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Admissible values of the travelling wave speed in an asymmetrically supported beam

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1 Introduction

This work is motivated by the appearance of travelling waves in various problems in many fields such as physics, biology, mechanics, engineering, etc. We study the existence of the homoclinic travelling wave solutions of the boundary value problem

$$\begin{cases} u_{tt} + u_{xxxx} + \alpha u^+ - \beta u^- + g(u) = 1, & x \in \mathbb{R},\ t > 0, \\ u(x,t) \to \dfrac{1}{\alpha},\ u_x(x,t) \to 0 \ \text{ for } |x| \to +\infty, \end{cases} \tag{1}$$

where $u = u(x,t)$, α, $\beta > 0$, $u^\pm = \max\{\pm u, 0\}$ and $g(1/\alpha) = 0$. The problem (1) can be used as a model of an asymmetrically supported bending beam or a generalized suspension bridge model.

As far as we know, the topic of the existence of travelling waves in suspension bridges was originally opened in McKenna and Walter (1990) (motivated by the collapse of Tacoma Narrows Bridge), then extended by Chen and McKenna (1997) and many others after that.

2 Techniques and methods in use

For the sake of simplicity, we omit the nonlinear term g in further considerations, i.e., we take $g \equiv 0$. In Holubová and Levá (2023) we have proved the existence of travelling wave solution of (1) for any wave speed $|c| \in (\sqrt[4]{100\beta/9}, \sqrt[4]{4\alpha})$. The result remains preserved for nonzero g under suitable assumptions. However, the lower bound for wave speed values is possibly too restrictive. The aim of our work is to find the optimal lower bound, i.e., to determine the admissible values of the wave speed c.

The main used technique is the variational method, in particular Mountain Pass Theorem (MPT). The travelling wave solution of (1) can be written in the form $u(x,t) = y(x-ct) = y(s)$. We take $z = z(s) = y(s) - 1/\alpha$ and define a functional

$$I(z) = \frac{1}{2}\int_{\mathbb{R}} \left((z'')^2 - c^2(z')^2\right) \mathrm{d}s + \frac{\alpha}{2}\int_{z>-1/\alpha} z^2 \,\mathrm{d}s + \frac{\beta}{2}\int_{z\le -1/\alpha} z^2 \,\mathrm{d}s + \frac{1}{2}\left(\frac{\beta}{\alpha} - 1\right)\int_{z\le -1/\alpha} (2z + 1/\alpha)\,\mathrm{d}s$$

for $z \in H^2(\mathbb{R}) = W^{2,2}(\mathbb{R})$ with the standard norm. The function z provides a weak solution of (1) if and only if z is the critical point of I. The known critical point is 0 which is the local minimum of I. Other critical point is gained by MPT.

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3 Main result

We restrain ourselves only to $\beta < \alpha$ to ensure I having the right (Mountain Pass) geometry. Existence of some $e \in H^2(\mathbb{R})$ far enough from 0 such that $I(e) < 0$ is also required. We consider $e = Av$ with $A \in \mathbb{R}^+$ large enough and v fixed element in $H^2(\mathbb{R})$. First, we take the following transforms: $x = sc/\sqrt{2}$, $\bar{\alpha} = 4\alpha/c^4$, $\bar{\beta} = 4\beta/c^4$ and $\bar{\alpha} = r\bar{\beta}$ with fixed $r \in (1, +\infty)$.

Now, let us define functional $J(\bar{\beta}, v) = \int_{\mathbb{R}} \left((v'')^2 - 2(v')^2 + r\bar{\beta}(v^+)^2 + \bar{\beta}(v^-)^2\right) \mathrm{d}x$. If we find some β_0 and v_0 such that $J(\beta_0, v_0) = 0$ then for any $\bar{\beta} < \beta_0$ we have $J(\bar{\beta}, v_0) < 0$ and thus $I(e = Av_0) < 0$. Our aim is to find the optimal (maximal) value of β_0, i.e.,

$$\beta^* = \beta^*(r) = \sup\{\beta \in (0,1) : \exists v \in H^2(\mathbb{R}) : J(\beta, v) = 0\} \leq 1.$$

Next, we define $\beta_l^* = \beta_l^*(r) := \sup_{v \in S_{l,r}} \int_{-l}^{l} (2(v')^2 - (v'')^2)\, \mathrm{d}x$ with $v \in H_0^2(-l, l)$ and $S_{l,r} = \left\{ v \in H_0^2(-l,l) : \int_{-l}^{l} (r(v^+)^2 + (v^-)^2)\, \mathrm{d}x = 1 \right\}$. Then $\beta^*(r) = \sup_{l>0} \beta_l^*(r) = \lim_{l \to +\infty} \beta_l^*(r)$.

Let us now consider the Dirichlet boundary value problem

$$\begin{cases} v^{(4)} + 2v'' + \alpha v^+ - \beta v^- = 0, & x \in (-l, l), \\ v(\pm l) = v'(\pm l) = 0, \end{cases} \tag{2}$$

with $l > 0$ and denote by Σ_l^D the corresponding Fučík spectrum of (2), i.e., the set of all $(\alpha, \beta) \in \mathbb{R}^2$ such that (2) has a nontrivial solution.

Our main result reads as follows.

Theorem 1 *Let $\alpha > \beta > 0$ be arbitrary but fixed and let β^* be given by*

$$\beta^*(r) = \sup_{l>0} \max\left\{\beta > 0 : (r\beta, \beta) \in \Sigma_l^D\right\}.$$

Then the travelling wave solution of (1) *exists for any wave speed c satisfying*

$$|c| \in \left(\sqrt[4]{\frac{4\beta}{\beta^*(\alpha/\beta)}}, \sqrt[4]{4\alpha} \right).$$

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