A Generalized Mandelbrot Set Based On Distance Ratio

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ABSTRACT

The iteration of complex function can generate beautiful fractal images. This paper presents a novel method based on the iteration of the distance ratio with two points, which generates a generalized Mandelbrot set according to distance ratio convergence times. This paper states the definition of distance ratio and its iteration. Then taking the complex function $f(z)=z^{\alpha}+c$ for example, it discusses the visual structure of generalized Mandelbrot with various exponent and comparing it with Mandelbrot set generated by escape time algorithm. When exponent $\alpha>1$, the outer border of DRM is same as Mandelbrot set, but has complex inner structure; when $\alpha<0$, the inner border of DRM is same as Mandelbrot set, DRM is the "outer" region and complement set of Mandelbrot set, the two sets cover the whole complex plane.

Keywords

Fractal; Distance Ratio; complex mapping; Mandelbrot set

1. INTRODUCTION

Since the introduction of fractal by Mandelbrot [Man77a], fractals have experienced considerable success in quantifying the complex structure exhibited by many natural patterns and have captured the imaginations of both scientists and artists [Spe03a]. Many researchers perform research on the generation method of fractal and draw many beautiful images [Fal90a].

Besides the escape time algorithm, many researchers propose other methods for generating fractal structure. [Hoo91a] proposes epsilon cross and star trails methods to render the interior structure of Mandelbrot set. [Cal96a] presents the algorithm of rendering pseudo-3D effects of fractal in complex plane, and on the basis of which presents two artistic rendering methods for Newton fractals [Cal99a]. [Roc00a] proposes a generalized Mandelbrot Set based on bicomplex number in 3D space and discusses its properties. But seldom are these approaches used to generate fractal structure in convergent region of complex mapping.

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This paper proposes a new method based on iteration distance ratio with two points, which renders fractal images by convergent times of distance ratio. Comparing with escape time algorithm, it consists of abundant details in the convergent region and more complex structure can be generated. It also can be used as a new method to render interior or exterior structure of escape time fractal. This paper states the visual characteristic of generalized Mandelbrot set generated by above method, and compares its image with escape time fractal.

The paper is organized as follows: Section 2 states the definition of distance ratio and rendering method; Section 3 generates a generalized Mandelbrot Set for $f(z)=z^{\alpha}+c$ and analyzes its visual characteristic; Finally, section 4 summarizes the paper.

2. ITERATION OF DISTANCE RATIO

This section states the definition of distance ratio and the method of generating fractal structure by using distance ratio convergent times.

Let $f: C \rightarrow C$ be an analytic function, for any $z_1 \neq z_2 \in C$, define the distance ratio L as follow:

$$L = \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} \quad (1)$$

Perform the iteration on the initial points z_1 and z_2, let $z_1^{(n+1)} = f(z_1^{(n)})$, $z_2^{(n+1)} = f(z_2^{(n)}), n = 0,1,\cdots$, and define the distance ratio after n-times iteration $L^{(n)}$ as:

$$L^{(n)} = \frac{\left|z_1^{(n+1)} - z_2^{(n+1)}\right|}{\left|z_1^{(n)} - z_2^{(n)}\right|} \qquad (2)$$

If mapping f is contraction, the iteration of initial points will converge to the fixed-point. Thus, the distance ratio will also converge and we have following propositions:

Proposition 1: If mapping f has a fixed-point z^* and $A(z^*)$ is the attraction basin of z^*, for any z_1 and z_2 in $A(z^*)$, it follow $\lim_{k\to\infty} L^{(k)} = \left|f'(z^*)\right|$

Proposition 2: For a specified threshold $\varepsilon, \exists k^* \in N$, if $k>k^*$, $\left|L^{(k)} - \left|f'(z^*)\right|\right| < \varepsilon$

Proposition 1 states that distance ratio will converge by iterating; Proposition 2 states that distance ratio will converge within limited times for a certain threshold. According to these propositions, we can generate a new type fractal set. We call this kind of set the Distance Ratio Fractal, in short **DRF.**

If classify iterated points according to their distance ratio convergent times and render **DRF** in parameter space, we can give the definition of generalized Mandelbrot set based on Distance Ratio (**DRM**) as follows:

$$DRM = \{c \in C : \{\lim_{k\to\infty} L_c^{(k)} \to f'(z^*)\}\} \qquad (3)$$

Supposing c is a complex number and ε is a small real number, we say that c is k-level if c satisfies the following condition:

$$\begin{aligned} \left|L_c^{(k-1)} - f'(z^*)\right| &> \varepsilon \\ \left|L_c^{(k)} - f'(z^*)\right| &\le \varepsilon \end{aligned} \qquad (4)$$

The set consisting of all the k-level points is termed **zone**(k). The points in **zone**(k) approach the derivative after at least k iterations. The DRM can be divided into some **zone**(k), like:

$$DRM = \bigcup_{k=1}^{\infty} zone(k) \qquad (5)$$

Below is the rendering method of DRM. Its main idea is to travel the rendering region, so as to compute the convergent times of iterated points, then color these points according to their convergent times, and finally a fractal image is obtained.

