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DEPARTMENT OF MATHEMATICS

MASTER'S THESIS

Applications of the Lovász local lemma and related methods

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Prohlašuji, že jsem svou diplomovou práci vypracovala samostatně, za použití pouze podkladů uvedených v přiloženém seznamu literatury.

V Plzni, dne ....

Karolína Hylasová

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Abstrakt:

V této práci se zabýváme aplikacemi Lovászova lokálního lemmatu a s ním souvisejících metod. Popíšeme postupný vývoj těchto metod a ukážeme konkrétní příklady jejich užití na příkladech z oblasti výzkumu nezávislých transverzál a hypergrafů.

Klíčová slova: Lovászovo lokální lemma, komprese entropie, hypergraf, nezávislá transverzála

Abstract:

In this thesis we investigate applications of the Lovász local lemma and its related methods. We are going to describe the gradual development of these methods and show the specific examples of its use in the field of research on independent transversals and hypergraphs.

Keywords: Lovász local lemma, entropy compression, hypergraph, independendet transversal

Contents

Introduction

The Lovász local lemma is effective proof technique frequently used to prove the existence of some object without showing its properties. Throughout the years many modifications of the original formulation were presented.

The goal of this thesis is to provide brief introduction to use of the Lovász local lemma and its related methods and demonstrate their functioning on certain examples.

Chapter 2 is dedicated to independent transversals. We examine results so far obtained in this field, focusing mainly on the sufficient conditions of the existence of an independent transversal in a graph. Also we present the algorithm for finding such independent transversal in a graph and applying the entropy compression we show that this algorithm terminates after a polynomial number of steps.

Chapter 3 is focused on generalization of graphs - hypergraphs. We define basic terms required to understand the basis of the hypergraph theory and show that although the Lovász local lemma and its related methods are very effective and powerful tools, sometimes we can get better results without its usage.

Chapter 1

Lovász local lemma and related methods

The Lovász local lemma was introduced by Erdős and Lovász who used it in their article about hypergraph coloring [12]. Since then the Lovász local lemma has become one of the most powerful tools in the probabilistic methods and is used to prove results in graph or hypergraph coloring, satisfiability problems, etc. The Lovász local lemma is a non-constructive method of proving the existence of a specific object without showing how it looks like. In 2009 Moser and Tardos [28] came with a breakthrough and using the entropy compression proved an algorithmic version of the Lovász local lemma which gave us a surprisingly simple randomized algorithm.

Recently Rosenfeld [31] came up with a new proof technique that can be applied to the same problems as the Lovász local lemma or the entropy compression. The Rosenfeld method gives more simple proof bounds similar as the entropy compression, but we lose the constructive feature of the entropy compression by its application.

The goal of the first chapter is to provide an overview of the Lovász local lemma and related methods and to show some applications on specific examples for an easier understanding of how effective and simple these proof techniques are.

1.1 Preliminaries

Firstly, we will state some basic definitions from the probability theory which will be required for the formulations of the Lovász local lemma in following sections.

A *probability space* is a triple (Ω, Σ, P), where Ω is a sample space (the set of all possible outcomes), Σ is a *σ-algebra* on Ω (a nonempty collection of subsets of Ω closed under complement and countable unions) and P is a probability on Σ. An element of Σ, i.e. a subset of a sample space to which the probability is assigned, is called *event*. A *random variable* is a mapping of possible outcomes from Ω to a measurable space.

Two events are *independent* if the occurrence of one event does not affect the probability of the other event. A set of events is *mutually independent* if the probability of each event stays the same no matter which of the other events occur. We say that an event $A \in \mathcal{A}$ is mutually independent from all the other events except for at most k of them, i.e. except for events $\{B_1, \ldots, B_k\} \in \mathcal{A}$, when all events in the set $A \backslash \{B_1, \ldots, B_k\}$ are mutually independent.

Let $\mathcal{P}$ be a finite set of random variables in a probability space Ω and let $\mathcal{A}$ be a finite set of events in a probability space Ω that are determined by the values of some subset $S \subseteq \mathcal{P}$ of random variables, i.e. if we know the values of S we can tell whether the events occur or not. Assign an evaluation v_P to every random variable $P \in \mathcal{P}$. If for any evaluation v_P of random variables from the subset $S \subseteq \mathcal{P}$ the corresponding event $A \in \mathcal{A}$ occurs, we say that event A is *violated.*

1.2 Lovász local lemma

In this section, we briefly summarize the gradual development of the symmetric Lovász local lemma, show its general form and how the symmetric one easily follows from it, and in the final subsection we will show some applications of symmetric version of the Lovász local lemma to known problems from satifiability and hypergraph coloring.

The Lovász local lemma states that if we have a set of events in which each of them occurs with probability $p \in (0{,}1)$ and is mutually independent of the others with the exception of a few, then there is a nonzero probability that none of the events will occur. The Lovász local lemma is used to prove an existence of a certain object by showing that none of the "bad events" (events that would prevent its existence) will occur.

1.2.1 Symmetric Lovász local lemma

There exist a few different formulations of so called *symmetric Lovász local lemma* which is the simplest but most used version of LLL and is applied to symmetric events - the events that are invariant under some permutation of the underlying probability space. The symmetric Lovász local lemma was first introduced by Erdős and Lovász [12], the motivation for their formulation was the proof of 2-colorability for hypergraphs (we will focus on this application of the symmetric Lovász local lemma in subsection 1.2.3).

Theorem 1.1 [12]. *Let $\mathcal{A}$ be a finite set of events such that each event occurs with probability at most p and is mutually independent from all the other events except for at most d of them. If*

$$4pd \leq 1,$$

then

$$\mathbb{P}\left(\bigwedge_{A\in\mathcal{A}} \overline{A}\right) > 0.$$

Under the term "Symmetric Lovász local lemma" is nowadays known the following version which was published by Spencer [33] to derive lower bounds for Ramsey functions.

Theorem 1.2 (**Symmetric Lovász local lemma**) [33]. *Let $\mathcal{A}$ be a finite set of events such that each event occurs with probability at most p and is mutually independent from all the other events except for at most d of them. If*

$$ep(d+1) \leq 1,$$

where $e = 2{,}718\ldots$ is Euler's number, then

$$\mathbb{P}\left(\bigwedge_{A\in\mathcal{A}} \overline{A}\right) > 0.$$

More general conditions under which the Symmetric Lovász local lemma holds were given by Shearer [32]. His approach shows how large the probability p of each event A can be, so that the final conclusion $\mathbb{P}(\bigwedge_{A\in\mathcal{A}} \overline{A}) > 0$ is still guaranteed. But despite the generality of Shearer's result, the Symmetric Lovász local lemma provides simpler and more practical conditions.

Theorem 1.3 (**Shearer's Lemma**)[32]. *Let $\mathcal{A}$ be a finite set of events such that each event occurs with probability at most p and is mutually independent from all the other events except for at most d of them. If*

$$\begin{cases} p < \frac{1}{2} & \text{for } d = 1, \\ p < \frac{(d-1)^{d-1}}{d^d} & \text{for } d \geq 2, \end{cases}$$

then

$$\mathbb{P}\left(\bigwedge_{A\in\mathcal{A}} \overline{A}\right) > 0.$$

1.2.2 Asymmetric Lovász local lemma

To formulate the stronger general version of Lovász local lemma (the "asymmetric" version) we will first need to define a structure which describes a dependency of events in a probability space. Let $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ be a finite set of events in a probability space. A *dependency graph* $G_{\mathcal{A}}$ for $\mathcal{A}$ is a graph with its vertex set formed by all events from $\mathcal{A}$ and in which the event $A \in \mathcal{A}$ is mutually independent from all but its neighbors. The neighborhood of an event $A \in \mathcal{A}$, i.e. other events which may depend on A, is denoted by $\Gamma_{\mathcal{A}}(A)$.

Theorem 1.4 (**Asymmetric Lovász local lemma**) [12]. *Let* $\mathcal{A} = \{A_1, \dots, A_n\}$ *be a set of events with dependency graph* $D = (V,E)$. *If there exists an assignment of reals* $x : \mathcal{A} \to (0,1)$ *such that*

$$\forall A \in \mathcal{A}: \quad \mathbb{P}(A) \leq x_A \prod_{B\in\Gamma_{\mathcal{A}}(A)} (1 - x_B),$$

then

$$\mathbb{P}\left(\bigwedge_{A\in\mathcal{A}} \overline{A}\right) \geq \prod_{A\in\mathcal{A}} (1 - x_A) > 0.$$

Let us show that the symmetric version (Theorem 1.2) follows immediately from Asymmetric Lovász local lemma (Theorem 1.4).

