Ambiguous Digitizations by Dilation

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ABSTRACT

This article deals with ambiguous surface digitizations by dilation in n-dimensional space. The digitization of a sufficiently regular surface is separating but not necessarily minimal. We will determine conditions under which the supercover and the grid intersection digitizations are discrete surfaces. It will also be proven that non-overlapping domains do not solve the problem of simple points in digitizations. No matter how the digitization domain is chosen there will occur ambiguous cases which have to be treated differently.

Keywords: n-dimensional raster graphics, digitization, discrete surfaces, simple points

1 Introduction

In raster graphics continuous objects are represented by discrete sets. Usually they are considered to be the digitization of existing or synthetic continuous objects. These discrete objects can only partially reflect the properties of the original continuous objects. Ideally, important features or properties of a continuous object should be found in appropriate discrete counterparts. The preservation of properties has been studied on various classes of objects [Veerl93, Agraw97]. Our previous work [Linck01] determined conditions under which the separation property of surfaces is preserved. Another important aspect of digitizations is the occurrence of ambiguities. Algorithms usually force uniqueness by half-open digitization domains [Cohen95, Cohen96] and special rules for the remaining ambiguous cases. Although such problems can easily be overcome in practice, these special cases cause major problems in the mathematical study of digitizations.

This paper is outlined as follows: The next section recalls basic definitions. Section 3 deals with simple points in digitized hyperplanes. We determine conditions under which simple points do not occur. In Section 4, we study digitizations by dilations with half-open domains and prove that they will always cause special cases.

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2 Basic Definitions

In raster graphics n-dimensional digital images are n-dimensional arrays of integer values. If these values are only 0 and 1 we speak of *binary digital images*. *Discrete objects* are subsets of $\mathbb{Z}^n$, that is the set of points which are associated with the value 1. An element $z \in \mathbb{Z}^n$ is known as a *grid point*. Its *Voronoi set* $\mathbb{V}(z)$, the set of all points in $\mathbb{R}^n$ which are at least as close to z as to any other grid point, forms a closed axes-aligned n-dimensional unit cube with center z. These sets are called *pixels* in 2D and *voxels* in 3D. Two n-dimensional Voronoi sets can share a point, a straight line segment, up to an $(n-1)$-dimensional cube. This fact leads to a definition of neighborhood.

Two grid points $z, z' \in \mathbb{Z}^n$ are said to be *k-neighbors* $(0 \leq k \leq n-1)$ if $\dim(\mathbb{V}(z) \cap \mathbb{V}(z')) \geq k$. A sequence $(z_0, \ldots, z_l)$ of points of an object $A \subseteq \mathbb{Z}^n$ is said to be a *k-arc* from z_0 to z_l in A if successive elements are k-neighbors. An object $A \subseteq \mathbb{Z}^n$ is *k-connected* if there exists a k-arc from z to z' for any points $z, z' \in \mathbb{Z}^n$. A *k-component* of $A \subseteq \mathbb{Z}^n$ is defined as a maximal k-connected non-empty subset of A. A discrete object $A \subseteq \mathbb{Z}^n$ is said to be *k-separating* if the background $\mathbb{Z}^n \setminus A$ consists of exactly two k-components. A k-separating object A is *minimal* if $A \setminus \{z\}$ is not k-separating for all $z \in A$. A k-separating surface is a minimal k-separating object. For point sets A, B the *dilation* $A \oplus B$ is given by $A \oplus B = \{a + b : a \in A, b \in B\}$.

