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Bifurcations in Nagumo Equations on Graphs and Fiedler Vectors

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Abstract
Reaction-diffusion equations serve as a basic framework for numerous dynamic phenomena like pattern formation and travelling waves. Spatially discrete analogues of Nagumo reaction-diffusion equation on lattices and graphs provide insights how these phenomena are strongly influenced by the discrete and continuous spatial structures. Specifically, Nagumo equations on graphs represent rich high dimensional problems which have an exponential number of stationary solutions in the case when the reaction dominates the diffusion. In contrast, for sufficiently strong diffusion there are only three constant stationary solutions. We show that the emergence of the spatially heterogeneous solutions is closely connected to the second eigenvalue of the Laplacian matrix of a graph, the algebraic connectivity. For graphs with simple algebraic connectivity, the exact type of bifurcation of these solutions is implied by the properties of the corresponding eigenvector, the so-called Fiedler vector.

Keywords Nagumo equation · Dynamical systems on graphs · Bifurcations · Algebraic connectivity · Fiedler vectors

Mathematics Subject Classification 35K57 · 39A12 · 39A28 · 05C40

1 Introduction

The need to describe dynamics of several natural phenomena in spatially heterogeneous domains has attracted lot of attention to dynamical systems on graphs over the last few decades. Motivated by the design of populations and natural habitats, epidemiological [15] and evolutionary [16] models on structured social networks have been developed signifi-

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cantly. Mathematically, the heterogeneous discrete spatial structure allows, e.g., to form a large number of spatially heterogeneous configurations, patterns, and large amount of other dynamical phenomena. Reaction-diffusion systems on lattices and graphs have been used as a guiding example of this behavior [13,17,27].

In this paper we focus on the emergence of heterogeneous stationary solutions of the Nagumo reaction-diffusion differential equation on a graph $\mathcal{G} = (V, E)$. We denote by V the set of vertices of $\mathcal{G}$, by E its set of edges, and by $\mathcal{N}(i) = \{j \in V : (i, j) \in E\}$ the neighborhood of a vertex $i \in V$. Our model has the form

$$u_i'(t) = d \sum_{j\in\mathcal{N}(i)} (u_j(t) - u_i(t)) + \lambda g(u_i(t)), \quad i \in V, \quad t \in \mathbb{R}, \tag{1.1}$$

in which $d > 0$ is the diffusion parameter, the sum corresponds to the graph Laplacian describing the linear diffusion on $\mathcal{G}$. The parameter $\lambda > 0$ represents the reaction strength, g is considered to be the cubic bistable nonlinearity (the general case is discussed later in Sect. 5)

$$g(u) = u(u - a)(1 - u), \quad a \in (0, 1). \tag{1.2}$$

The graph differential equation (abbreviated by GDE) (1.1) is a network analogue of the well-known Nagumo reaction-diffusion PDE [22]

$$u_t = du_{xx} + \lambda g(u), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{1.3}$$

which serves as a primary example in the modeling of the spatial competition between two stable states $u \equiv 0$ and $u \equiv 1$, see (1.2). Moreover, it is often used to illustrate key dynamic phenomena, e.g., the existence and stability of traveling waves. The spatial Euler discretization of (1.3) yields the Nagumo lattice differential equation (LDE)

$$u_i'(t) = d\,(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)) + \lambda g(u_i(t)), \quad i \in \mathbb{Z}, \quad t \in \mathbb{R}, \tag{1.4}$$

which has much richer set of equilibria [10,17] and enables intricate behavior of traveling waves, including new phenomena like pinning [4,12,13,18] and the existence of non-monotone (multichromatic) traveling waves [9,11].

Considering infinite graphs, the GDE (1.1) can be seen as a generalization of the LDE (1.4), which is its special case with $\mathcal{G} = \mathbb{Z}$. Moreover, the analysis of the structure of equilibria of the GDE (1.1) on cyclic graphs $\mathcal{G} = \mathcal{C}_n$, $n \in \mathbb{Z}$, allowed to describe the behavior of periodic equilibria of the LDE (1.4) [10] as well as to construct the non-monotone traveling waves connecting these periodic equilibria [11].

We restrict our attention to the GDE (1.1) on finite and connected graphs with $n = |V|$. It is well-known that for $\lambda \ll d$ there are only three constant stationary solutions $u \equiv 0$, $u \equiv a$, and $u \equiv 1$ corresponding to the fixed points of (1.2), whereas for $\lambda \gg d$ there are 3^n stationary solutions out of which 2^n are stable. It has been shown that for each graph $\mathcal{G}$ and fixed d and a there exists a value $\underline{\lambda} > 0$ such that there are no spatially heterogeneous solutions for $\lambda < \underline{\lambda}$. However, there are only rough estimates for the threshold $\underline{\lambda}$, [25]. In this paper we show that the emergence of these first non-constant solutions occurs via three different mechanisms in the neighborhood of a critical value λ_B. Its exact value and the bifurcation type depend on the interplay between the nonlinearity (1.2) and two graph characteristics – the value of algebraic connectivity λ_2 of $\mathcal{G}$ and properties of the corresponding eigenvector ϕ, the so-called Fiedler vector. Our computations consequently provide closer approximations of the value $\underline{\lambda}$.

In order to formally introduce these two graph attributes we define an $n \times n$ Laplacian matrix L of graph $\mathcal{G}$ by $L = D - A$ in which $D = \mathrm{diag}\{\deg(v_1), \ldots, \deg(v_n)\}$ is the diagonal matrix of vertex degrees and A is the adjacency matrix of $\mathcal{G}$. The matrix L is positive semidefinite and in the case of connected graphs, the eigenvalues of L satisfy

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \lambda_n.$$

The eigenvalue λ_2 is the above mentioned algebraic connectivity and provides estimates both for edge and vertex connectivities of a graph [5]. There are numerous applications of algebraic connectivity [20,21] and lot of attention has been devoted to its properties [1]. The eigenvectors ϕ corresponding to λ_2 are called the Fiedler vectors [6]. The Fiedler vector has many interesting applications. For example, it splits the set of vertices V into two connected nodal domains based on the signs of its entries [6], is connected to the fitness landscapes and can be used for graph colorings [2]. Beside the listed papers, these notions are illustrated in Sect. 4 which provides examples of Laplacian matrices and Fiedler vectors of specific graphs.

Our results contribute to several studies involving the role of spectral properties of graphs on dynamical systems. Most advanced can be found in the epidemiological models on networks, see, e.g., [3], where the interplay of eigenvalues of adjacency matrices and the degree distributions is investigated. More specifically, the value of λ_2 is known to be important in the theory of random walks and linear diffusion processes on graphs [19], synchronization of Kuramoto models [23], and Lotka-Volterra models on graphs [24]. Focusing on the graphs with simple algebraic connectivity λ_2 we show that the emergence of spatially heterogeneous solutions of the GDE (1.1) depends not only on the nonlinearity g and the algebraic connectivity λ_2 but also on the structural properties of the Fiedler vector ϕ.