For every $A \in \mathcal{A}$ set $x_A = \frac{1}{d+1}$. If we substitute this value in the condition of Theorem 1.4, we get that

$$\begin{aligned} x_A \prod_{B\in\Gamma_{\mathcal{A}}(A)} (1 - x_B) &= \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{d_D(A)} \\ &\geq \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{d} \geq \frac{1}{(d+1)e} \geq p. \end{aligned}$$

Therefore, it follows that

$$\mathbb{P}\left(\bigwedge_{A\in\mathcal{A}} \overline{A}\right) > 0.$$

1.2.3 Applications of LLL

Let us examine two particular examples of using the Symmetric Lovász local lemma. From the beginning both k-SAT and 2-colorability of hypergraphs were an important part of the development of the Lovász local lemma.

The k-SAT problem

A *literal* is a boolean variable (a variable which has only two possible values - TRUE or FALSE) or its negation. A *clause* is a finite set of literals and boolean operators (conjuction $\wedge$, disjunction $\vee$, implication $\Rightarrow$, equivalence $\Leftrightarrow$).

A *boolean formula* φ is a finite sequence of symbols such that

(1) a boolean variable x_i is formula,

(2) if x_1, x_2 are formulas, then also $\neg x_1, x_1 \wedge x_2, x_1 \vee x_2, x_1 \Rightarrow x_2, x_1 \Leftrightarrow x_2$ are formulas.

A formula φ is in the *conjunctive normal form* (CNF) if every clause is a disjunction of literals and the formula φ is a conjuction of these clauses. If every clause contains exactly k literals we say that the formula φ is in k-CNF. For example

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$$

is in 3-CNF.

An evaluation of φ is the replacement of each variable by a value TRUE or FALSE. If there exists an assignment of variables such that the formula φ evaluates to TRUE then φ is called *satisfiable.*

We say that two clauses *overlap* if they contain the same variable x_i, regardless of whether it is x_i or its negation $\neg x_i$.

The k-SAT problem, or the "k-satisfiability problem", is to decide if the given k-CNF formula is satisfiable.

We already stated an example of 3-CNF formula. Every SAT problem, i.e. the satisfiability problem for arbitrary formulas, can be easily reduced to the 3-SAT. Since SAT was the first problem proven to be NP-complete [8], the 3-SAT is NP-complete as well.

Lemma 1.1 [21]. *Let φ be a k-CNF formula. If each of its clauses overlaps with at most 2^{k-2} clauses, then φ is satisfiable.*

Proof. Randomly replace each variable x_i with TRUE or FALSE. Number each clause and let $I = \{1, 2, \ldots, l\}$ be the set of indexes of these clauses. For the i-th clause of φ define the "bad" event A_i which means that the i-th clause is not satisfied (in the case of CNF this can only happen when all literals inside the i-th clause are replaced by FALSE).

The probability that the event A_i will occur is

$$\mathbb{P}(A_i) = p = \frac{1}{2^k}.$$

An event A_i is mutually independent from all events which do not share same variables as A_i. By assumption of the Lemma that means that every event A_i is mutually independent from all the other events except for at most 2^{k-2} of them.

If we substitute these values into Theorem 1.1 we get that

$$4pd \leq 4\,\frac{1}{2^k}\,2^{k-2} = 1.$$

Therefore the assumption is satisfied, thus, with the nonzero probability none of bad events will occur, so we can find an evaluation of φ for which φ evaluates as TRUE, i.e. the k-CNF formula is satisfiable.

□

2-colorability of hypergraphs

The second example which we will focus on is the coloring of hypergraphs, which are objects that we will study in more detail in Chapter 3.

A *hypergraph* $H = (V,E)$ is a generalization of a graph in which every edge, called *hyperedge*, connects an arbitrarily large number of vertices. A hypergraph H is said to be k-uniform if each of its hyperedges contains exactly k vertices. A hyperedge *intersects* some other hyperedge if they have at least one common vertex.

A *coloring* of a hypergraph H is a mapping $f : V(H) \to \mathbb{N}$. A *proper* coloring of H is an assignment of colors to each vertex of H such that no hyperedge is monochromatic, i.e. contains all vertices of the same color. A hypergraph H is called 2-*colorable* if it has a proper 2-coloring.

Theorem 1.5 [12]. *Let $H = (V,E)$ be a k-uniform hypergraph in which every hyperedge intersects at most $\frac{2^{k-1}}{e} - 1$ other hyperedges. Then H is 2-colorable.*

Proof. Suppose that we have two colors, say RED and BLUE. Randomly color each vertex of a hypergraph H with one of these colors. Let A_i denote the "bad event" that the hyperedge e_i is monochromatic. Then the probability that the event A_i will occur is

$$\mathbb{P}(A_i) = p = \frac{2}{2^k} = 2^{1-k}.$$

Each event A_i is independent on the event A_j, if the hyperedges e_i and e_j are disjoint. Thus, from the assumption we get that every event A_i is mutually independent from all the other events except for at most $\frac{2^{k-1}}{e} - 1$ of them.

It implies that

$$ep(d+1) \leq e\,2^{1-k}\left(\frac{2^{k-1}}{e} - 1 + 1\right) = 1.$$

Therefore, the condition of the Symmetric Lovász local lemma (Theorem 1.2) is satisfied and with nonzero probability non of the events A_i will occur, which means that the hypergraph H is 2-colorable. $\square$

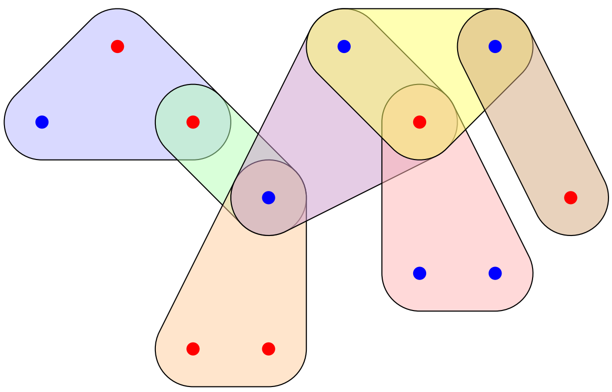


Figure 1.1: An example of 2-colored hypergraph.

1.3 Algorithmic Lovász local lemma

The Lovász local lemma is indeed a powerful tool in the probabilistic method, but for a long time it was only a non-constructive proof technique. In 1991, Beck [4] was the first who suggested the possible existence of an algorithmic version of the Lovász local lemma. On the example of 2-colorability of a k-uniform hypergraph Beck demonstrated that there exists a polynomial time algorithm that finds a certain 2-coloring of a hypergraph, but only if every hyperedge intersects, i.e. shares at least one of its vertices, with at most $2^{\frac{k}{48}}$ other hyperedges. Therefore, his approach required stronger conditions than the Lovász local lemma. This discovery started the gradual improvement and further search for a constructive proof of the Lovász local lemma. For instance Alon [1] simplified and randomized Beck's algorithm and also improved its required dependency to at most $2^{\frac{k}{8}}$, but much more work has been done in the search for algorithmic Lovász local lemma such as [34, 9, 27]. The biggest breakthrough came when Moser [26] presented a constructive proof of the Lovász local lemma that worked without further required restrictions and after that together with Tardos [28] extended his previous result in such a way that it can be used with almost all known applications of the general version of the Lovász local lemma.

1.3.1 Moser-Tardos algorithm

The Moser-Tardos constructive proof of the Lovász local gives surprisingly simple randomized algorithm, which not only shows the existence of an evaluation for which no "bad event" occurs, but also finds such an evaluation.

As discussed in Section 1.1 every event from the finite set of events $\mathcal{A}$ in a probability space Ω is determined by random variables from $\mathcal{P}$, i.e. values of variables determining event $A \in \mathcal{A}$ tell us whether the event A occurs or not. There exists a minimal subset $S \subseteq \mathcal{P}$ that determines each event A; let us denote this subset as $\mathrm{vbl}(A)$. Therefore in a dependency graph $G_{\mathcal{A}}$ two events $A, B \in \mathcal{A}, A \neq B$, may only be joined by an edge if $\mathrm{vbl}(A) \cap \mathrm{vbl}(B) \neq \emptyset$.

At the beginning of the algorithm we randomly assign an evaluation v_P to each variable $P \in \mathcal{P}$. We check whether any event $A \in \mathcal{A}$ is violated by current evaluation. If that is the case then we arbitrarily choose one of the violated events A and assign a new evaluation v_P to all variables from $P \in \mathrm{vbl}(A)$ on which A depends. Therefore, all variables from $\mathcal{P} \backslash \mathrm{vbl}(A)$ remain unchanged. We call this process *resampling* of the event A. As long as there exists any violated event A, we continue resampling such events.