Digitization is the transformation of a continuous object into a discrete one. An important class are digitizations by dilation. A grid point belongs to the digitization, if and only if a fixed set, the so-called *domain*, translated to this grid point hits the continuous objects. The *digitization by dilation* $\Delta_{\oplus}^{D}$: $\wp(\mathbb{R}^n) \rightarrow \wp(\mathbb{Z}^n)$ *with domain* $D \subseteq \mathbb{R}^n$ is defined as $\Delta_{\oplus}^{D}(A) = \{z \in \mathbb{Z}^n : A \cap D_z \neq \emptyset\}$ for every continuous object $A \subseteq \mathbb{R}^n$. A digitization by dilation with domain $D = \mathbb{V}(0)$ is called *supercover digitization* and indicated by Δ_{SC}. Let L_i $(1 \leq i \leq n)$ be the straight line segment obtained by intersecting the i-th coordinate axis with $\mathbb{V}(0)$. The digitization with domain $D = \bigcup_{i=1}^{n} L_i$ is called *grid intersection digitization* Δ_{GI}.

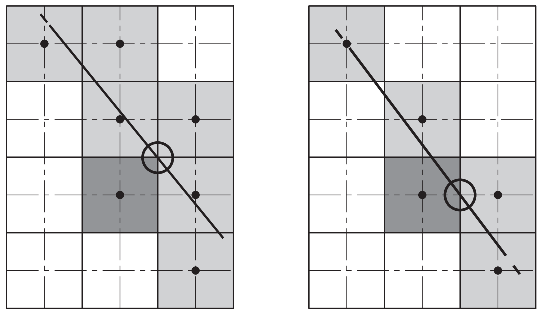

Figure 1: The supercover and the grid intersection digitization of a straight line is not necessarily minimal.

3 Simple Points in Surface Digitizations

A continuous or discrete surface without boundary can be characterized as a minimal separating set: If S is a surface without boundary, then its complement S^C consists of exactly two components and the removal of any surface point destroys this property.

Theorem 1. *A k-separating set $S \subseteq \mathbb{Z}^n$ $(0 \leq k < n)$ is minimal iff every point $z \in S$ is a k-neighbor of each k-component of S^C.*

Proof. Let $S \subseteq \mathbb{Z}^n$ be a k-separating. The two k-component of S^C shall be denoted by A and B. Let us first assume that S is minimal. By definition $A \cup B \cup \{z\}$ must be a k-connected set for every $z \in S$. Hence, there exists a k-path between every two points in $A \cup B \cup \{z\}$. In particular, there are k-paths from z to every point in A and B. Consequently, z is a k-neighbor of A and B. Conversely, if $z \in S$ is a k-neighbor of A and B then the set $A \cup B \cup \{z\}$ is trivially k-connected. □

Cohen-Or et al. [Cohen95, Cohen96] and Andres et al. [Andre97] investigated the supercover digitization of straight lines in 2D and planes in 3D. They came to the conclusion that the supercover of some lines or planes are not minimal. Both groups proposed refined digitizations schemes to overcome such situations. As shown in the 2D examples (Fig. 1) the supercover and the grid intersection digitization are not necessarily minimal. The supercover of a straight line in $\mathbb{R}^2$ is a 1-separating surface, except if the line passes through a corner of one, and thus four, pixel. In this case the result is not minimal because each of these four pixels has more than two 1-neighbors. Andres called these configurations *bubbles* [Andre99]. We will refer to them as *simple points*. In general, a simple point of a discrete object can be removed without changing the topological properties of the object. In the case of surface digitizations, a simple point is one that can be removed from the digitization without violating the separation property.

Simple points occur in both schemes because two neighbored digitization domains can have common points. Let z and z' be two different points in $\mathbb{Z}^n$. If z and z' are 0-neighbors then their Voronoi sets $\mathbb{V}(z)$ and $\mathbb{V}(z')$, which are equal to the domain of the supercover translated to these points, are not disjoint. For n-neighbored grid points the translations of the domain $D = \bigcup_{i=1}^{n} L_i$ consists of a common point, i.e. $D_z \cap D_{z'} \neq \emptyset$.