Without loss of generality we shall consider the GDE (1.1) with normalized diffusion[1], $d = 1$,

$$u_i'(t) = \sum_{j \in \mathcal{N}(i)} (u_j(t) - u_i(t)) + \lambda g(u_i(t)), \quad i \in V, \quad t \in \mathbb{R}. \tag{1.5}$$

Finding stationary solutions of (1.5) is then equivalent to the problem of solving the nonlinear algebraic equation

$$F(u, \lambda) := -Lu + \lambda G(u) = 0, \tag{1.6}$$

in which $G(u) := (g(u_1), g(u_2), \ldots, g(u_n))^\top$.

Denoting $\mathtt{a} := (a, \ldots, a)^\top$ and $\lambda_B := \frac{\lambda_2}{a(1-a)}$ we study the solutions which bifurcate from $(\mathtt{a}, \lambda_B)$ as a smooth curve $\gamma : (u(s), \lambda(s))$, $s \in (-\eta, \eta)$, $\eta > 0$. At point $(\mathtt{a}, \lambda_B)$, the first derivative of F with respect to u

$$F_u(\mathtt{a}, \lambda_B) = -L + \lambda_2 I \tag{1.7}$$

is singular. We show that the bifurcation occurs in one of three possible ways – the transcritical bifurcation,

$$\gamma_1 : \begin{cases} u(s) = \mathtt{a} + \phi s + O(s^2), \\ \lambda(s) = \frac{\lambda_2}{a(1-a)} + cs + O(s^2), \qquad c \neq 0, \qquad s \in (-\eta, \eta), \quad \eta > 0, \end{cases} \tag{1.8}$$

the supercritical pitchfork bifurcation,

$$\gamma_2 : \begin{cases} u(s) = \mathtt{a} + \phi s + O(s^2), \\ \lambda(s) = \frac{\lambda_2}{a(1-a)} + cs^2 + O(s^3), \qquad c > 0, \qquad s \in (-\eta, \eta), \quad \eta > 0, \end{cases} \tag{1.9}$$

1 Alternatively we could fix $\lambda = 1$, which is more common, but we prefer to fix the diffusion parameter because of the direct natural connection between the value of λ and the algebraic connectivity λ_2.

or the subcritical pitchfork bifurcation,

$$\gamma_3 : \begin{cases} u(s) = \mathtt{a} + \phi s + O(s^2), \\ \lambda(s) = \frac{\lambda_2}{a(1-a)} - cs^2 + O(s^3), \qquad c > 0, \qquad s \in (-\eta, \eta), \quad \eta > 0. \end{cases} \tag{1.10}$$

Our main result states that all graphs with the simple algebraic connectivity λ_2 can be sorted into three groups. In each of these classes, different combinations of bifurcations (1.8)–(1.10) occur based on the parameter a of the nonlinearity g. This classification depends on the values of cubes of entries of the Fiedler vector $\sum_{i=1}^{n} \phi_i^3$, and a scalar product involving the second Hadamard (element-wise) power of ϕ which we denote by

$$\phi^{\circ 2} = \left(\phi_1^2, \phi_2^2, \ldots, \phi_n^2\right)^{\top}$$

and the Moore-Penrose pseudoinverse $(-L + \lambda_2 I)^{+}$ of the singular matrix $F_u(\mathtt{a}, \lambda_B) = -L + \lambda_2 I$.

Theorem 1.1 *Let $\mathcal{G} = (V, E)$ be a graph with a simple algebraic connectivity λ_2 and let ϕ be its Fiedler vector. Then there exists a unique smooth curve $\gamma : (u(s), \lambda(s))$ of hetereogeneous stationary solutions of (1.5) emanating from $\left(\mathtt{a}, \frac{\lambda_2}{a(1-a)}\right)$. Moreover,*

1. *if $\sum_{i=1}^{n} \phi_i^3 \neq 0$ then γ has the form of*
   (a) *γ_1 provided $a \neq \frac{1}{2}$, or*
   (b) *γ_2 provided $a = \frac{1}{2}$,*
2. *if $\sum_{i=1}^{n} \phi_i^3 = 0$ and $\left((-L + \lambda_2 I)^{+}\phi^{\circ 2}, \phi^{\circ 2}\right) \geq 0$ then γ has the form of γ_2 for all $a \in (0, 1)$,*
3. *if $\sum_{i=1}^{n} \phi_i^3 = 0$ and $\left((-L + \lambda_2 I)^{+}\phi^{\circ 2}, \phi^{\circ 2}\right) < 0$ then there exists $\delta \in \left(0, \frac{1}{2}\right)$ such that γ has the form of*
   (a) *γ_2 provided $\left|a - \frac{1}{2}\right| < \delta$, or*
   (b) *γ_3 provided $\left|a - \frac{1}{2}\right| > \delta$.*

The statement of Theorem 1.1 is visualised in Fig. 1. As its by-product we get that $\lambda_B = \frac{\lambda_2}{a(1-a)}$ is an upper bound for $\underline{\lambda}$.

Corollary 1.2 *Let $\mathcal{G} = (V, E)$ be a graph with a simple algebraic connectivity λ_2. Then $\underline{\lambda} \leq \lambda_B = \frac{\lambda_2}{a(1-a)}$. Moreover, if the assumptions of cases 1a or 3b in Theorem 1.1 are satisfied then the strict inequality holds.*

The paper is organized as follows. In Sect. 2 we use the Crandall-Rabinowitz theorem to show the existence of a unique smooth branch emanating from $(\mathtt{a}, \lambda_B)$. In Sect. 3 we prove the exact bifurcation type for each of the three cases in Theorem 1.1. Each is then illustrated by an example in Sect. 4. Sect. 5 provides a generalization to arbitrary nonlinearity g and the general role of Fiedler vectors is emphasized by an example. In Sect. 6, numerical simulations are performed to indicate that case 1 of Theorem 1.1 is prevalent among all graphs and we conclude with few conjectures.