Algorithm 1 Algorithmic Lovász local lemma

for all $P \in \mathcal{P}$ **do**
    $v_P \leftarrow$ a random evaluation of P;
while $\exists A \in \mathcal{A} : A$ is violated when $(P = v_p : \forall P \in \mathcal{P})$ **do**
    pick an arbitrary violated event $A \in \mathcal{A}$;
    for all $P \in \mathrm{vbl}(A)$ **do**
        $v_P \leftarrow$ a new random evaluation of P;
return $(v_p)_{P \in \mathcal{P}}$

Surprisingly with assumptions on the events this algorithm terminates after the finite number of steps. It means that the algorithm reaches an evaluation of all variables for which none of the events from $\mathcal{A}$ is violated if the conditions of the Lovász local lemma are satisfied. The following theorem by Moser and Tardos [28] shows the maximum number of steps it takes to reach the required evaluation, i.e. the number of resamplings we must do to ensure that no bad event occurs.

Theorem 1.6 (**Algorithmic Lovász local lemma**) [28]. *Let $\mathcal{P}$ be a finite set of mutually independent random variables in a probability space. Let $\mathcal{A}$ be a finite set of events determined by variables from $\mathcal{P}$ and with dependency graph $D = (V,E)$. If there exists an assignment of reals $x : \mathcal{A} \to (0,1)$ such that*

$$\forall A \in \mathcal{A} : \quad \mathbb{P}(A) \leq x(A) \prod_{B \in \Gamma_{\mathcal{A}}(A)} (1 - x(B)),$$

then there exists an assignment of values to the variables $\mathcal{P}$ not violating any of the events in $\mathcal{A}$. Moreover the randomized algorithm described above resamples an event $A \in \mathcal{A}$ at most an expected $\frac{x(A)}{1-x(A)}$ times before it finds such an evaluation. Thus the expected total number of resampling steps is at most

$$\sum_{A \in \mathcal{A}} \frac{x(A)}{1 - x(A)}.$$

1.3.2 Entropy compression

While examining Moser's first algorithm from [26] Terence Tao in his blog post [37] analyzed in detail the method that Moser used to prove the Algorithmic Lovász local lemma. He described it by the term *entropy compression* - a method that shows that the given algorithm terminates.

Since its discovery, the entropy compression was used with known results proved by the Lovász local lemma and often obtained the stronger bounds than which was already achieved. See [13] for its application in acyclic edge coloring.

The use of the entropy compression is overall simple. When we apply the entropy compression to a certain problem, we must be able to record the history of its process in a way, such that from the current state of the process we will be able to reconstruct the state of the process of any past time, and such that the amount of recorded information is less than the amount of information which would be randomly generated in these steps. The amount of recorded information at any step of the process never exceeds the amount of randomly generated information, therefore, the algorithm certainly terminates.

The principle of the entropy compression method can be briefly summarized by the following theorem:

Theorem 1.7 (**Entropy compression**) [29]. *Consider an algorithm* Φ *and an input* Q *of independent and identically distributed uniformly sampled bits. Suppose that* Φ *is such an algorithm that for each step* $t = 0, 1, 2, \ldots$ *the following holds:*

- Φ *can be modified to maintain a bit string* R_t *recording its history after each step* t*, such that the random bits of* Q *read so far can be recovered from* R_t;
- R_t *has length at most* $r_0 + t\Delta r$ *after step* t;
- $q_0 + t\Delta q$ *random bits have been read after step* t.

If $\Delta r < \Delta q$*, then the step after which* Φ *terminates is at most*

$$\frac{r_0 - q_0}{\Delta q - \Delta r}$$

in expectation.

Chapter 2

Existence of an independent transversal

An independent transversal of a graph is an independent set of vertices in which there is exactly one vertex from each part of partition of its vertex set. In this chapter we will look at the history of the problem of finding an independent transversal in a graph and construct an algorithm to find an independent transversal. We then apply the entropy compression to this algorithm and prove one of the earlier results from the search for the sufficient conditions for the existence of an independent transversal in a graph.

2.1 Introduction

Let G be a graph (simple, finite and undirected) with maximum degree Δ and $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ a partition of its vertex set into r pairwise disjoint subsets called *parts* of $V(G)$. An *independent transversal* of G with respect to the partition $\{V_i\}_{i \in [r]}$ is an independent set of vertices which contains exactly one vertex from each part V_i.

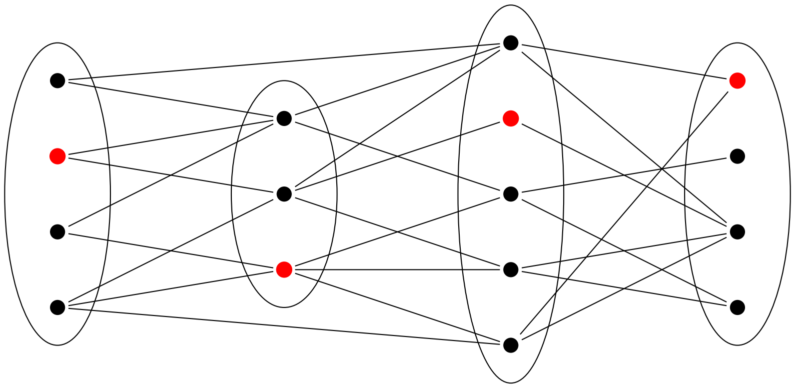


Figure 2.1: An independent transversal of G.

The problem of finding sufficient conditions for the existence of an independent transversal was first raised in 1972 by Erdős [11] who conjectured that if the size of each part is at least $\Delta + 1$ then G contains an independent transversal. Even though Graver found a proof for $r = 3$, Seymour constructed counterexamples for $r \leq 4$ and disproved the conjecture. Some attributed [24] the beginning of this problem to Bollobás, Erdős and Szemerédi [6], who in 1975 studied independent transversals in more detail. Alon [2] and independently Fellows [14] gave the first linear bounds in the terms of the maximum degree Δ of a graph G. Later Alon and Spencer [3] proved by using the Lovász local lemma that if the size of each part is at least $2e\Delta$ then G has an independent transversal. Afterwards Haxell [19] improved their bound to 2Δ.

Theorem 2.1 [19]. *If G is a graph with maximum degree at most Δ whose vertex set is partitioned into parts $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ of size $|V_i| \geq 2\Delta$, then G has an independent transversal.*

On the other hand Jin [20] and Yuster [39] gave examples of graphs with all parts of size $2\Delta - 1$ where the maximum degree Δ had to be a power of 2 which have no independent transversal. In 2006 Szabó and Tardos [36] also constructed graphs with size of parts $2\Delta - 1$ and no independent transversal but their construction works for every Δ. This means that 2Δ is the best possible value for the size of each part of G, therefore 2Δ is tight.

2.1.1 Average degree conditions

The conditions for existence of an independent transversal in a graph G can be defined depending on terms other than just the maximum degree of G.

Let G be a graph with partition $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ of its vertex set with size of each part at least s. The *degree* $d(V_i)$ of the part $V_i \in V(G)$ is the number of edges with exactly one vertex in V_i. The *maximum block average degree* of the partition of $V(G)$ is the maximum over all blocks V_i of $\frac{d(V_i)}{|V_i|}$.

Reed and Wood remarked in their article [30] that the Lovász local lemma implies a version of Theorem 2.1 in terms of an average degree. They showed that if a graph G has the maximum block average degree at most $\frac{s}{2e}$, then G has an independent transversal. This sufficient condition was later [10] improved to $\frac{s}{4}$. Wanless and Wood [38] used the Rosenfeld method and obtained more general formulation of the sufficient conditions of the existence of an independent transversal in graph G (their formulation was given in terms of hypergraphs).

Theorem 2.2 [38]. *Let G is a graph whose vertex set is partitioned into parts $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ of size $|V_i| \geq s$ and whose maximum block average degree is at most $\frac{s}{4}$. Then there exist $\left(\frac{s}{4}\right)^r$ independent transversals in G.*

Recently it was shown in the article [18] that the bound in Theorem 2.2 cannot be improved.

Loh and Sudakov [24] defined the *local degree* of a graph G with its vertex set partitioned into parts $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ as the maximum number of neighbors of a vertex $v \in V_i$, taken over all choices of V_i and $v \notin V_i$. Through the local degree they proved the existence of an independent transversal in G.

Theorem 2.3 [24]. *For every $\epsilon > 0$ there exists $\gamma > 0$ such that the following holds. If G is a graph with maximum degree at most Δ whose vertex set is partitioned into parts $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ of size $|V_i| \geq (1+\epsilon)\Delta$, and the local degree of G is at most $\gamma\Delta$, then G has an independent transversal.*

This result was later reformulated such that instead of the maximum degree the sufficient condition for the existence of an independent transversal in a graph is dependent on the maximum block average degree [22, 17].