Theorem 2. *Let H be a hyperplane in $\mathbb{R}^n$. Its supercover $\Delta_{SC}(H)$ is a minimal 0-separating set if and only if $z \oplus (0.5, \ldots, 0.5) \notin H$ for every grid point $z \in \mathbb{Z}^n$.*

Proof. Let H be a hyperplane in $\mathbb{R}^n$. A and B denote the two components of $\mathbb{Z}^n \setminus \Delta_{SC}(H)$. The first direction will be proven by reductio ad absurdum, the other part of the proof is straight forward.
We assume first that the supercover $\Delta_{SC}(H)$ is a minimal 0-separating set and that there exists a point $z \in \mathbb{Z}^n$ with $z \oplus (0.5, \ldots, 0.5) \in H$. Consequently, all 2^n grid points whose Voronoi sets contain the vertex $z \oplus (0.5, \ldots, 0.5)$ belong to the supercover $\Delta_{SC}(H)$. H hits at least two of these voxels only on their boundary. Without loss of generality we can assume that one of them is z. Otherwise we rename the axes accordingly. Now we obtain $H \cap \mathrm{int}(\mathbb{V}(z)) = \emptyset$, where int denotes the *interior* of a closed set. Because of Theorem 1 the point z has the 0-neighbors $a \in A$ and $b \in B$ in each component of the background. Since $a, b \notin \Delta_{SC}(H)$, we have $\mathbb{V}(a) \cap H = \emptyset$ and $\mathbb{V}(b) \cap H = \emptyset$. Parts of the boundary of the voxel $\mathbb{V}(z)$ are not hit by the hyperplane H, i. e. $\mathbb{V}(a) \cap \mathbb{V}(z) \cap H = \emptyset$ and $\mathbb{V}(b) \cap \mathbb{V}(z) \cap H = \emptyset$. In fact, H must separate the voxel $\mathbb{V}(z)$ into two continuous components where each of them contains one of these boundary segments $\mathbb{V}(a) \cap \mathbb{V}(z)$ and $\mathbb{V}(b) \cap \mathbb{V}(z)$. This is a contradiction to $H \cap \mathrm{int}(\mathbb{V}(z)) = \emptyset$.

Let us now assume that $z \oplus (0.5, \ldots, 0.5) \notin H$ for every grid point $z \in \mathbb{Z}^n$. Hence, H does not hit any vertex of any voxel $\mathbb{V}(z)$ that belongs to the supercover. Consequently, the hyperplane H separates every voxel $\mathbb{V}(z)$ into two continuous components. In $\mathbb{V}(z)$ there exists at least one pair of opposite vertices that belong to different components. Again without loss of generality, these vertices are $z \oplus (0.5, \ldots, 0.5)$ and $z \oplus (-0.5, \ldots, -0.5)$. Then, the points $z \oplus (1, \ldots, 1)$ and $z \oplus (-1, \ldots, -1)$ are 0-neighbors of z. One of them belongs to set A and the other to set B. This proves that $\Delta_{SC}(H)$ is minimal. □

This theorem says that simple points occur if and only if there exist two 0-neighbors $z, z' \in \mathbb{Z}^n$ such that the hyperplane H hits only the common boundary of the translated digitization domains D_z and $D_{z'}$ and not their interior. A similar relationship holds for the grid intersection digitization. The domain D of the grid intersection digitization was the bundle of the coordinate axes intersected with $\mathbb{V}(0)$. D^o denotes the intersection of D with $\mathrm{int}(\mathbb{V}(0))$, which is essentially D without the end points of each straight line segment.