2 Preliminaries and Unique Bifurcating Branch

We prove Theorem 1.1 in several steps. In this preliminary section, we introduce basic notations and concepts and prove the existence and uniqueness of the bifurcating branch. As indicated, the stationary solutions of (1.5) solve the algebraic problem

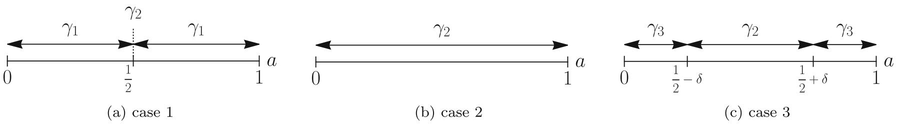

Fig. 1 Illustration of Theorem 1.1. Bifurcation schemes of spatially heterogeneous solutions from the constant solution $\mathrm{a} = (a, \dots, a)^\top$ at $\lambda_B = \frac{\lambda_2}{a(1-a)}$. The theorem classifies three cases, each of them groups together graphs with given properties of the Fiedler vector ϕ

$$F(u, \lambda) = -Lu + \lambda G(u) = 0, \tag{2.1}$$

in which

$$G(u) = (g(u_1), g(u_2), \dots, g(u_n))^\top ,$$

with the cubic nonlinearity g defined by (1.2). Firstly recall that the graph Laplacian matrix L has the one-dimensional kernel, provided the graph $\mathcal{G}$ is connected, and (see [2])

$$\operatorname{Ker} L = \operatorname{span} \left\{(1, 1, \dots, 1)^\top\right\}.$$

Therefore, $\left((0, 0, \dots, 0)^\top , \lambda\right)$, $\left((a, a, \dots, a)^\top , \lambda\right) = (\mathrm{a}, \lambda)$, and $\left((1, 1, \dots, 1)^\top , \lambda\right)$ with arbitrary $\lambda \in \mathbb{R}$ are homogeneous stationary solutions of (1.6), since $g(0) = g(a) = g(1) = 0$.

We are interested in the heterogeneous solutions (u, λ) with $u \neq \mathrm{a}$, which bifurcate from the solution branch (a, λ), $\lambda \in \mathbb{R}$. The necessary condition for (u_B, λ_B) to be a bifurcation point is that the derivative $F_u(u_B, \lambda_B)$ is singular, which follows from the implicit function theorem. Since

$$F_u(\mathrm{a}, \lambda) = -L + \lambda a(1 - a)I,$$

the eigenvalues μ_i of $F_u(\mathrm{a}, \lambda)$ are shifted eigenvalues of $-L$, specifically, $\mu_i = -\lambda_i + \lambda a(1 - a)$. The eigenvalue μ_2 of $F_u(\mathrm{a}, \lambda)$ vanishes at

$$\lambda_B = \frac{\lambda_2}{a(1 - a)},$$

and we consequently study the bifurcation from the point (a, λ_B). Our assumption on simplicity of λ_2 implies

$$\dim \operatorname{Ker} F_u\, (\mathrm{a}, \lambda_B) = 1, \qquad \operatorname{Ker} F_u\, (\mathrm{a}, \lambda_B) = \operatorname{span} \{\phi\}, \tag{2.2}$$

in which ϕ is the Fiedler vector (the eigenvector of L corresponding to λ_2). Without loss of generality, we assume from now on that

$$\|\phi\| = 1.$$

We prove the existence and uniqueness of the bifurcating branch from the following version of the Crandall–Rabinowitz local bifurcation theorem (see, e.g., [14, Thm. I.5.1, Cor. I.5.2]).

Theorem 2.1 *(Crandall–Rabinowitz) Let* $F \in C^k (\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ *with* $k > 1$, $F(0, \lambda) = 0$ *for all* $\lambda \in \mathbb{R}$, *and*

$$\dim \operatorname{Ker} F_u(0, \lambda_B) = 1, \qquad \operatorname{Ker} F_u(0, \lambda_B) = \operatorname{span} \{\phi\}, \qquad \|\phi\| = 1, \tag{2.3}$$

$$F_{u\lambda}(0, \lambda_B)\phi \notin \operatorname{Im} F_u(0, \lambda_B). \tag{2.4}$$

Then there is a nontrivial C^{k-1}*-curve* $\gamma : (u(s), \lambda(s))$, $s \in (-\eta, \eta)$, $\eta > 0$, *such that* $(u(0), \lambda(0)) = (0, \lambda_B)$, $F(u(s), \lambda(s)) = 0$ *for all* $s \in (-\eta, \eta)$, *and*

$$u'(0) = \phi. \tag{2.5}$$

Moreover, all solutions of $F(u, \lambda) = 0$ *in a neighborhood of the point* $(0, \lambda_B)$ *are on the trivial solution curve, or on* γ.

Applying this theorem to (2.1) we get the uniqueness of a smooth solution branch emanating from (a, λ_B).

Lemma 2.2 *Let the assumptions of Theorem 1.1 be satisfied. Then there exists a unique* C^∞*-curve* $\gamma : (u(s), \lambda(s))$ *of hetereogeneous stationary solutions of (1.5) emanating from* $\left(\mathrm{a}, \frac{\lambda_2}{a(1-a)}\right)$. *Moreover,* $u'(0) = \phi$.

Proof The statement is an immediate consequence of Theorem 2.1. The function $F(u, \lambda)$ defined by (1.6) is of class C^∞, $F(\mathrm{a}, \lambda) = 0$ for all $\lambda \in \mathbb{R}$, and (2.3) is verified by (2.2). Further, there is

$$F_{u\lambda}(\mathrm{a}, \lambda_B)\phi = a(1-a)\phi \in \mathrm{Ker}\, F_u(0, \lambda_B) = (\mathrm{Im}\, F_u(0, \lambda_B))^\perp .$$

Hence, the assumption (2.4) is also satisfied and the existence and uniqueness of the nontrivial C^∞-curve γ from Theorem 1.1 with $u'(0) = \phi$ follows from the application of Theorem 2.1 for $u - \mathrm{a}$. □

3 Transcritical, Sub- and Supercritical Pitchfork Bifurcation

The second part of the proof of Theorem 1.1 consists of careful investigation of particular situations in which the bifurcation is transcritical, sub- and supercritical. By Lemma 2.2 there exists C^∞-curve $\gamma : (u(s), \lambda(s))$, $s \in (-\eta, \eta)$, such that $(u(0), \lambda(0)) = (\mathrm{a}, \lambda_B)$, $\lambda_B = \frac{\lambda_2}{a(1-a)}$,

$$F(u(s), \lambda(s)) = 0 \quad \text{for all} \quad s \in (-\eta, \eta), \tag{3.1}$$

and $u'(0) = \phi$ by (2.5), which bifurcates from the homogeneous solution (a, λ), $\lambda \in \mathbb{R}$, at the point (a, λ_B).