2.2 Proof by Lovász local lemma

One of the breakthrough results achieved in the search for sufficient conditions for existence of an independent transversal in a graph was proved with the use of the symmetric Lovász local lemma. Alon [2, 3] using Lemma 1.2 obtained a simple proof which showed that an independent transversal in a graph exists as long as the size of each part is at least $2e\Delta$:

Theorem 2.4 [2, 3]. *If G is a graph with maximum degree at most Δ whose vertex set is partitioned into parts $V(G) = V_1 \cup V_2 \cup \cdots \cup V_n$ of size $|V_i| \geq 2e\Delta$, where $e = 2{,}718\ldots$ is Euler's number, then G has an independent transversal.*

Proof. Assume that every V_i has size exactly $\lceil 2e\Delta \rceil$, other wise we can replace each V_i by its subset of size $\lceil 2e\Delta \rceil$ and replace graph G by its induced subgraphs on these new parts.

We will independently choose one random vertex in every V_i. The bad event A_e means that we chose both endvertices of edge e. This can happen with probability $\lceil 2e\Delta \rceil^{-2}$.

Every event A_e where e is the edge with its endvertices in V_i and V_j is mutually independent of all other events except for those whose corresponding edges have their endvertices in $V_i \cup V_j$. Therefore, every event A_e is independent of all other events $\{A_i\}$ except at most $2\lceil 2e\Delta \rceil \Delta - 1$ of them. Applying the symmetric Lovász Local lemma we get that

$$\frac{2e\Delta}{\lceil 2e\Delta \rceil} < 1,$$

i.e. with nonzero probability none of the bad events A_e occur, so we can find independent transversal in graph G. □

2.3 Proof by entropy compression

Bissacot, Fernández, Procacci and Scoppola [5] improved the Lovász local lemma and used this approach to refine Alon's condition to 4Δ. In this section we will prove their result with the entropy compression.

Theorem 2.5 [5]. *Let $G = (V,E)$ be a graph with maximum degree Δ and $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ a partition of vertex set $V(G)$ into r pairwise disjoint parts. Suppose that for each set V_i we have $|V_i| \geq 4\Delta$. Then, there exists an independent transversal.*

The result obtained in this thesis is weaker than the result in Theorem 2.5, but it illustrates the application of the entropy compression.

Firstly, we need to define some auxiliary terms that will be useful in our approach. The *global ordering* is a fixed ordering of all vertices within each part V_i. On the other hand the *local ordering* is a fixed ordering of all neighbours of any vertex v_i.

Without loss of generality assume that no vertex is connected to another vertex inside the same part, we can neglect these edges, because they do not affect the existence of an independent transversal in any way.

Now, we can construct Algorithm 2 which will find an independent transversal in a graph G. The main goal of the following algorithm is to record its process into a binary string R in such a way, that from R and from the current selection of the vertices of final independent transversal we can reconstruct a history of the algorithm at any step of its execution. Each time we randomly select a vertex $v \in V_i$ from such a part V_i, in which no other vertex was selected. Because in an independent transversal there are no two vertices connected by edge, we consider a bad event as the selection of a vertex which is connected by edge with any other vertex already selected into an independent transversal. If the bad event occurs, we write down this "wrong" selection into our record R. Otherwise we write down in the binary encoding that no of the bad events occur.

Algorithm 2 Finding an independent transversal in a graph G

for $i \in [n]$, such that i is the smallest index for which V_i has no selected vertex **do**
    randomly select $v_i \in V_i$
    if $\{v_i,v_j\} \in E(G)$ for some already selected v_j **then**
        write down to R : 1, binary encoded global ordering of v_i and local ordering of v_j
        unselect v_i and v_j
    else
        write down 0 to R

For the reconstruction of the history of the previous algorithm we need to read the record R twice and each time we encode either part V_i or selected vertex v_i. In the first reading we start at the beginning of the record R and gradually assign each action (recorded at each step of the algorithm) to a certain part V_i. On the contrary, in the second reading we start at the end of the record R and find out which vertex in a part assigned to a certain action was selected.

Example 2.1. Let us now look at the first few steps of one particular example of the application of the algorithm used to find an independent transversal in a graph.

Define a graph $G = (V,E)$ with its vertex set partitioned into five parts $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$ with each part of size at least 4Δ. Without loss of generality assume that no edge joins two vertices contained in the same part V_i.

The graph G is shown in the following Figure 2.2 with global ordering of all vertices in every part V_i. In the steps of the algorithm all vertices will be represented by dots, but the global ordering remains fixed during the whole execution of the algorithm.

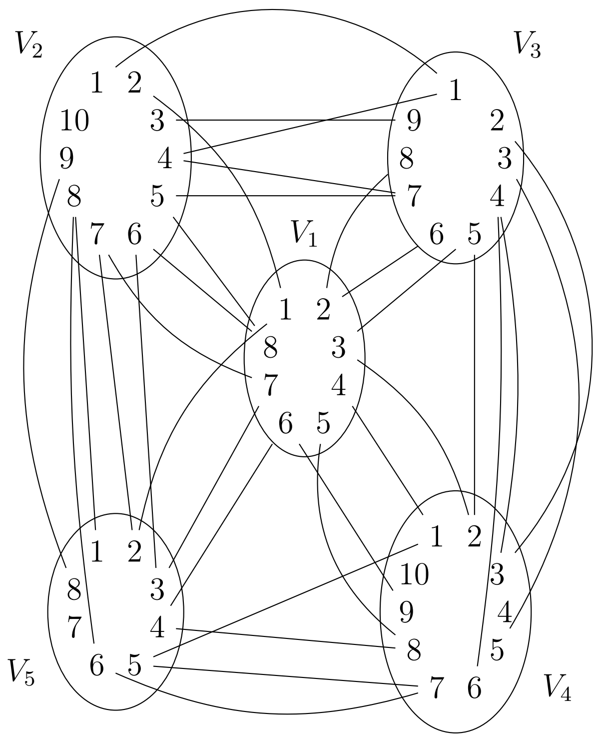
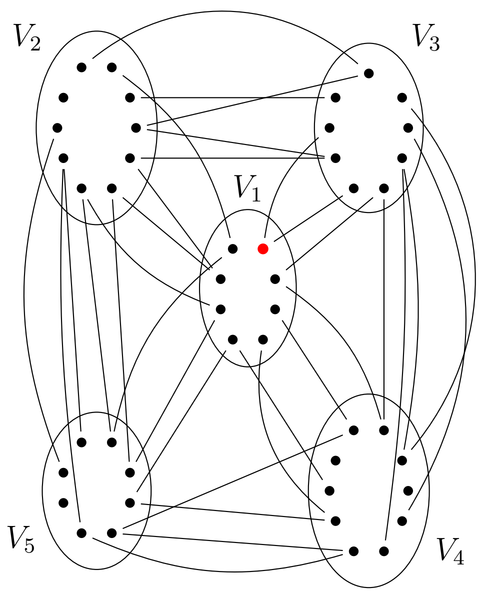


Figure 2.2: Graph G with the global ordering of all vertices inside each part V_i.

1st step

We start the algorithm in the part with the lowest index, i.e. in the part V_1. Randomly select a vertex from part V_1 and write 0 to the record R (with only one selected vertex there cannot be any wrong selection).

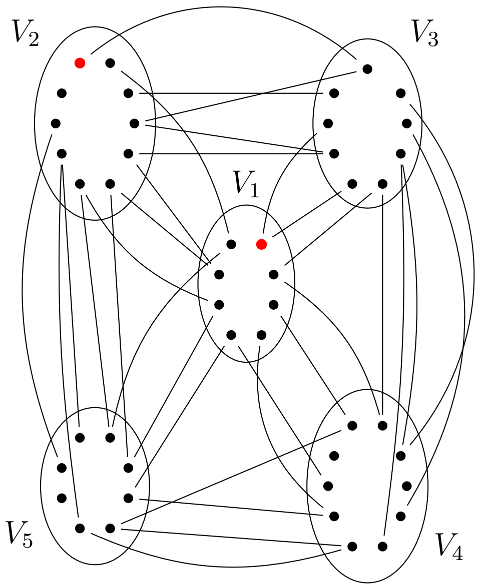


Record R :

0

2nd step

Continue the part V_2 and randomly select one of its vertices. The selected vertex is not connected with previously selected vertex from V_1, therefore again write 0 to R.