Let rendering region D be a rectangle; top left corner be (-1.5, 1.2); bottom right corner be (1.5,-1.2); maximum iteration times n=10000; threshold ε=0.0000001. Steps of the algorithm are as follows:

1. Select point c in rendering region D; construct mapping f(z, c); let z_1=(0,0) z_2=(0.0001,0);

2. If $k<n$, perform the iteration $z_1^{(k)} = f(z_1^{(k-1)}, c)$, $z_2^{(k)} = f(z_2^{(k-1)}, c)$, else go to step 7;

3. Computing the distance ratio $L^{(k)} = \frac{\left|z_1^{(k+1)} - z_2^{(k+1)}\right|}{\left|z_1^{(k)} - z_2^{(k)}\right|}$;

4. If $\left|L^{(k)} - L^{(k-1)}\right| > \varepsilon$, k=k+1, go to step 3;

5. Color point c according to its iteration times;

6. Repeat steps 2-5, till all points in D are computed.

3. Image of DRM

Taking complex mapping f(z)=z^α+c as example, this section generates images of DRM and states conjectures on their visual characteristics.

A lot of research has been done about Mandelbrot set for f(z)=z^α+c generated by escape time algorithm. [Guj91a] generates many images of M-set, and states some conjectures on relationship between image structure and exponent value. [Wan00a] analyzes the image properties of negative exponent value and provides the strict mathematics prove. But previous work does not render images when -1<α<1(Escape time algorithm is not able to render fractal images when -1<α<1). The following will analyze the images under different conditions, while α>1, 0<α<1, and a<0.

3.1 DRM for α>1

Fig.1 shows the DRM obtaining by setting α=2. Its border is same as M-set generated by escape time algorithm, but there is very complex structure in its inner stable region. Its visual characteristic is listed as follows:

1. The border of DRM is exactly same as M-set generated by escape time algorithm.

2. There is layer structure in central quasi-circle and surrounding periodic buds.

3. Each layer is composed of some small areas like petals.

4. The inner petals are relatively in big size and small number; while outer petals are on the contrary, and the outer, the larger number, to infinity.

The layer structure is a typical characteristic of DRM. As points in same layer have same convergent times k, each layer is a zone(k). The portion marked by a square in Fig.1 is zoomed in and shown in Fig.2. From it we can see clear layer and petal structure.

It is the convergence region of the mapping that is computed by the distance ratio iteration method. The border of DRM is the boundary between divergent and convergent, so DRM and M-set have the same shape. Fig.3 and Fig.4 are generated when α=4 and α=1.4. Compared with images shown in [Guj91a], they have the same border. Therefore, distance ratio iteration method can be used as a new method for rendering inner structure of M-set when α>1.