The parametrization of the curve γ has the following Taylor expansion at $s = 0$:

$$\gamma : \begin{cases} u(s) = \mathrm{a} + \phi s + O(s^2), \\ \lambda(s) = \frac{\lambda_2}{a(1-a)} + \lambda'(0)s + \frac{1}{2}\lambda''(0)s^2 + O(s^3), \qquad s \in (-\eta, \eta). \end{cases} \tag{3.2}$$

Following [8,14], we derive precise formulas for $\lambda'(0)$ and $\lambda''(0)$ in the parametrization of the bifurcating curve (3.2). We distinguish among three possibilities, specifically, $\lambda'(0) \neq 0$ (transcritical); $\lambda'(0) = 0$ and $\lambda''(0) < 0$ (subcritical); $\lambda'(0) = 0$ and $\lambda''(0) > 0$ (supercritical).[2] Let us firstly derive the formula for $\lambda'(0)$. Differentiating (3.1) twice with respect to s at $s = 0$ we obtain

$$\begin{aligned} &F_{uu}(\mathrm{a}, \lambda_B)[u'(0), u'(0)] + 2F_{u\lambda}(\mathrm{a}, \lambda_B)u'(0)\lambda'(0) + F_{\lambda\lambda}(\mathrm{a}, \lambda_B)(\lambda'(0))^2 \\ &\quad + F_u(\mathrm{a}, \lambda_B)u''(0) + F_\lambda(\mathrm{a}, \lambda_B)\lambda''(0) = 0. \end{aligned} \tag{3.3}$$

[2] We omit the case $\lambda'(0) = 0$ and $\lambda''(0) = 0$, since the behavior of bifurcation depends on higher order terms for which we do not obtain representing formulas. Consequently, the case $\left|a - \frac{1}{2}\right| = \delta$ is missing in the statement of Theorem 1.1, case 3.

By the definition of F (1.6) we obtain

$$\begin{aligned} &F_\lambda(\mathtt{a}, \lambda_B) = 0, \quad F_{uu}(\mathtt{a}, \lambda_B) = \frac{2\lambda_2(1-2a)}{a(1-a)}\mathbb{I}_3, \\ &F_{u\lambda}(\mathtt{a}, \lambda_B) = a(1-a)\mathbb{I}_2, \quad F_{\lambda\lambda}(\mathtt{a}, \lambda_B) = 0, \end{aligned} \tag{3.4}$$

the symbol $\mathbb{I}_k$ denotes the identity tensor, i.e., a $(k-1)$-linear mapping defined for $v^{(1)}, v^{(2)}, \ldots, v^{(k-1)} \in \mathbb{R}^n$ by

$$\left(\mathbb{I}_k\left[v^{(1)}, v^{(2)}, \ldots, v^{(k-1)}\right]\right)_i = v_i^{(1)} v_i^{(2)} \ldots v_i^{(k-1)}, \qquad i = 1, 2, \ldots, n,$$

which can be reformulated via the so-called Hadamard (element-wise) product

$$v^{(1)} \circ v^{(2)} = \left(v_1^{(1)} v_1^{(2)}, v_2^{(1)} v_2^{(2)}, \ldots, v_n^{(1)} v_n^{(2)}\right)^\top$$

of $v^{(1)}, v^{(2)} \in \mathbb{R}^n$ as

$$\mathbb{I}_k\left[v^{(1)}, v^{(2)}, \ldots, v^{(k-1)}\right] = v^{(1)} \circ v^{(2)} \circ \ldots \circ v^{(k-1)}.$$

Therefore, we can use $u'(0) = \phi$ (cf. (2.5)) and simplify (3.3) into

$$\frac{2\lambda_2(1-2a)}{a(1-a)}\phi^{\circ 2} + 2\lambda'(0)a(1-a)\phi + F_u(\mathtt{a}, \lambda_B)u''(0) = 0, \tag{3.5}$$

with $\phi^{\circ 2} = \phi \circ \phi = \left(\phi_1^2, \phi_2^2, \ldots, \phi_n^2\right)^\top$ being the second Hadamard power of the Fiedler vector ϕ. Since $\phi \in \operatorname{Ker} F_u(\mathtt{a}, \lambda_B) = (\operatorname{Im} F_u(\mathtt{a}, \lambda_B))^\perp$ and $\|\phi\| = 1$, the scalar multiplication of (3.5) by ϕ yields

$$\frac{\lambda_2(1-2a)}{a(1-a)} \sum_{i=1}^n \phi_i^3 + \lambda'(0)a(1-a) = 0.$$

Consequently, we get the formula

$$\lambda'(0) = \frac{\lambda_2(2a-1)}{a^2(1-a)^2} \sum_{i=1}^n \phi_i^3. \tag{3.6}$$

If this quantity is non-zero, the transcritical bifurcation occurs at $(\mathtt{a}, \lambda_B)$.

Lemma 3.1 *(case 1a) Let the assumptions of Theorem 1.1 be satisfied with $\sum_{i=1}^n \phi_i^3 \neq 0$ and $a \neq \frac{1}{2}$. Then γ has the form of γ_1 given by (1.8).*

Proof The statement follows from the Taylor expansion (3.2) and the formula (3.6). In (1.8) we set

$$c := \lambda'(0) = \frac{\lambda_2(2a-1)}{a^2(1-a)^2} \sum_{i=1}^n \phi_i^3 \neq 0.$$

□

In all other cases in Theorem 1.1 we have

$$\lambda'(0) = 0, \tag{3.7}$$

thanks to either $\sum_{i=1}^{n} \phi_i^3 = 0$ or $a = \frac{1}{2}$. The type of bifurcation (sub-/supercritical) is then given by the sign of $\lambda''(0)$. Computing the third derivative of (3.1) at $s = 0$ we obtain

$$\begin{aligned} &F_{uuu}(\mathtt{a}, \lambda_B)[u'(0), u'(0), u'(0)] + 3F_{uu\lambda}(\mathtt{a}, \lambda_B)[u'(0), u'(0)]\lambda'(0) \\ &\quad +3F_{u\lambda\lambda}(\mathtt{a}, \lambda_B)u'(0)(\lambda'(0))^2 + F_{\lambda\lambda\lambda}(\mathtt{a}, \lambda_B)(\lambda'(0))^3 + 3F_{uu}(\mathtt{a}, \lambda_B)[u'(0), u''(0)] \\ &\quad +3F_{u\lambda}(\mathtt{a}, \lambda_B)u''(0)\lambda'(0) + 3F_{u\lambda}(\mathtt{a}, \lambda_B)u'(0)\lambda''(0) + 3F_{\lambda\lambda}(\mathtt{a}, \lambda_B)\lambda'(0)\lambda''(0) \\ &\quad +F_u(\mathtt{a}, \lambda_B)u'''(0) + F_\lambda(\mathtt{a}, \lambda_B)\lambda'''(0) = 0. \end{aligned} \tag{3.8}$$