Record R :

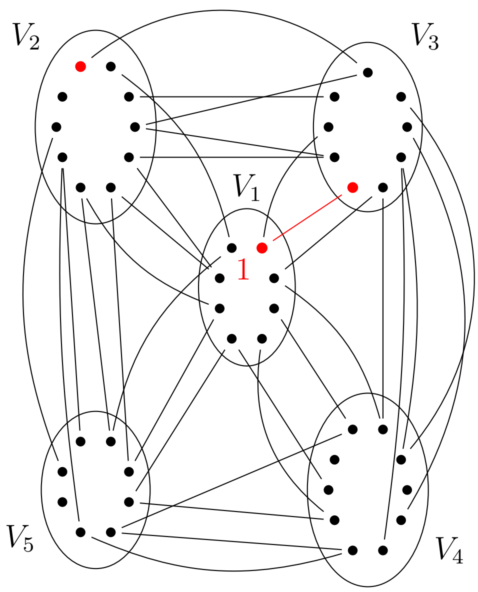
0

0

3rd step

After selection in V_3, there are two selected vertices that are joined by the edge - the currently selected vertex from V_3 and the already selected vertex from V_1. Therefore, write to R number 1 (representation of the wrong selection), binary encoded global index of the vertex from V_3 and binary encoded local index of the vertex from V_1.

Furthermore, delete the selected vertices from V_1 and V_3, thus these parts are now without selected vertices.



Record R :

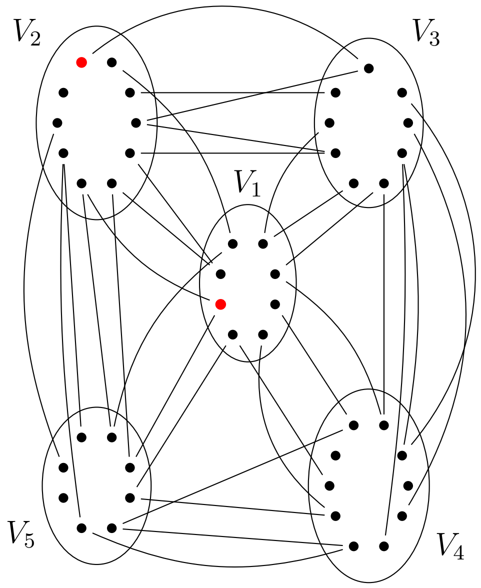
0

0

1,110,1

4th step

After deleting the vertices from wrong selection, the part with the lowest index is again V_1. Therefore, randomly select a vertex from V_1 and write 0 to the R because there is no edge connecting this vertex with other already selected vertices.



Record R :

0

0

1,110,1

0

After the fourth step, the execution of Algorithm 2 will continue in the same way until it finds an independent transversal in the graph G.

We will prove that Algorithm 2 terminates (finds an independent transversal in G) after finite number of steps assuming weaker bound for size of each part V_i than in Theorem 2.5.

Theorem 2.6. *Let $G = (V,E)$ be a graph with maximum degree Δ whose vertex set is partitioned into r pairwise disjoint parts $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$. Suppose that for each part V_i we have $|V_i| \geq 8\Delta$, then G has an independent transversal.*

Proof. We want to prove by the entropy compression that the Algorithm 2 terminates, i.e. it finds an independent transversal. Firstly, we demonstrate that we can pair together right and wrong selection throughout the whole execution of the Algorithm 2. Let us denote the size of each part V_i by x. In the record R we can pair together each bad selection (starting with 1) and one previous 0. That is so, because every bad selection of vertex v_i had to be preceded by the selection of vertex v_j which is joined by an edge with the newly selected vertex v_i, otherwise the bad selection would not happen.

Now we will use the properties of the entropy compression and show that the Algorithm 2 actually terminates. To binary encode arbitrarily selected vertex from each part V_i (in its global ordering) we need $1 + \lfloor \log_2 x \rfloor$ bits. However, to binary encode selections through our algorithm we need 1 bit to specify the correctness of the selection, $1 + \lfloor \log_2 x \rfloor$ bits for global index of selected vertex and $1 + \lfloor \log_2 \Delta \rfloor$ for local index of conflict vertex, i.e. our record R requires

$$\begin{cases} 1 \quad \text{bit} & \text{for right selection} \\ \Big(1 + (1 + \lfloor \log_2 x \rfloor) + (1 + \lfloor \log_2 \Delta \rfloor)\Big) \quad \text{bits} & \text{for wrong selection} \end{cases}$$

If we pair right and wrong selections, as mentioned above, the resulting pair will be binary encoded by at most

$$\Big(1 + (1 + \lfloor \log_2 x \rfloor) + (1 + \lfloor \log_2 \Delta \rfloor)\Big) + 1$$

bits. On the other hand, binary encoding of two randomly selected vertices needs at most $2(1 + \lfloor \log_2 x \rfloor)$. In order to use the entropy compression we must compare these two binary encodings and show that the binary encoding of the record R requires less bits than the binary encoding of random selection.

By the assumption of Theorem 2.6 each part V_i is of size

$$x \geq 8\Delta.$$

Thus, it holds that

$$\log_2 x \geq 3 + \log_2 \Delta.$$

Thanks to the properties of the floor function

$$\lfloor \log_2 x \rfloor \geq \lfloor 3 + \log_2 \Delta \rfloor$$

$$\lfloor \log_2 x \rfloor \geq 3 + \lfloor \log_2 \Delta \rfloor$$

Therefore,

$$\lfloor \log_2 x \rfloor > 2 + \lfloor \log_2 \Delta \rfloor.$$

In order to get the requiring comparison, we make some trivial addition

$$2 + 2\lfloor \log_2(x) \rfloor > 4 + \lfloor \log_2(x) \rfloor + \lfloor \log_2(\Delta) \rfloor.$$

And finally, if we arrange expressions into brackets according to binary encodings, we get that

$$2\Big(1 + \lfloor \log_2(x) \rfloor\Big) > \Big(1 + (1 + \lfloor \log_2(x) \rfloor) + (1 + \lfloor \log_2(\Delta) \rfloor)\Big) + 1.$$

Therefore, the binary encoding of the record R of the Algorithm 2 is smaller than the binary encoding of the random selection as long as the size of each part V_i is at least 8Δ.

Which means that we are able to binary encode the whole process of the Algorithm 2 in such a way that we need less bits for the record R than for binary encoding of each randomly selected vertex from an independent transversal separately and at the same time from the final record R we are able to reconstruct the whole process of Algorithm 2. Therefore, by Theorem 1.7 the algorithm terminates, i.e. we will find an independent transversal of the graph G.

□

Chapter 3

Shrinking hypergraphs

The Lovász local lemma and its related methods are effective proof techniques which are quite simple to apply. But sometimes we can achieve better results without these techniques and in this chapter we will look at one such example.

Klimošová and Thomassé in their article [23] showed that for every ℓ, there exists d_ℓ such that every every 3-edge connected graph with minimum degree d_ℓ can be edge partitioned into paths with length ℓ which divides the number of edges of this graph. One of the tools used in their article is the following Lemma 3.1 which was proved using entropy compression (see Section 3.1 for definitions).

Lemma 3.1 . *Let $H = (V,E)$ be a hypertree with hyperedges of size at most three. It is possible to shrink all hyperedges of size three so that the resulting graph is a spanning tree T such that*

$$d_T(v) \geq \frac{d_H(v)}{100}$$

for every $v \in V$.

In this chapter we will formulate an improvement of this lemma valid for hypertrees with hyperedges of any size (not only for hypertrees with hyperedges of size at most 3) and which provides a better final bound for degrees of vertices in the resulting spanning tree.

3.1 Definitions

A *hypergraph* is a pair $H = (V,E)$, where V is a finite set and $E \subseteq 2^V$. Elements of V are called *vertices*, and the elements of set E are called *hyperedges* of hypergraph H.

The *size* of a hyperedge $e \in E$ is the number of vertices which form this hyperedge. If all hyperedges of hypergraph H are exactly of size k, then the hypergraph H is called *k-uniform.* Therefore a graph is a 2-uniform hypergraph.

Hypergraph H which does not contain loops (i.e. hyperedges of size 1) or repeated hyperedges (i.e. hyperedges which are formed by identical vertex set) is called *simple.* In this chapter we will only consider simple hypergraphs so from now on the adjective simple will be omitted.

