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Unbounded stationary solutions of semidiscrete Nagumo equation & Functional equation containing inverse

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1 Introduction

We study semidiscrete Nagumo equation on lattice

$$\frac{\partial u_i}{\partial t} = D\,(u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i(1 - u_i)(u_i - a), \qquad t > 0,\ i \in \mathbb{Z}, \tag{1}$$

where $D, \lambda > 0$ stand for diffusion and reaction rates and $a \in (0, 1)$ is a viability parameter. Nagumo equation serves as a basic model of competition between two stable states and is also among the simplest equations which exhibit important dynamic phenomena like pattern formation and traveling waves. In contrast to continuous Nagumo equation, the equation (1) has a large number of stationary patterns. We describe and analyze an uncountable set of equivalence classes of unbounded stationary solutions.

In the second part, we show that a task to determine a certain type of unbounded stationary solutions of the equation (1) is equivalent to proving an existence and uniqueness of non-negative solution of the functional equation

$$\frac{f(x) + f^{-1}(x)}{2} = \varphi(x) \tag{2}$$

where $\varphi(x)$ is derived from the reaction term of the equation (1).

2 Results

Stationary solutions of the equation (1) can be understand as real double sequences $\{u_n\}_{n\in\mathbb{Z}}$. Based on properties of these sequences we distinguish between two types - two-sided and one-sided unbounded stationary solutions. Since the system is autonomous in $i \in \mathbb{Z}$ stationary solutions can be shifted arbitrarily in i. Thus, we construct equivalence classes. We restrict ourself to stationary solutions with $u_k > 1$ for all $k \in \mathbb{Z}$.

First, we describe an uncountable set of equivalence classes of unbounded stationary solutions which satisfy

$$\lim_{i\to\pm\infty} u_i = +\infty.$$

Such stationary solution can be determined by a pair $(\alpha,\, \beta) \in \mathcal{U}$, where

$$\mathcal{U} := \left\{(v,\, w) \in \mathbb{R}^2 : v > 1 \wedge \varphi^{-1}(v) \le w \le \varphi(v)\right\}, \tag{3}$$

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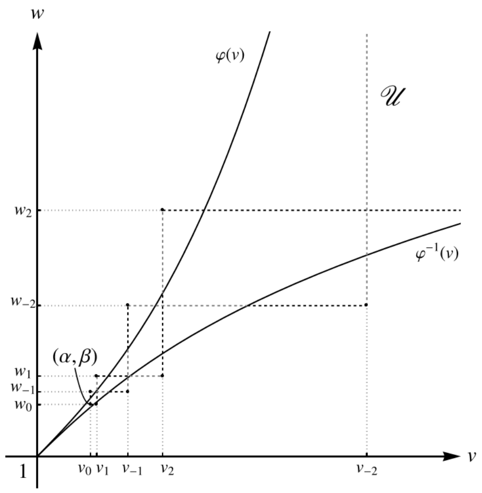

Figure 1: The region $\mathcal{U}$ given by (3) and a construction of two-sided unbounded solution by mirror schemes with initial values $(\alpha,\,\beta)$.

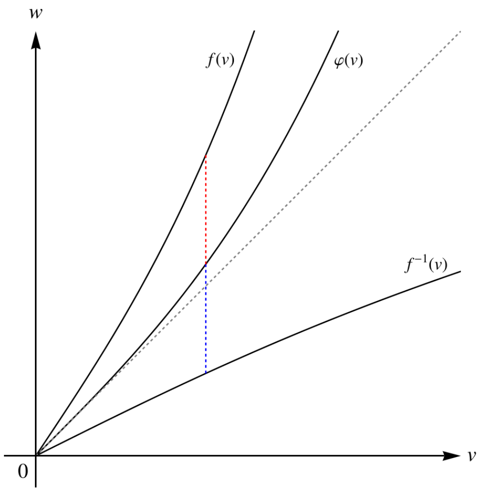

Figure 2: Solutions of the equation (2) with $\varphi(x)$ given by (4).

with

$$\varphi(v) := v - \frac{\lambda}{2D} v(1-v)(v-a). \tag{4}$$

The pair $(\alpha,\,\beta)$ is unique if $(\alpha,\,\beta) \in \operatorname{int}\mathcal{U}$ but in the opposite case $(\alpha,\,\beta)$ and $(\beta,\,\alpha)$ represent same stationary solution. For this purpose we use so called mirror schemes. Here, mirroring is meant by mapping a point in $\mathbb{R}^2$ on the opposite side of the curve $\varphi(x)$ (or $\varphi^{-1}(x)$) in the axis direction and preserving the distance, see Fig. 1.

Next we show that there exists another class of unbounded stationary solution of (1) which are unbounded at a one side but converge to 1 on the other side. Unlike two-sided unbounded stationary solutions, the element which represents equivalence class of such stationary solution may belong only to one of two curves in $\mathbb{R}^2$. This result is equivalent to the statement that the equation (2) has exactly two solution $f_1(x)$ and $f_2(x) = f_1^{-1}(x)$ in the first quadrant. In the proof of this, we do not use the exact form of the function $\varphi(x)$ given by (4) but rather properties of its derivatives. Thus, the existence and uniqueness result for (2) holds for a whole class of functions $\varphi(x)$ with common properties.

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References

Stehlik P., Volek J., Hesoun J. (2023) Mirroring maps and unbounded stationary solutions of lattice equations. *Submitted for publication.*