Theorem 3. *Let H be a hyperplane in $\mathbb{R}^n$. Its grid intersection digitization $\Delta_{GI}(H)$ is not a minimal $(n-1)$-separating set iff there exist two $(n-1)$-neighbors $z, z' \in \mathbb{Z}^n$ such that $(D_z \cap D_{z'}) \subseteq H$, $D_z^o \cap H = \emptyset$ and $D_{z'}^o \cap H = \emptyset$.*

Proof. Let H be a hyperplane in $\mathbb{R}^n$. H_A and H_B denote the two continuous components of $\mathbb{R}^n \setminus H$ associated with A and B, the two discrete components of $\mathbb{Z}^n \setminus \Delta_{GI}(H)$. Again, the first part will be proven by reductio ad absurdum.
Let us assume that $\Delta_{GI}(H)$ is not a minimal $(n-1)$-separating set then there must exist a point $z \in \Delta_{GI}(H)$ that has no $(n-1)$-neighbors in, say, the component A. Without loss of generality, we suppose that the first coordinate axis is the major axis of H, which is the coordinate axis with the largest coefficient in the hyperplane equation [Klett85], and the points $z = (0, \ldots, 0)$ and $z' = (1, 0, \ldots, 0)$ belong to $\Delta_{GI}(H)$. This means that H hits D_z and $D_{z'}$. Since the first axis is the major axis, H can only hit the point $(0.5, 0, \ldots, 0)$ and no points of D_z^0 and $D_{z'}^0$.
Conversely, we assume that there exist two $(n-1)$-neighbors $z, z' \in \mathbb{Z}^n$ such that $(D_z \cap D_{z'}) \subseteq H$, $D_z^o \cap H = \emptyset$ and $D_{z'}^o \cap H = \emptyset$. Without loss of generality, we suppose that $z = (0, \ldots, 0)$ and $z' = (1, 0, \ldots, 0)$ and that the component H_A contains the point z and $z' \in H_B$. The hyperplane hits the point $(0.5, 0, \ldots, 0)$, which belongs to both domains D_z and $D_{z'}$. Further H does not hit any other point of the domains. The only $(n-1)$-neighbor of z which is contained in the continuous component H_B is z', but z' does belong to $\Delta(H)$. Hence, z has no $(n-1)$-neighbor in B and $\Delta_{GI}(H)$ is not minimal. □

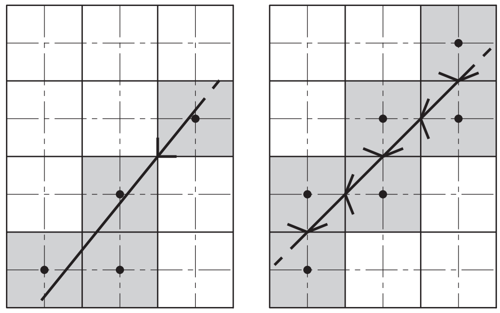

Figure 2: Critical situation for the modified supercover and grid intersection digitization.

As a consequence of this section simple points in the supercover and the grid intersection digitization of a hyperplane $H \subseteq \mathbb{R}^n$ can be easily determined. A point $z \in \Delta_{SC}(H)$ is simple if H hits the digitization domain D_z only in non-unique points, i.e. in points which also belong to the domain of a neighbored point.

4 Avoiding Simple Points

Digitization algorithms try to avoid simple points by using half-open digitization domains, so-called *reduced voxels* [Cohen95, Cohen96]. However, there are still ambiguous situations which have to be treated as special cases. We want to illustrate these situations in 2D. A digitization with the half open unit square as domain (Fig. 2, left) will produce thinner results than the supercover digitization, but there are cases in which the digitization is too thin. This happens when a line with slope in the first quadrant passes through a vertex of a pixel. Adding one more point to the domain will solve the problems for this class of straight lines, but in other cases the resulting digitization will not be minimal because of the overlapping domains. On the other hand (Fig. 2, right), the modified grid intersection with two half-open straight line segments as domain fails if a line with slope 1 hits the mid-points between two grid points. For the grid intersection digitization of hyperplanes it suffices to determine the intersections with the grid lines along the major [Klett85]. Algorithms force uniqueness in ambiguous cases by giving one of the axes a higher priority.