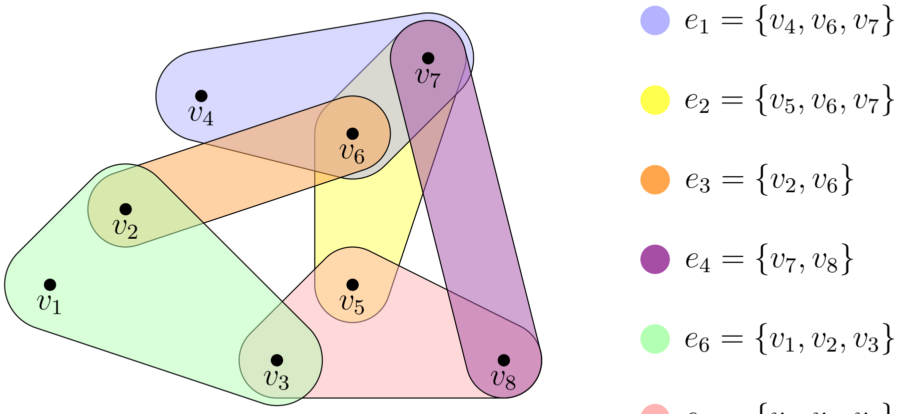


Figure 3.1: A hypergraph H with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and set of hyperedges $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.

The number of hyperedges which contain vertex $v \in V$ is called the *degree* of the vertex and is denoted by $d_H(v)$.

A connected hypergraph is defined analogously to a connected graph, thus the hypergraph H is called *connected* if there is path between every pair of vertices, i.e. in the hypergraph H there exists a finite sequence of vertices in which every two vertices are connected by hyperedge in H and in which all vertices, therefore also all hyperedges, are distinct.

To *shrink* a hyperedge e means to replace it by an edge e' (with two endvertices, i.e. $|e'| = 2$) such that $e' \subsetneq e$. If we shrink every hyperedge of a hypergraph H then the resulting graph G_H is called a *shrinked* graph of a hypergraph H.

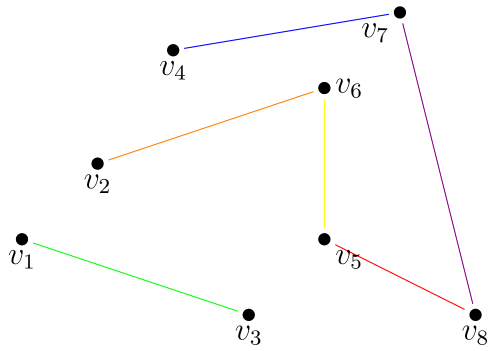


Figure 3.2: A shrinked graph G_H obtained by shrinking every hyperedge of a hypergraph H.

A hypergraph $H = (V,E)$ is a *hypertree* if there is a tree T with vertex set V such that every edge $e \in E$ induces a subtree in T (T is then called the *underlying vertex tree* of E).

Lovász [25] proved that it is always possible to shrink a hypertree to a tree.

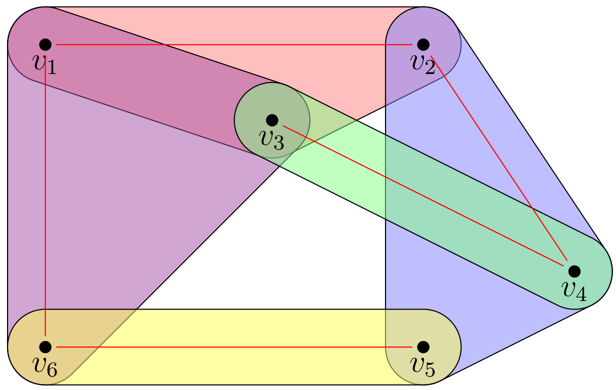


Figure 3.3: A hypertree H with highlighted tree T.

There are many ways how we can define directed hypergraphs, however, we will stick to the definition used in [16]. A *directed hypergraph* is a pair $\vec{H} = (V,\vec{E})$, where V is a finite set of vertices and $\vec{E}$ is a set of so called *hyperarcs.* The hyperarc is a subset $e \subseteq V$ of $\vec{H}$ with designated *head* $v \in e$ and the remaining vertices $e - v$ are called the *tails.*

Every hypergraph $H = (V,E)$ can be transformed into a directed hypergraph by choosing an *orientation,* i.e. the assignment of a head to each hyperedge from E.

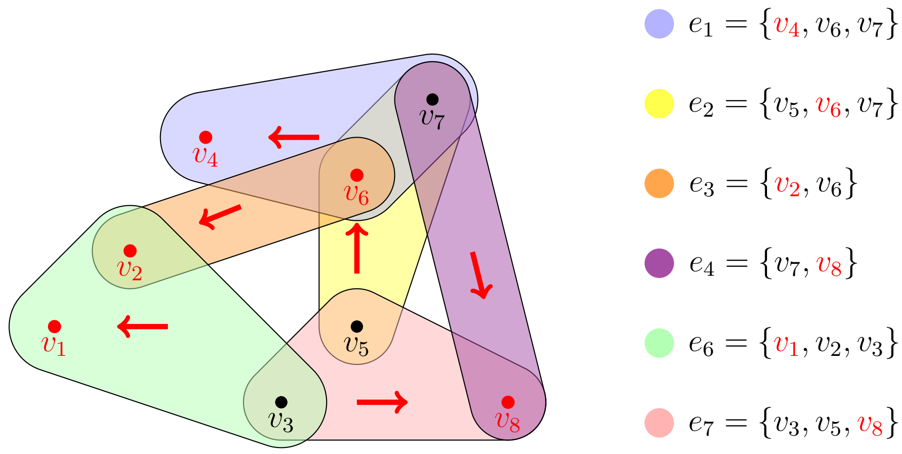


Figure 3.4: One of the possible orientations of hypergraph H with highlighted head of each hyperarc.

The *indegree* of vertex v in a directed hypergraph $\vec{H}$, which is denoted by $d^{IN}_{\vec{H}}(v)$, is the number of hyperarcs whose head is v. Similarly the *outdegree* of vertex v, denoted as $d^{OUT}_{\vec{H}}(v)$, is the number of hyperarcs in which the vertex v is a tail.

From the definition of a directed hypergraph we can see that every hyperarc contributes to the indegree of exactly one vertex and also to the outdegree of $k-1$ vertices, where k is the size of this hyperarc.

3.2 Orientation with boundaries

In the previous section we defined an orientation of a hypergraph H, i.e. an oriented hypergraph $\vec{H}$ which we obtain in such a way that we choose one vertex from every hyperedge of a hypergraph H and this vertex we will designate as its head vertex.

In the proof of Lemma 3.4 below we will need to create an orientation $\vec{H}$ of a hypergraph H in such a way that an indegree $d^{IN}_{\vec{H}}(v)$ of every vertex v in $\vec{H}$ will be at least as large as a given value. Before formulating the required orientation lemma we mention some theorems, which will be essential for its proof. Firstly, we state a lemma allowing us to obtain an orientation $\vec{H}$ of a hypergraph H with indegrees $d^{IN}_{\vec{H}}(v)$ equal to predetermined values:

Lemma 3.2 [16]. [1] *Let $H = (V, E)$ be a hypergraph and let $f : V \to \mathbb{Z}^+$ be a mapping of the vertex set V of H into the set of non-negative integers. Then there is an orientation $\vec{H}$ of H such that $d^{IN}_{\vec{H}}(v) = f(v)$ for every $v \in V$ if and only if*

(i) $\sum_{v \in V} f(v) = e(H)$,

(ii) $\sum_{v \in F} f(v) \geq e(F)$ *for every* $F \subseteq V$,

where $e(F)$ denotes the number of hyperedges in the subset $F \subseteq V$ and $e(H)$ is the number of all hyperedges in H.

An orientation of graphs is a special case of hypergraph orientation for which we have the following theorem:

Theorem 3.1 [15]. *The graph $G = (V, E)$ has an orientation satisfying*

$$d^{IN}_{\vec{G}}(v) \geq f(v)$$

where $f(v)$ denotes an integer function defined on the vertices of an undirected graph G, if and only if for every subset $F \subseteq V$ it holds that

$$\sum_{v \in F} f(v) \leq e^*(F),$$

where $e^*(F)$ denotes the number of edges incident to F.

This theorem was originally formulated in terms of a lower bound for an outdegree $d^{OUT}_{\vec{H}}(v)$ of vertices v from vertex set V of graph G, but if we reverse the orientation of every edge in graph G, i.e. each head will become tail and vice versa, then this holds also for indegrees.

Now we have everything necessary to formulate and prove the required lemma about an orientation $\vec{H}$ of a hypergraph H that satisfies predetermined lower bounds of indegrees of every vertex in $\vec{H}$. We can extend Theorem 3.1 to hypergraphs. For the proof we use a modification of an approach used in [7]:

Lemma 3.3. *Let $H = (V, E)$ be a hypergraph and let $f : V \to \mathbb{Z}^+$ be a mapping of the vertex set V of H into the set of non-negative integers. Assume that*

$$\sum_{v \in F} f(v) \leq e^*(F), \tag{3.1}$$

[1] Lemma 3.2 was first formulated in [16] but we will use its equivalent formulation from [7].

where $f(v)$ is defined on the vertices of an undirected graph H and $e^*(F)$ denotes the number of hyperedges incident to F, holds for every $F \subseteq V$. Then there is an orientation $\vec{H}$ of H such that

$$d_{\vec{H}}^{IN}(v) \geq f(v) \tag{3.2}$$

for every $v \in V$.