Using (2.5), (3.4), (3.7), and

$$\begin{aligned} &F_{uuu}(\mathtt{a}, \lambda_B) = -\frac{6\lambda_2}{a(1-a)}\mathbb{I}_4, \quad F_{uu\lambda}(\mathtt{a}, \lambda_B) = 2(1-2a), \\ &F_{u\lambda\lambda}(\mathtt{a}, \lambda_B) = 0, \quad F_{\lambda\lambda\lambda}(\mathtt{a}, \lambda_B) = 0, \end{aligned} \tag{3.9}$$

we simplify (3.8) into

$$-\frac{6\lambda_2}{a(1-a)}\phi^{\circ 3} + \frac{6\lambda_2(1-2a)}{a(1-a)}(\phi \circ u''(0)) + 3a(1-a)\lambda''(0)\phi + F_u(\mathtt{a}, \lambda_B)u'''(0) = 0, \tag{3.10}$$

in which $\phi^{\circ 3}$ is the third Hadamard power of ϕ. By the scalar multiplication of (3.10) with ϕ we obtain

$$-\frac{2\lambda_2}{a(1-a)}\|\phi\|_4^4 + \frac{2\lambda_2(1-2a)}{a(1-a)}(\phi^{\circ 2}, u''(0)) + a(1-a)\lambda''(0) = 0,$$

in which we apply $(\phi \circ u''(0), \phi) = (\phi^{\circ 2}, u''(0))$ and $\phi \in \text{Ker}\, F_u(\mathtt{a}, \lambda_B) = (\text{Im}\, F_u(\mathtt{a}, \lambda_B))^{\perp}$, $\|\phi\| = 1$ once again. The formula for $\lambda''(0)$ now reads as

$$\lambda''(0) = \frac{2\lambda_2}{a^2(1-a)^2}\left(\|\phi\|_4^4 + (2a-1)(\phi^{\circ 2}, u''(0))\right). \tag{3.11}$$

In the symmetric case $a = \frac{1}{2}$ we can reduce this formula to get the supercritical bifurcation at $(\mathtt{a}, \lambda_B)$.

Lemma 3.2 *(case 1b) Let the assumptions of Theorem 1.1 be satisfied with $\sum_{i=1}^{n} \phi_i^3 \neq 0$ and $a = \frac{1}{2}$. Then γ has the form of γ_2 given by (1.9).*

Proof The formula (3.6) implies that $\lambda'(0) = 0$ for $a = \frac{1}{2}$. We then obtain from (3.11) that

$$\lambda''(0) = 32\lambda_2\|\phi\|_4^4 > 0.$$

Therefore, the statement follows again from the Taylor expansion (3.2) which yields the curve (1.9) with $c := \frac{1}{2}\lambda''(0) = 16\lambda_2\|\phi\|_4^4 > 0$. □

In the remaining cases, $\lambda''(0)$ in (3.11) can be both positive and negative.

Lemma 3.3 *(cases 2, 3) Let the assumptions of Theorem 1.1 be satisfied with $\sum_{i=1}^{n} \phi_i^3 = 0$. Then,*

(i) if $\left((-L + \lambda_2 I)^+\phi^{\circ 2}, \phi^{\circ 2}\right) \geq 0$ then γ has the form of γ_2 given by (1.9),
(ii) if $\left((-L + \lambda_2 I)^+\phi^{\circ 2}, \phi^{\circ 2}\right) < 0$ then there exists $\delta \in \left(0, \frac{1}{2}\right)$ such that γ has the form of

(a) γ_2 given by (1.9) if $\left|a - \frac{1}{2}\right| < \delta$, or
(b) γ_3 given by (1.10) if $\left|a - \frac{1}{2}\right| > \delta$.

Proof Since $\sum_{i=1}^n \phi_i^3 = 0$, the formula (3.6) yields that $\lambda'(0) = 0$ again. We thus obtain from (3.5)

$$F_u(\mathrm{a}, \lambda_B)u''(0) = \frac{2\lambda_2(2a-1)}{a(1-a)}\phi^{\circ 2}. \tag{3.12}$$

From the decomposition $\mathbb{R}^n = \operatorname{Ker} F_u(\mathrm{a}, \lambda_B) \oplus \operatorname{Im} F_u(\mathrm{a}, \lambda_B)$ we can write

$$u''(0) = \alpha\phi + \psi, \qquad \alpha \in \mathbb{R}, \quad \psi \in \operatorname{Im} F_u(\mathrm{a}, \lambda_B),$$

and (3.12) then yields that

$$F_u(\mathrm{a}, \lambda_B)u''(0) = F_u(\mathrm{a}, \lambda_B)\psi = \frac{2\lambda_2(2a-1)}{a(1-a)}\phi^{\circ 2}.$$

Then applying the Moore–Penrose pseudoinverse matrix $F_u^+(\mathrm{a}, \lambda_B)$ of $F_u(\mathrm{a}, \lambda_B)$ we get

$$\psi = F_u^+(\mathrm{a}, \lambda_B)F_u(\mathrm{a}, \lambda_B)\psi = \frac{2\lambda_2(2a-1)}{a(1-a)}F_u^+(\mathrm{a}, \lambda_B)\phi^{\circ 2}$$

and thus,

$$u''(0) = \alpha\phi + \frac{2\lambda_2(2a-1)}{a(1-a)}F_u^+(\mathrm{a}, \lambda_B)\phi^{\circ 2}. \tag{3.13}$$

Furthermore, if we put (3.13) into (3.11) and use $(\phi^{\circ 2}, \phi) = \sum_{i=1}^n \phi_i^3 = 0$, we get

$$\lambda''(0) = \frac{2\lambda_2}{a^2(1-a)^2}\left(\|\phi\|_4^4 + \frac{2\lambda_2(2a-1)^2}{a(1-a)}\left(F_u^+(\mathrm{a}, \lambda_B)\phi^{\circ 2}, \phi^{\circ 2}\right)\right).$$

Let us denote further

$$\sigma := \left(F_u^+(\mathrm{a}, \lambda_B)\phi^{\circ 2}, \phi^{\circ 2}\right) = \left((-L + \lambda_2 I)^+\phi^{\circ 2}, \phi^{\circ 2}\right).$$

Note that the value of σ depends only on the graph properties and does not depend on $a \in (0, 1)$. Therefore,

$$\lambda''(0) = \frac{2\lambda_2}{a^2(1-a)^2}\left(\|\phi\|_4^4 + \frac{2\lambda_2(2a-1)^2\sigma}{a(1-a)}\right). \tag{3.14}$$

If $\sigma > 0$, there is immediately $\lambda''(0) > 0$. Then the former statement of lemma follows from the Taylor expansion (3.2) setting $c := \frac{1}{2}\lambda''(0) > 0$ in (1.9).