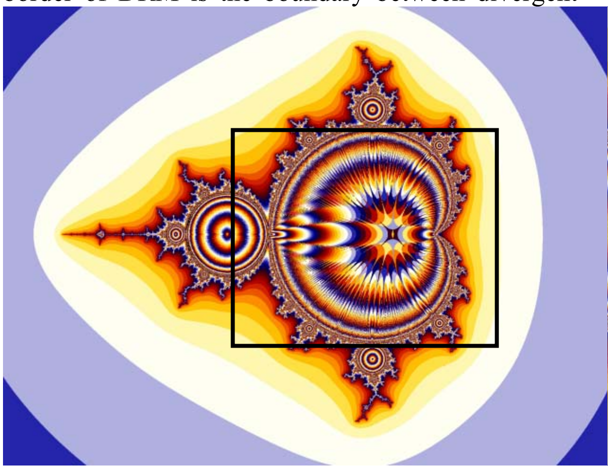

Figure 1. α=2

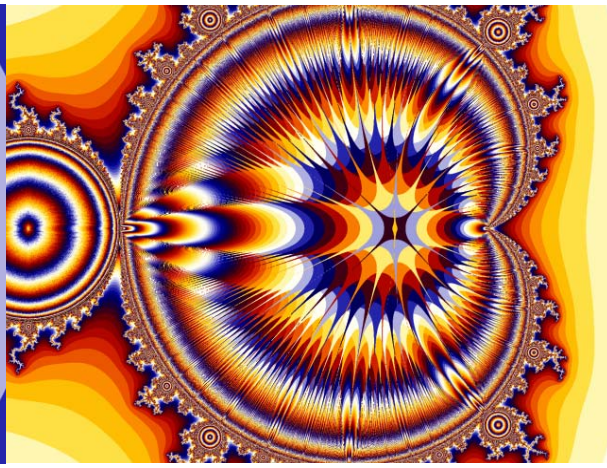

Figure 2. α=2 zoomed

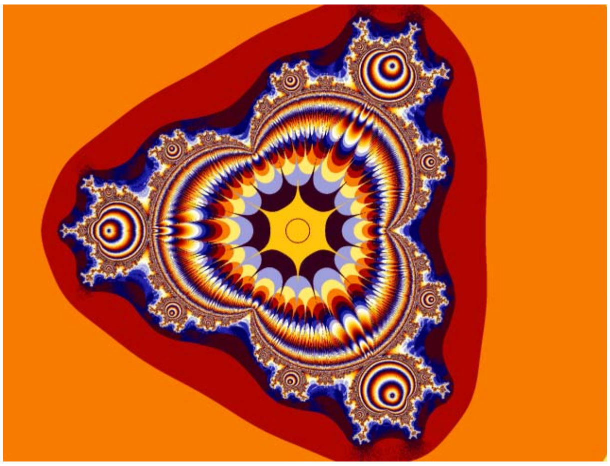

Figure 3. α=4

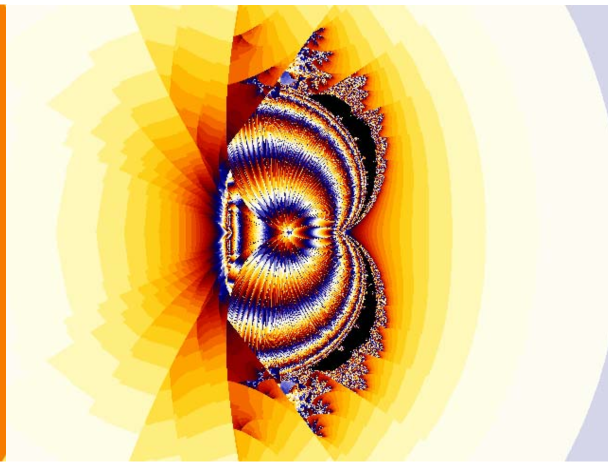

Figure 4. α=1.4

3.2 DRM for 0<α<1

Fig.5 illustrates that M-set generated by escape time algorithm is a kind of simple geometry image, because the attractive region of the mapping changes when -1<α<1. But DRM still has complex structure.

Fig.6 shows partly DRM obtained by setting α=0.1. It looks like a flower with five petals. There are fewer petals at outer layers with the minimum of one petal. When it goes to inner layers, the number of petals increases, so that a complex structure is formed. Fig.7-10 are gradually zoomed in images of a square area shown in Fig.6. From Fig.7 we can see the above-mentioned layer petal structure, in the center of which there is a spindle core like a “pistil”, shown in Fig.8. When the right part of the spindle is zoomed in, we can see a pseudo-3D “spray hole” in Fig.10, which keeps spraying the petal structures out from the hole.