Proof. Let $g : V \to \mathbb{Z}^+$ be a mapping such that

(i) $\sum\limits_{v \in F} g(v) \leq e^*(F)$ for every $F \subseteq V$,

(ii) $g(v) \geq f(v)$ for every $v \in V$,

(iii) $\sum\limits_{v \in V} g(v)$ is the maximum among all functions that satisfy (i) and (ii).

We will show that $\sum_{v \in V} g(v) = e(H)$, where $e(H)$ denotes the number of hyperedges of H. Then if we substitute $d_{\vec{H}}^{IN}(v)$ for $g(v)$ there is an orientation $\vec{H}$ of a hypergraph H for which $d_{\vec{H}}^{IN}(v) \geq f(v)$ for every vertex $v \in V$.

Let $X \subseteq V$ be a set with maximum cardinality for which $\sum_{v \in X} g(v) = e^*(X)$. Firstly, we want to show that $V \backslash X$ does not contain any vertex w with the function $g(w)$ equal to the degree of w in the hypergraph H.

It is possible that X is the empty set. If $X = V$ then we are done, so assume that $|X| < |V|$. If w is a vertex in $V \backslash X$ for which $g(w) = d_H(w)$, where $d_H(w)$ is a degree of w in H, then by maximality of $|X|$ we obtain

$$\sum_{v \in V \cup \{w\}} g(v) < e^*(X \cup \{w\}) \leq e^*(X) + e^*(\{w\}) = \sum_{v \in X} g(v) + d_H(w) =$$
$$= \sum_{v \in X} g(v) + g(w) = \sum_{v \in V \cup \{w\}} g(v),$$

which is a contradiction. Therefore $V \backslash X$ contains no vertex w with $g(w) = d_H(w)$.

Now we will prove that there exists a function $h(v)$ which satisfies properties (i) and (ii), but for every vertex $v \in V$, $h(v) > g(v)$ which contradicts the maximality of $\sum\limits_{v \in V} g(v)$.

Let z be a vertex in $V \backslash X$ and let $h : V \to \mathbb{Z}^+$ be a mapping such that $h(v) = g(v)$ for all $v \in V \backslash \{z\}$ and $h(z) = g(z) + 1$. Suppose that there exists a subset $F \subseteq V$ such that $\sum_{v \in F} h(v) > e^*(F)$. Then $z \in F$ and

$$e^*(F) \geq \sum_{v \in F} g(v) = \sum_{v \in F} h(v) - 1 > e^*(F) - 1,$$

which implies that $\sum_{v\in F} g(v) = e^*(F)$. But by the maximality of $|X|$ and because F is not contained in X (as $z \in F$ and $z \in V\backslash X$) we get that

$$\sum_{v\in F\cup X} g(v) = \sum_{v\in F} g(v) + \sum_{v\in X} g(v) - \sum_{v\in F\cap X} g(v) \geq$$
$$\geq e^*(F) + e^*(X) - e^*(F\cap X) \geq e^*(F\cup X),$$

which is again a contradiction.

Therefore for all subsets $F \subseteq V$ it holds that $\sum_{v\in F} h(v) \leq e^*(F)$ and also for every vertex $v \in V$ it holds that $h(v) \geq g(v) \geq f(v)$, so the function h satisfies properties (i) and (ii). However $\sum_{v\in V} h(v) = \sum_{v\in V} g(v) + 1$, i.e. $\sum_{v\in V} h(v) > \sum_{v\in V} g(v)$ which is a contradiction with maximality of $\sum_{v\in V} g(v)$. Hence $|X| = |V|$ and the proof is done. □

We will show an example of a function $f(v)$ that satisfies the assumption of Lemma 3.3 and so there exists an orientation $\vec{H}$ of a hypergraph H in which every vertex $v \in V$ has an indegree at least as large as $f(v)$ (in adition, this function will be later helpful for proving the Lemma 3.4):

Theorem 3.2. *Let $H = (V,E)$ be a hypergraph with hyperedges of size at most k, then there exists an orientation $\vec{H}$ of H in which*

$$d^{IN}_{\vec{H}}(v) \geq \lfloor \frac{d_H(v)}{k} \rfloor$$

for every vertex $v \in V$.

Proof. Let $F \subseteq V$ be a subset of vertices of H. Hyperedge on the vertex set X is incident to F, if $X \cap F \neq \emptyset$, i.e. if F contains at least one vertex from X. Since hyperedges of H are of size at most k every subset $F \subseteq V$ may contain at most k vertices from their vertex set. Therefore, every hyperedge of H can be counted into $e^*(F)$ (the number of hyperedges incident to F) at most k times.

The inequality (3.1) in Lemma 3.3 certainly holds for $f(v) = \frac{d_H(v)}{k}$.

It follows that

$$\sum_{v\in F} \lfloor \frac{d_H(v)}{k} \rfloor \leq e^*(F),$$

i.e. the assumption (3.1) also holds for the integer-valued function $\sum_{v\in F} \lfloor \frac{d_H(v)}{k} \rfloor$. Thus there exists an orientation $\vec{H}$ of H in which the indegree of every vertex $v \in V$ is

$$d^{IN}_{\vec{H}}(v) \geq \lfloor \frac{d_H(v)}{k} \rfloor$$

and we are done. □

3.3 Heterochromatic spanning tree

An *edge coloring* of a graph G is a mapping $f : E(G) \to \mathbb{N}$, where the assigned integers are called colors. A graph G is said to be *edge colored* if G has assigned edge coloring. If no two adjacent edges have the same color, then an edge coloring is called *proper*.

A subgraph of an edge colored graph is called *heterochromatic* (also called *rainbow*) if it does not contain any two edges with the same color. A spanning tree of a graph G whose edges have different colors is said to be a *heterochromatic spanning tree*.

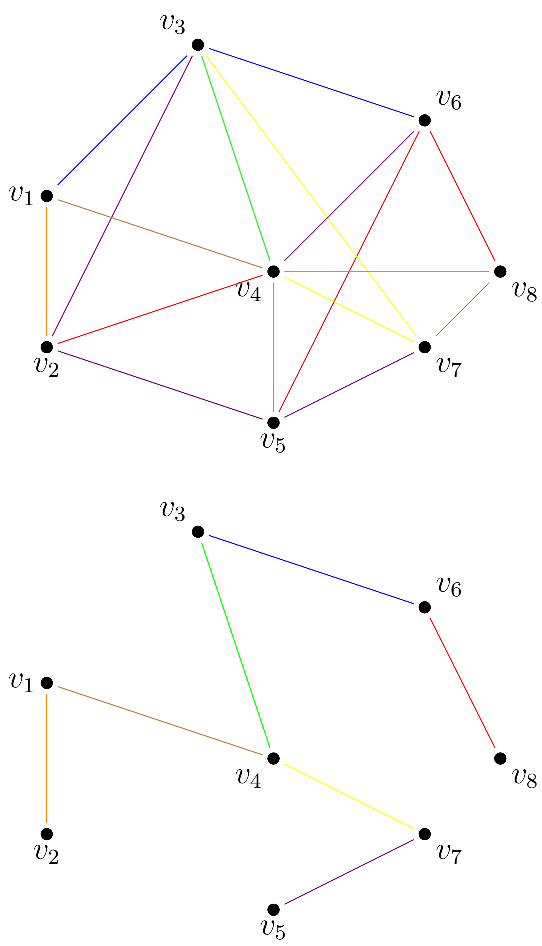


Figure 3.5: An edge colored graph G and its heterochromatic spanning tree T.

One of the tools which we will use in the proof of our improvement of Lemma 3.1 is a necessary and sufficient condition for the existence of a heterochromatic spanning tree in an edge colored connected graph:

Theorem 3.3 [35]. *An edge colored connected graph G of order n has a heterochromatic spanning tree, if and only if, for any r colors $(1 \leq r \leq n-2)$, the removal of all edges colored with these r colors from G results in a graph having at most $r+1$ components.*

3.4 Improvement of Lemma 3.1

Klimošová and Thomassé [23] improved Lovász result [25] and presented Lemma 3.1. For the proof they use the entropy compression. It turns out that in this case we can get better bounds using other methods.

We will extend Lemma 3.1 by allowing arbitrary hyperedge size, while proving a stronger bound for the degrees of vertices in the resulting graph.