Assume now that $\sigma < 0$ and define the function

$$r(a) = -\frac{2\lambda_2(2a-1)^2\sigma}{a(1-a)}.$$

The function $r(a)$ is continuous, strictly decreasing for $a \in \left(0, \frac{1}{2}\right)$, strictly increasing for $a \in \left(\frac{1}{2}, 1\right)$, $r\left(\frac{1}{2}\right) = 0$, and $r(a) \to \infty$ for $a \to 0+$ or $a \to 1-$. Consequently, there exist two intersections of $r(a)$ with the constant $\|\phi\|_4^4$, which can be computed as

$$a_i = \frac{1}{2} \mp \frac{\|\phi\|_4^2}{2\sqrt{\left(\|\phi\|_4^4 - 8\lambda_2\sigma\right)}} \in (0, 1), \qquad i = 1, 2.$$

Moreover, $r(a) > \|\phi\|_4^4$ for $a \in (0, a_1) \cup (a_2, 1)$, or $r(a) < \|\phi\|_4^4$ for $a \in (a_1, a_2)$. If we denote

$$\delta := \frac{\|\phi\|_4^2}{2\sqrt{\left(\|\phi\|_4^4 - 8\lambda_2\sigma\right)}} < \frac{1}{2}, \tag{3.15}$$

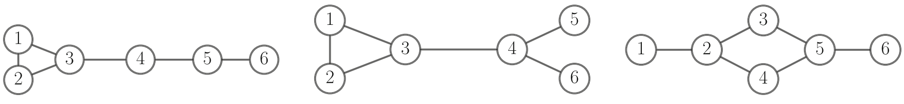

Fig. 2 Three graphs illustrating different bifurcation behavior of the GDE (1.5) in Sect. 4

we obtain from (3.14) that $\lambda''(0) > 0$ for $\left|a - \frac{1}{2}\right| < \delta$, or $\lambda''(0) < 0$ for $\left|a - \frac{1}{2}\right| > \delta$. Thus, the latter statement of lemma follows again from (3.2) putting $c := \frac{1}{2}\lambda''(0) > 0$ in (1.9) for $\left|a - \frac{1}{2}\right| < \delta$, or $c := -\frac{1}{2}\lambda''(0) > 0$ in (1.10) for $\left|a - \frac{1}{2}\right| > \delta$. ◻

This lemma is the last ingredient to the proof of our main result.

Proof of Theorem 1.1 The existence and uniqueness of the smooth bifurcating branch is proved by Lemma 2.2. Furthermore, Lemmas 3.1 – 3.3 verify the particular cases 1 – 3 depending on graph properties, i.e., the sum $\sum_{i=1}^{n} \phi_i^3$ and the scalar product $\left((-L + \lambda_2 I)^{+}\phi^{\circ 2}, \phi^{\circ 2}\right)$, and on the critical value $a \in (0, 1)$ from the bistable nonlinearity. This finishes the proof of Theorem 1.1. ◻

Remark 3.4 Note that the proofs of Lemmas 3.1 – 3.3 provide exact values of c in bifurcation curves (1.8)–(1.10) and the value of δ (see (3.15)) which determines the behavior in case 3. We omitted these detailed expressions in the statement of Theorem 1.1 for the sake of clarity.

4 Examples and Numerical Simulations

In this section we illustrate Theorem 1.1 by three examples of the Nagumo GDE (1.5) on graphs with six vertices (see Fig. 2). Each corresponds to one of the three different bifurcation behavior described by Theorem 1.1.

In order to visualize the spatially heterogeneous solutions in higher dimensions ($\mathbb{R}^6$ in these three cases), we compute numerically and depict the coordinates

$$P_\phi(u) = (u, \phi) \tag{4.1}$$

of a solution u in the subspace generated by the normalized Fiedler vector ϕ, which determines the direction of bifurcation curves, see (1.8)–(1.10).

Example 4.1 (case 1) Let us consider the graph $\mathcal{G}_1$ in Fig. 2a. The Laplacian matrix

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

has the second eigenvalue $\lambda_2 \doteq 0.325$ with multiplicity one and the Fiedler vector $\phi \doteq (-0.419, -0.419, -0.283, 0.081, 0.419, 0.621)^\top$ satisfying

$$\sum_{i=1}^{6} \phi_i^3 \doteq 0.144.$$

Consequently, Theorem 1.1, case 1, implies that there is the transcritical bifurcation at $\left(\mathrm{a}, \frac{\lambda_2}{a(1-a)}\right)$ if $a \neq \frac{1}{2}$ and the supercritical pitchfork bifurcation for $a = \frac{1}{2}$. See Fig. 3a and b for a numerical illustration.

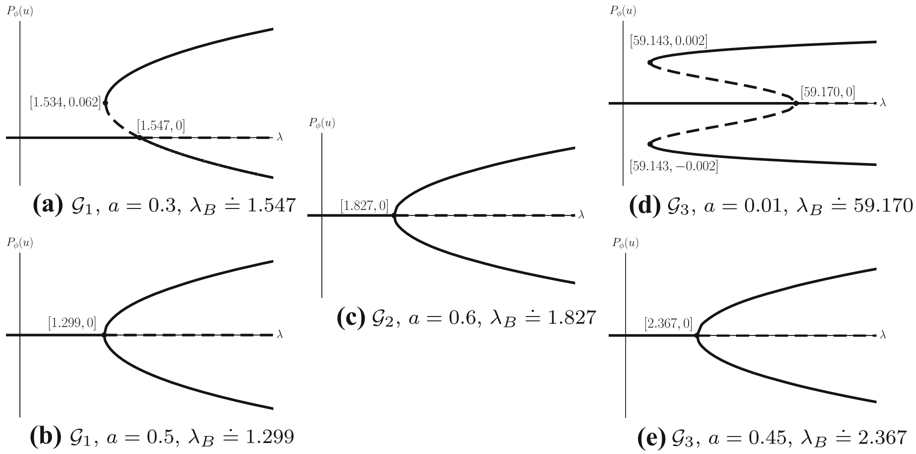

Fig. 3 Bifurcation diagrams of the GDE (1.5) on graphs $\mathcal{G}_1 - \mathcal{G}_3$ at (a, λ_B), see Examples 4.1 – 4.3. The value a is fixed and $P_\phi(u)$ given by (4.1) is the coordinate of solution u in the projection to the subspace generated by the Fiedler vector ϕ. The solutions lying on the solid lines have $n-1$ negative eigenvalues, those on the dashed lines have $n-2$ negative eigenvalues

Example 4.2 (case 2) Let the graph $\mathcal{G}_2$ be given by Fig. 2b. The Laplacian matrix

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

also has a simple algebraic connectivity $\lambda_2 \doteq 0.438$. However, the Fiedler vector $\phi \doteq (-0.465, -0.465, -0.261, 0.261, 0.465, 0.465)^\top$ satisfies

$$\sum_{i=1}^{6} \phi_i^3 = 0.$$

Since $\left((-L + \lambda_2 I)^+ \phi^{\circ 2}, \phi^{\circ 2}\right) \doteq 0.369 > 0$, Theorem 1.1, case 2, yields that there is the supercritical pitchfork bifurcation at $\left(\mathrm{a}, \frac{\lambda_2}{a(1-a)}\right)$ for all $a \in (0, 1)$. See Fig. 3c for a numerical bifurcation diagram.

