads the scissors (blunts them), the scissors reload the paper (cuts it), and the paper reloads the stone (wraps it). One round beat, so the whole round is canceled, and all players remain playing (Wikipedia, 2009**Σφάλμα! Το αρχείο προέλευσης**

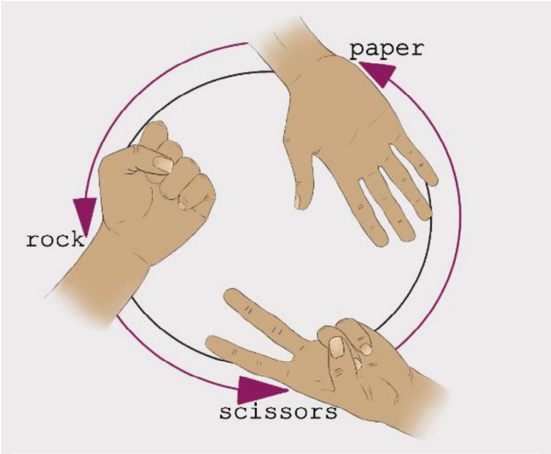

Figure 2. Rock, scissors, paper (source: own)

της αναφοράς δεν βρέθηκε.).

Now let's see what properties this binary relation has (x overrides y).

Reflexivity: (x ρ x) - so *x* should overcharge itself. This does not apply according to the rules, because if the symbols are the same, the wheel is canceled. It follows that the session is not reflexive, but even antireflexive.

Symmetry: [(x ρ y) ⇒ (y ρ x)] - if the element *x* beats the element *y*, then the element *y* beats the element *x*. Since it is a one-way circle, the property also does not apply - for example, the stone is overwhelmed by scissors, but the scissors do not overwhelm the stone.

Antysymetry: [(x ρ y) ∧ (y ρ x) ⇒ (x = y)] - that is, the relation is symmetric only when the elements x and y are equal. But right part of this implication cannot happen. From the logic functions we know, that 0 (cannot happen) implicates 1 (true), which in the and means that this relation is antisymetric.

Transitivity: [(x ρ y) ∧ (y ρ z) ⇒ (x ρ z)] - if *x* overloads y and at the same time *y* overloads *z*, then it follows that *x* overloads *z*. As we mentioned, it is one-way circle, and therefore this property is also invalid. For example, paper overwhelms stone and stone overwhelms scissors, but paper does not overwhelm scissors.

As a result, the "stone, scissors, paper" relation is a relation that is antireflective, antisymmetric, and not transitive.

Pony

Pony is a well-known card game that originated in the post-war years. Its original name "Mau-mau" comes from the wing of the liberation movement in Kenya. This game is most often played with original Pony cards (32 cards - 7, 8, 9, 10, bottom, top, king, ace; colors: heart, green, acorns and balls (see Figure 3)), and the main task for players (usually 2 to 6) is to get rid of their cards as soon as possible (Wikipedia, 2019**Σφάλμα! Το αρχείο προέλευσης της αναφοράς δεν βρέθηκε.**).

Figure 3. Pony cards (source: own)

The rules of the game are as follows: one player deals 4 cards to everyone and then places one in the middle of the table, places the rest of the deck on the table to draw cards (the player must take a card from the deck if he cannot or does not want to discard any card he holds in hand). Each player can play cards so that on some number, he can only play the same number (different types of colors) or any number (same colors). For example, if there is a green 9 on the table, the next player can play any nine on this card (acorn, heart, or ball), or any green card (7, 8, 10, bottom, top, king, ace) (Wikipedia, 2019).

However, these rules do not apply at all to the so-called special cards (seventh, top, ace (see Figure 4)), which have special effects: seventh - if the seventh is played, the next player must take two cards from the deck. Or if the players agree to "recharge" each other, if seven falls and the next player has another seven in his hand, he can "overcharge" the first one, and thus the next player takes the cards from the deck, this time four. Top - can be played at any time and on any card (except sevens and aces) and the player who used the upper determines in which color the next player will continue. For example, player A plays the top and opts for acorns, which means that the next player must play any acorn card or use another top to change colors again. Ace - like seven, if a player plays an ace, the next player does not play. The exception is again "overcharging" when, if an ace is played and the next player owns it between their cards, the first player can "overcharge" and thus "stand". (Wikipedia, 2019)

Figure 4. Special Pony cards (source: own)

Now let's try to define the properties of the Raining session, in this case the session defined by the term "can play on" (we can overcharge in the game):

Reflexivity: (x ρ x) - so I can play the *x* card again on the *x* card. this is generally the case, and in the case of overcharging and the session is therefore reflexive. However, it is necessary to mention here that each card is located exactly once in each deck, so this situation cannot occur in practice (when playing with one set of cards).