The next theorem proves that *ambiguity cannot be avoided*. No matter how the domain is chosen, there exists always hyperplanes whose digitization will either be not minimal or not separating. It states that it is impossible to find non-overlapping digitization domains such that the digitization is compatible with all motions that map $\mathbb{Z}^n$ onto $\mathbb{Z}^n$. We finally need to introduce the notion *tessellation* of the space $\mathbb{R}^n$, which is a collection of sets that cover the space without gaps or overlaps.

Theorem 4. *There exists no tessellation $\{D_z : z \in \mathbb{Z}^n\}$ of $\mathbb{R}^n$ which is compatible with the group of motions in $\mathbb{Z}^n$.*

Proof. Let $M_{\mathbb{Z},n}$ denote the set of all motions in the Euclidean space $\mathbb{R}^n$ that map $\mathbb{Z}^n$ onto $\mathbb{Z}^n$ and let $\{D_z : z \in \mathbb{Z}^n\}$ be a tessellation of $\mathbb{R}^n$. The tessellation D is compatible with $M_{\mathbb{Z},n}$ iff $\sigma(D_z) = D_{\sigma(z)}$ for every z.
We assume that $D \subseteq \mathbb{V}(0)$ and that the collection $\{D_z : z \in \mathbb{Z}^n\}$ consists of $\mathbb{R}^n$ with mutually disjoint sets. Let $x \in D$ be a point in D which is also located on the *boundary* of the $\mathbb{V}(0)$, $x \in \mathrm{bd}(\mathbb{V}(0))$. Without loss of generality, x can be written as $x = (0.5, x_2, \dots, x_n)$. Any reflection in the Euclidean space is a motion. The reflection τ on the hyperplane that is orthogonal to the first coordinate and passes through the point $(0.5, 0, \dots, 0)$ clearly maps $\mathbb{Z}^n$ onto $\mathbb{Z}^n$, so $\tau \in M_{\mathbb{Z},n}$, and $\tau(x) = x$. Because of $x \in D_{(0,\dots,0)}$, the condition $\tau(0) = (1, 0, \dots, 0)$ and the compatibility, we obtain $\tau(D_0) = D_{(1,0,\dots,0)}$ which is a contradiction. Thus, there exists not tessellation with the assumed properties. □

Consequently, a modified supercover with a half open domain will not be compatible with $M_{\mathbb{Z},n}$, no matter how the domain is chosen. Non-overlapping domains $D \subseteq \mathbb{V}(0)$ are incompatible if they have points on the boundary of $\mathbb{V}(0)$. This is particularly true for a grid intersection digitization with a set of half-open line segments as domain. Thus half-open domains will not avoid ambiguous cases for these digitizations. There are always locations of hyperplanes such that their digitization is either too thin or too thick, that is not separating or not minimal, respectively.

The number of voxels in a digitized object is countable and so is the number of intersection of a surface with points that cause simple points under digitization. Consequently, the surface can be translated by a vector of arbitrarily small length such that the resulting surface does not hit any of these critical points in $\mathbb{R}^n$. There exists a small enough $\epsilon > 0$, such that $\Delta_D(S \oplus \bar{\epsilon})$, the digitization of the translate of a surface $S \subseteq \mathbb{R}^n$ by a vector $\bar{\epsilon}$ ($|\bar{\epsilon}| < \epsilon$), is equal to the digitization $\Delta_{D'}(S)$ with the half-open domain $D' = D \cap [-0.5, 0.5)^n$.

As a consequence, for the general study of digitizations, their properties and their effect on given classes of objects, it is not necessary to deal with every special case or ambiguity caused by common points of two domains.

5 Summary And Future Work

In this article we studied simple points in the supercover and the grid intersection digitization of hyperplanes in n-dimensional space. It has been proven that ambiguity in surface digitizations cannot be avoided and the use of half-open domains will always cause special cases which have to be treated differently. This paper is part of a series on surface digitizations [Linck00, Linck01] and its result is important for our further research. We are currently investigating strategies to eliminate simple points in digitized surfaces.

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