Example 4.3 (case 3) Let us consider the graph $\mathcal{G}_3$ in Fig. 2c represented by the Laplacian matrix

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The algebraic connectivity $\lambda_2 \doteq 0.586$ is simple and the Fiedler vector $\phi \doteq (-0.653, -0.271, 0, 0, 0.271, 0.653)^\top$ also satisfies

$$\sum_{i=1}^{6} \phi_i^3 = 0.$$

In contrast to $\mathcal{G}_2$ in Example 4.2 we have $\left((-L+\lambda_2 I)^+\phi^{\circ 2}, \phi^{\circ 2}\right) \doteq -0.018 < 0$. Therefore, we can use (3.15) to compute $\delta \doteq 0.451$. Consequently, Theorem 1.1, case 3, yields that there is the supercritical pitchfork bifurcation at $\left(\mathtt{a}, \frac{\lambda_2}{a(1-a)}\right)$ for all $a \in \left(\frac{1}{2}-\delta, \frac{1}{2}+\delta\right) \doteq (0.049, 0.951)$ and the subcritical pitchfork bifurcation for all $a \in \left(0, \frac{1}{2}-\delta\right) \cup \left(\frac{1}{2}+\delta, 1\right) \doteq (0, 0.049) \cup (0.951, 1)$. See Fig. 3d and e for two numerical bifurcation diagrams illustrating both situations.

5 Generalization

The ideas and techniques of the proof of Theorem 1.1 in Sect. 3 can be straightforwardly applied to the problem (1.5) with arbitrary nonlinearity $g(u)$. The following result gives a more detailed insight about the way the derivatives of g at its stationary points, the algebraic connectivity of a graph, and the structure of its Fiedler vector determine the bifurcations of spatially heterogeneous solutions for small λ. On the other hand, we lose the interesting dependence on a parameter a, see Theorem 1.1, case 3, in the general setting.

Theorem 5.1 *Let $\mathcal{G} = (V, E)$ be a graph with a simple algebraic connectivity λ_2 and ϕ be its Fiedler vector. Let g satisfy for some $a \in \mathbb{R}$:*

(g_1) *g is sufficiently smooth (at least of class C^3),*
(g_2) *$g(a) = 0$,*
(g_3) *$g'(a) \neq 0$.*

Then there exists a unique smooth curve

$$\gamma : \begin{cases} u(s) = \mathtt{a} + \phi s + O(s^2), \\ \lambda(s) = \frac{\lambda_2}{g'(a)} + c_1 s + c_2 s^2 + O(s^3), \qquad c_1, c_2 \in \mathbb{R}, \qquad s \in (-\eta, \eta), \quad \eta > 0, \end{cases} \tag{5.1}$$

of hetereogeneous stationary solutions of (1.5) emanating from $\left(\mathtt{a}, \frac{\lambda_2}{g'(a)}\right)$. Moreover,

1. *if $\sum_{i=1}^n \phi_i^3 \neq 0$, then*
   (a) *if $g''(a) \neq 0$, then $c_1 \neq 0$, or*
   (b) *if $g''(a) = 0$, then $c_1 = 0$ and*
      (i) *$c_2 > 0$ provided $g'''(a) < 0$, or*
      (ii) *$c_2 < 0$ provided $g'''(a) > 0$,*
2. *if $\sum_{i=1}^n \phi_i^3 = 0$, then $c_1 = 0$ and*
   (a) *$c_2 > 0$ provided $\frac{3\lambda_2 (g''(a))^2}{g'(a)} \left((-L+\lambda_2 I)^+\phi^{\circ 2}, \phi^{\circ 2}\right) > g'''(a)\|\phi\|_4^4$, or*
   (b) *$c_2 < 0$ provided $\frac{3\lambda_2 (g''(a))^2}{g'(a)} \left((-L+\lambda_2 I)^+\phi^{\circ 2}, \phi^{\circ 2}\right) < g'''(a)\|\phi\|_4^4$.*

To illustrate the application of Theorem 5.1, we present the following example in which we show that the properties of the Fiedler vector ϕ can determine the type of bifurcation at $\left(\mathtt{a}, \frac{\lambda_2}{g'(a)}\right)$ also for symmetric bistable nonlinearities.

Example 5.2 Our main result, Theorem 1.1, implies that the bifurcation for the symmetric bistable nonlinearity $g_{\text{cub}}(u) = u(1-u)\left(u - \frac{1}{2}\right)$ at $(\mathtt{a}, 4\lambda_2)$ is always supercritical (given by γ_2 in (1.9), see Fig. 2) once we consider graphs with a simple algebraic connectivity. Note that

$$g_{\mathrm{cub}}\left(\tfrac{1}{2}\right) = 0, \qquad g'_{\mathrm{cub}}\left(\tfrac{1}{2}\right) = \frac{1}{4} > 0, \qquad g''_{\mathrm{cub}}\left(\tfrac{1}{2}\right) = 0, \qquad g'''_{\mathrm{cub}}\left(\tfrac{1}{2}\right) = -6 < 0. \tag{5.2}$$

We use Theorem 5.1 to prove that this is not the case if we consider a more complicated quintic (but still symmetric) bistable nonlinearity

$$g_{\mathrm{qui}}(u) = u(1-u)\left(u - \tfrac{1}{2}\right)\left(4 + 200\left(u - \tfrac{1}{2}\right)^2\right), \quad u \in \mathbb{R}, \tag{5.3}$$

in GDE (1.5). We can readily compute

$$g_{\mathrm{qui}}\left(\tfrac{1}{2}\right) = 0, \qquad g'_{\mathrm{qui}}\left(\tfrac{1}{2}\right) = 1 > 0, \qquad g''_{\mathrm{qui}}\left(\tfrac{1}{2}\right) = 0, \qquad g'''_{\mathrm{qui}}\left(\tfrac{1}{2}\right) = 276 > 0.$$

In contrast to $g'''_{\mathrm{cub}}\left(\tfrac{1}{2}\right) < 0$ in the cubic case (5.2), the inequality $g'''_{\mathrm{qui}}\left(\tfrac{1}{2}\right) > 0$ in combination with Theorem 5.1 ensures different types of bifurcations at $(\mathtt{a}, \lambda_2)$ based on the value of $\sum_{i=1}^n \phi_i^3$ in the quintic case (5.3).

First, let us consider $\mathcal{G}_2$ from Fig. 2b. The Fiedler vector satisfies $\sum_{i=1}^n \phi_i^3 = 0$ (see Example 4.2). Consequently, the bifurcation at $(\mathtt{a}, \lambda_2)$ is supercritical as well by Theorem 5.1, case 2a.

However, the situation changes if we consider the graph $\mathcal{G}_1$ from Fig. 2a. The Fiedler vector ϕ satisfies $\sum_{i=1}^n \phi_i^3 \neq 0$ (see Example 4.1). We can apply Theorem 5.1, case 1b(ii), to show that the bifurcation at $(\mathtt{a}, \lambda_2)$ is subcritical ($c_1 = 0$ and $c_2 < 0$ in (5.1)).