Symmetry: [(x ρ y) ⇒ (y ρ x)] - which describes a relation such that if we can play the *x* card with the *y* card, then we can play the *y* card with the *x* card. Here we come to a problem with special cards. Without special cards, symmetry would work in all cases, but if, for example, card *x* was red 8 and card *y* was red 7, then we get into a dispute, because on heart 7 the next player cannot immediately play heart 8.

Note: the relation could theoretically be symmetric provided that recharging is not allowed. It would be that the first player would play the heart 7, the second player would have to take two cards from the deck, so he would not play (he would not put any more card into the game). The third player could therefore play the given heart 8 as the next card - which would lead to the fact that the heart 8 would be played on the heart seven. The same would apply in the case of overcharging, but provided that the second player had no seven, which he could beat (then he would have to draw, and the situation would be similar).

With this note we can say the relation is symmetric.

Transitivity: [(x ρ y) ∧ (y ρ z) ⇒ (x ρ z)] - this assumption can be interpreted as meaning that we can play the *x* card on the *y* card, and then we can play the *y* card on the *z* card. And this implicates that *x* card can by played on *z* card. This implication we can refute it in several ways. For example, if we play x = ball 10 and y = acorn 10, then z = acorn 8. In this case, we can't play ball 10 on acorn 8 (this contradicts the rules of the game). Another example is the use of special cards, for example for x imagine green 9 and for y green ace, then z heart ace.

Here again we know that we can't play green 9 on heart ace.

As a result, we get here that this relation is reflexive, symmetric, but it is not transitive.

Public transport

We can see an interactive map of public transport in Pilsen with using link below or QR code: https://jizdnirady.pmdp.cz/provoz.

SCAN ME

Figure 5. QR code (map of transport in Pilsen)

Under the term public transport, we therefore imagine a system of public passenger transport lines. In the Pilsen region, there would be 3 tram lines, 9 trolleybus lines, 27 bus lines and 9-night lines. This system then uses exactly 312 stops spread throughout the city and in its suburban areas.

A relation defined by the term "able to transfer from line x to line y". Let us now determine the properties of this session.

Reflexivity: (x ρ x) - so we are interested if we go some line *x*, then we can change to the same line *x* at some stop. Of course, as soon as we get off at a stop on line *x*, we can definitely get on the same line again at the same place (either if we got out of the car just to let the other passengers out, or if we want to wait for the next flight). For example, if we take line 12 and get off at the Mikulášská stop. We can take line 12 again at the same stop. The session is therefore reflexive.

Symmetry: [(x ρ y) ⇒ (y ρ x)] - in our case, the principle is that if we can change from line *x* to line *y*, then we can also change from line *y* to line *x* at the same place. So as soon as line *x* stops at a stop where another line *y* stops, we can switch between them at will. There are 183 such stops where more than one line stops. So, for example, if we took tram line 4 and got off at the Pod Záhorskem stop, we could change to tram line 1 and we could also change from line 1 to line 4 at the Pod Záhorském stop. The relation is therefore symmetric.

Transitivity: [(x ρ y) ∧ (y ρ z) ⇒ (x ρ z)] - we are interested in whether we can change from line *x* to line *y* and from line *y* to line *z*, if we can also switch from line *x* to line *z*. We refute this assumption if, for example, we choose bus line 32 for *x*, trolleybus line 16 for *y* and tram line 2 for *z*. We can change from line 32 to line 16 at two stops (U Luny and U Teplárny). From line 16 we can then change to tram 2 at the Hlavní nádraží stop, but from line 32 we can never change directly to tram 2. The session is therefore not transitive.

Note This relation could be transitive if, for example, the predicate was "able to transfer from line x to line y at the same stop". In that case, all three elements would have to stop at the same stop and the session would be transitive. In this case, the relation would also be an equivalence relation. There are 123 such stops (including night lines) in Pilsen, where more than three lines stop.

The relation is therefore reflexive, symmetric but not transitive.

Conclusion

In this work, we focused mainly on defining the term binary relation and its subsequent application to examples formed by concepts from the world around us. However, we can define many terms as binary relations. Examples: owning a newer car model, making more money, being a teacher, being a neighbor, being a senior, winning, and many more.

In this work, we mention at least some of nothing and show how to work with them as mathematical concepts.

The main reason why was this article made is (with its non-ordinary examples) to expand the small amount of existing literature on the topic of binary relations, and especially on the topic of binary relations in real world. Also, to expand people awareness of this interesting topic.

Thanks to the whole article, even a simple layman can finally verify that mathematics blends into the daily lives of each of us.

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