Lemma 3.4. *Let $H = (V,E)$ be a hypertree with hyperedges of size at most k. It is possible to shrink all hyperedges of H so that the resulting graph is a spanning tree T such that for every vertex $v \in V$,*

$$d_T(v) \geq \frac{d_H(v)}{2k}. \tag{3.3}$$

Proof. In Theorem 3.2 we proved that there exists an orientation $\vec{H}$ of H in which for every vertex $v \in V$,

$$d^{IN}_{\vec{H}} \geq \lfloor \frac{\mathrm{d}_H(v)}{k} \rfloor.$$

We will use the properties of the floor function and show that for every vertex $v \in V$ with degree $d_H(v) \geq k$,

$$\lfloor \frac{d_H(v)}{k} \rfloor \geq \frac{d_H(v)}{2k}.$$

Firstly, we want to prove that for every real $x \geq 1$ it is true that

$$\lfloor x \rfloor \geq \frac{x}{2}.$$

The floor function represents the greatest integer less than or equal to the original real number, i.e. $\lfloor x \rfloor \geq x$. From the properties of the floor function we know that

$$\lfloor x \rfloor \geq x - 1.$$

Let $x \geq 2$, then surely is satisfied the inequality

$$2\lfloor x \rfloor \geq 2x - 2 \geq x.$$

Now let $1 < x < 2$. For the floor function of such a real number it is true that $\lfloor x \rfloor = 1$, thus

$$1 = \lfloor x \rfloor \geq \frac{x}{2}.$$

Therefore, every real number $x \geq 1$ satisfies the required inequality

$$\lfloor x \rfloor \geq \frac{x}{2}.$$

If we substitute $\frac{d_H(v)}{k}$ for x, we get for every vertex $v \in V$ with $d_H(v) \geq k$,

$$\lfloor \frac{d_H(v)}{k} \rfloor \geq \frac{d_H(v)}{2k}.$$

Therefore, the orientation $\vec{H}$ of H satisfies

$$d^{IN}_{\vec{H}}(v) \geq \frac{d_H(v)}{2k}$$

for every $v \in V$ with $d_H(v) \geq k$.

The next step is to show that if we replace each hyperarc with vertex set X from the orientation $\vec{H}$ with a star graph on X, then we obtain a graph G_H admitting a heterochromatic spanning tree T.

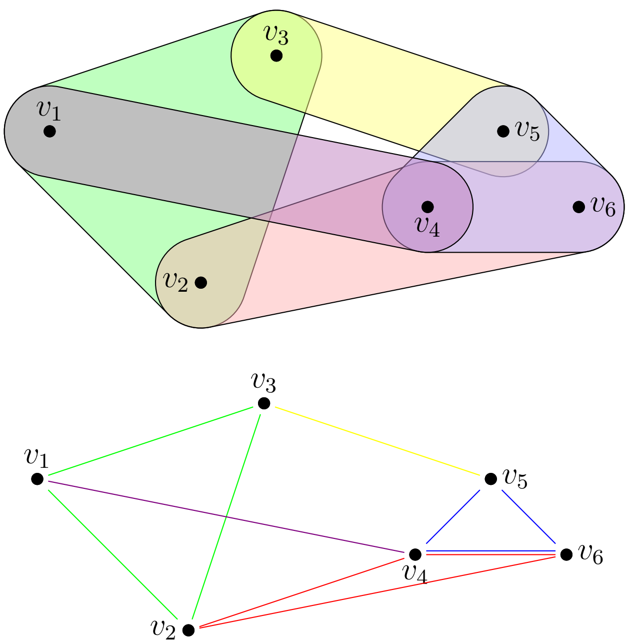
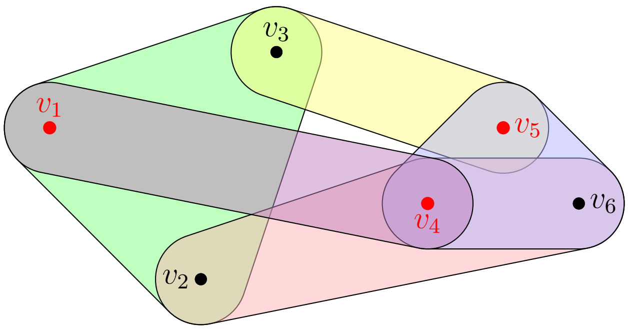


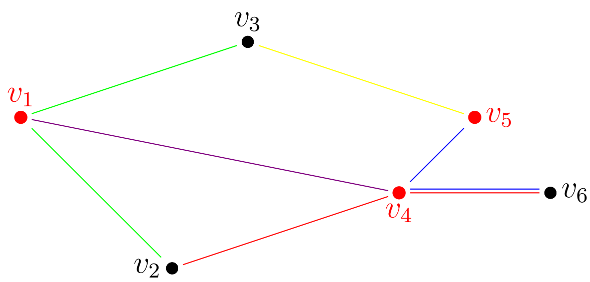
Figure 3.6: A representation of hyperedges by complete graphs.

Let $\widetilde{G}_H$ be the graph obtained from H by replacing every hyperedge with the vertex set X in H by a complete graph on X, i.e. we connect every vertex $v \in X$ with every other vertex from X, as illustrated in Figure 3.6. If we color edges of each complete graph on X according to its original hyperedge, we obtain edge colored graph $\widetilde{G}_H$ in which there surely exists a heterochromatic spanning tree, since H is the hypertree.

Now let us look at a different way of replacing hyperedges to obtain a graph. We can represent every hyperarc with vertex set X of a directed hypertree $\vec{H}$ by a star graph on X such that the central vertex of this star graph is the head vertex of the corresponding hyperarc, as illustrated in Figure 3.7. Let G_H denote the graph obtained from H by replacing every hyperedge of $\vec{H}$ with a star graph.



(a) A directed hypertree $\vec{H}$ with a highlighted head vertex in every hyperarc.



(b) A graph G_H in which every hyperarc is represented by a star graph with an central vertex in its head vertex.

Figure 3.7: A representation of hyperarcs by star graphs.

Color edges of every star graph of G_H in the same way as the edge coloring of complete graphs in $\widetilde{G}_H$, i.e. assign one color to all edges of the star graph on the vertex set X obtained by replacing hyperedge with the vertex set X. Therefore, each star graph obtained from different hyperedge will have distinct edge color.

In edge colored graph G_H there exists a heterochromatic spanning tree T. That is because we have already shown that $\widetilde{G}_H$ contains a heterochromatic spanning tree and since the number of components in Lemma 3.3 does not change if we are replacing hyperedges by complete graphs or any other connected subgraphs on the same vertex set, G_H must contain a heterochromatic spanning tree as well.

At last, we will prove that every vertex $v \in V$ has a degree

$$d_T(v) \geq \frac{d_H(v)}{2k}$$

in the heterochromatic spanning tree T obtained from G_H.

To obtain a heterochromatic spanning tree T we select exactly one edge from every edge colored star graph. Since every hyperarc on the vertex set X contributes to the indegree of its head vertex $v \in X$ in $\vec{H}$, the selection of an edge from star graph on X to the heterochromatic spanning tree T will contribute to the degree of central vertex $v \in X$ in T. Thanks to the properties of the orientation $\vec{H}$ every vertex v for which $d_H(v) \geq k$ in $\vec{H}$ has indegree $d^{IN}_{\vec{H}} \geq \frac{d_H(v)}{2k}$, therefore, vertex v will certainly have $d_T(v) \geq \frac{d_H(v)}{2k}$.

For the vertices with degree less than k the inequality holds automatically, because the resulting heterochromatic spanning tree T is a connected graph, which means that each of its vertices must have degree at least 1.

Therefore, each vertex has degree

$$d_T(v) \geq \frac{d_H(v)}{2k}$$

in the heterochromatic spanning tree T .

We obtained spanning tree T from hypertree H by replacing every hyperedge from H by exactly one edge in T. Therefore, we shrinked H to T.

$\square$

Conclusion

In this thesis, we studied Lovász local lemma and related methods. We provided a brief summary of this methods and its applications.

In Chapter 2 we introduced the problem of finding a sufficient condition for the existence of an independent transversal in a graph. We constructed an algorithm for finding an independent transversal in a graph with maximum degree Δ with its vertex set partitioned into parts of size $|V_i| \geq 8\Delta$ and afterwards we proved with application of the entropy compression that this algorithm terminates, therefore, it finds an independent transversal in such a graph.

In Chapter 3, we introduced the basic definitions from the hypergraph theory and dealt with the shrinking of hypergraphs. We showed that for a result from [23] originally proved by the entropy compression it is possible to derive a stronger and more general statement by other method.

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