Remark 5.3 Considering the common cubic bistable nonlinearity $g(u)$ given by (1.2), Theorem 5.1 provides also information about bifurcations from constant solutions $(0, 0, \ldots, 0)^\top$ and $(1, 1, \ldots, 1)^\top$. However, the negative derivatives $g'(0) = -a < 0$ and $g'(1) = a-1 < 0$ imply that the bifurcations occur at $\lambda_B = -\frac{\lambda_2}{a} < 0$ and $\lambda_B = \frac{\lambda_2}{a-1} < 0$. Consequently, they are irrelevant since we assume $\lambda > 0$. The same holds for general bistable nonlinearities, e.g., the quintic $g_{\mathrm{qui}}(u)$ given by (5.3) from Example 5.2.

6 Graph Theoretical Aspects and Conjectures

Our main result Theorem 1.1 and the examples from Sect. 4 lead us to two natural questions. How common are the graphs with a simple algebraic connectivity λ_2? How common are the three cases of Theorem 1.1 among those? We do not have definite answers but the following insights and numerical experiments not only connect our results to very interesting questions in algebraic graph theory but also indicate that graphs with simple λ_2 are prevalent and among those, graphs with $\sum_{i=1}^n \phi_i^3 \neq 0$ (i.e., case 1 in Theorem 1.1) dominate.

Our main result Theorem 1.1 is based on the application of Crandall-Rabinowitz theorem and rely thus on the simplicity of λ_2. Despite the broad applicability of both the algebraic connectivity λ_2 and Fiedler vectors ϕ, there exist only scarce results regarding simplicity of λ_2. It can be shown that special classes of graphs have a simple algebraic connectivity λ_2 (paths P_n, complete graphs without an edge $K_n \setminus e$, graphs $C_1 - v - C_2$ with a cutting vertex v and Perron components C_1, C_2, all minimal asymmetric graphs, etc.) and there are other classes which have λ_2 with higher multiplicity (complete graphs K_n, cyclic graphs C_n, star graphs S_n, bipartite graphs $B_{m,n}$, etc.).

However, there is no proven connection between the simplicity of λ_2 and other graph characteristics. To illustrate, let us discuss one of the most natural candidate - graph asymmetry. A graph $\mathcal{G}$ is symmetric if there exists a nontrivial (i.e., nonidentical) graph automorphism

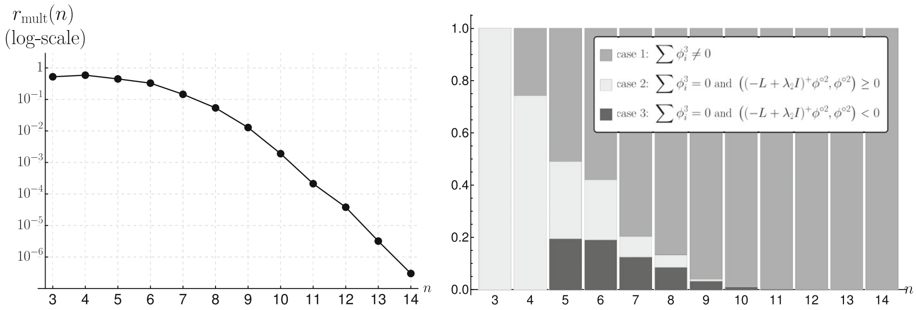

Fig. 4 Numerical simulations illustrating the descending ratio $r_{\text{mult}}(n)$ of graphs with n vertices whose algebraic connectivity λ_2 is not simple (left panel) and the distribution of cases from Theorem 1.1 among graphs with simple λ_2 (right panel)

$a : V \to V$ such that if $(u, v) \in E$ then $(a(u), a(v)) \in E$. Graphs with only trivial automorphisms are called asymmetric. All above examples of graphs with λ_2 with higher multiplicity are symmetric graphs. However, symmetric graphs can have simple λ_2 as well, pathgraphs or a graph $K_n - v - K_n$ are two elementary examples.

On the other hand, not all asymmetric graphs have simple λ_2. In this case we must search deeper among larger and more complicated graphs. The so-called strongly regular graphs can be constructed from Latin squares [7, Chapter 10]. All small Latin squares (and consequently small strongly regular graphs) have several automorphisms. However, starting from dimension 7, there exist Latin squares without nontrivial automorphisms. Such Latin squares of dimension 7 can be used to construct a graph on 49 vertices, which is strongly regular – it is 18-regular, every two adjacent vertices have 7 common neighbors, every two non-adjacent vertices have 6 common neighbors. More importantly, this graph is due to lack of automorphism of the Latin square also asymmetric and the corresponding algebraic connectivity λ_2 has multiplicity 18.

To conclude, despite the fact that graphs with λ_2 of higher multiplicity are common, it seems likely that their ratio tends to zero as the graph size $n = |V|$ tends to infinity. The left panel in Fig. 4 shows the ratio of graphs with simple algebraic connectivity among all graphs with given $|V|$. Theoretically, [26] proved that almost all random graphs have adjacency matrices with simple spectrum. Given the relationship between adjacency and Laplacian matrices, this result suggests that a similar statement could hold for Laplacian matrices as well.

Our numerical simulations (see the right panel in Fig. 4) indicate that the majority of graphs has not only a simple algebraic connectivity but also satisfies $\sum_{i=1}^{n} \phi_i^3 \neq 0$, i.e., case 1 in Theorem 1.1.

Going back to the Nagumo GDE (1.5) with the specific cubic nonlinearity (1.2), there are still delicate questions to be investigated. In the case of parameters leading to the supercritical bifurcation, the numerical results (e.g., Example 4.2) indicate that there are no spatially heterogeneous solutions for $\lambda < \lambda_B = \frac{\lambda_2}{a(1-a)}$ which would imply that $\underline{\lambda} = \lambda_B$. Note that our proof provides only a local information of one of the solution curves and thus the following statement is a mere conjecture.

Conjecture 1 et $\mathcal{G} = (V, E)$ be a graph with a simple algebraic connectivity λ_2 and let the assumptions of Theorem 1.1 cases 1b, 2 or 3a hold. Then $\underline{\lambda} = \lambda_B = \frac{\lambda_2}{a(1-a)}$.

In other cases, the numerical experiments (e.g., Examples 4.1 and 4.3) suggest the existence of saddle-node bifurcations at a value $\underline{\lambda} < \lambda_B$, see Fig. 3. Our paper does not provide any technique on the localization of $\underline{\lambda}$ in this case. Similarly, the bifurcation behavior for graphs with λ_2 of higher multiplicity remains to be described. Crandall-Rabinowitz theorem cannot be applied in this case.

Finally, the reverse question could be of interest as well. If $\lambda \gg d$, the problem (1.1) has 3^n solutions. Define $\overline{\lambda}$ as an infimum of all values of λ for which there are 3^n stationary solutions. What is the value of $\overline{\lambda}$? There are only rough estimates on $\overline{\lambda}$, [25].

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