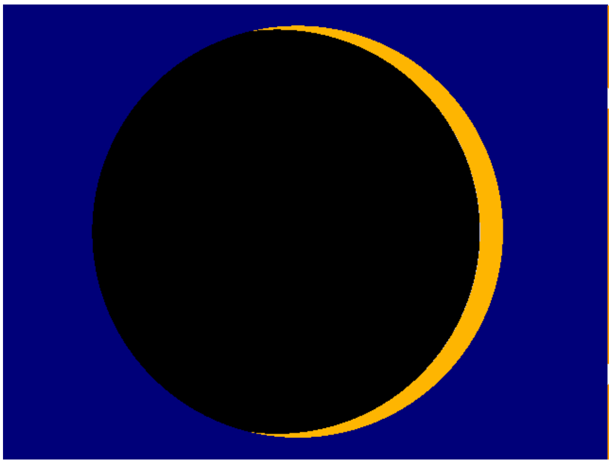

Figure 5. Mandelbrot set for α=0.1

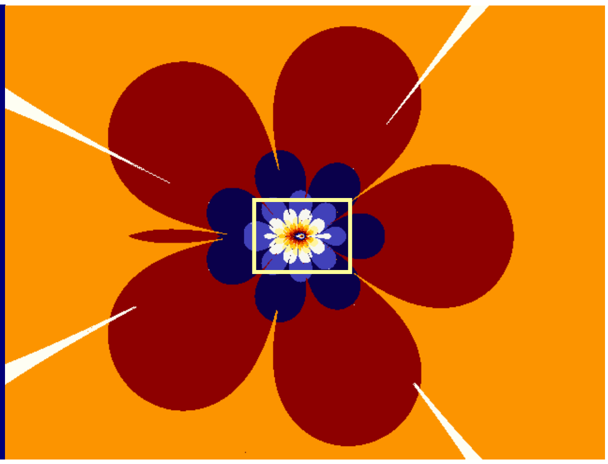

Figure 6. DRM for α=0.1

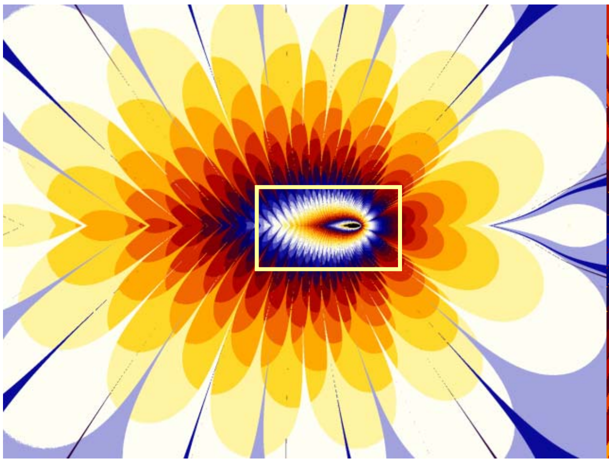

Figure 7. DRM for α=0.1 zoomed

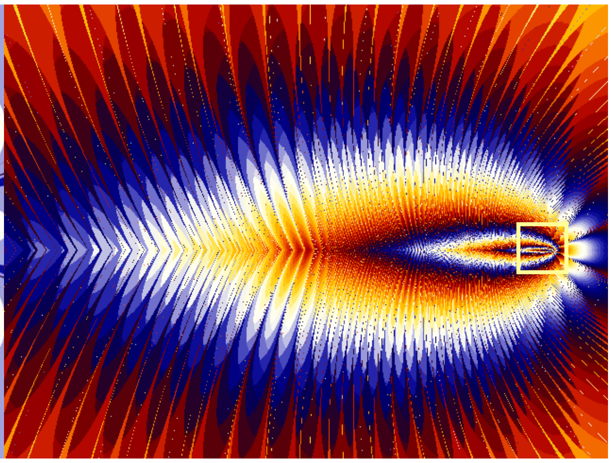

Figure 8. DRM for α=0.1 zoomed

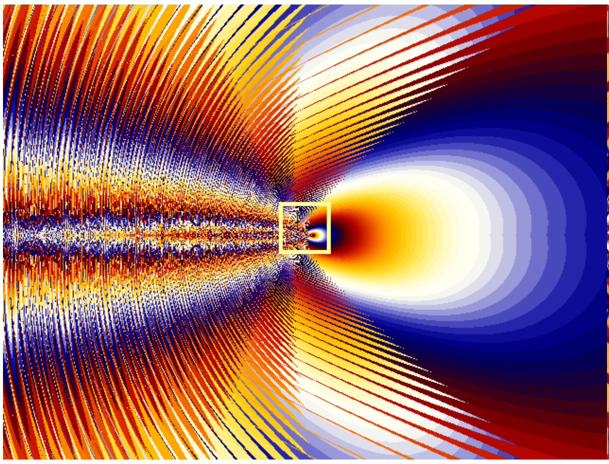

Figure 9. DRM for α=0.1 zoomed

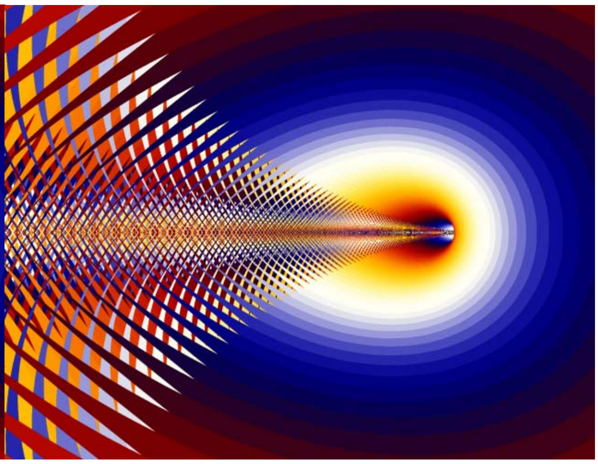

Figure 10. DRM for α=0.1 zoomed

3.3 DRFM for α<0

There are different structures in DRM when α<0. Fig.13 shows the DRM by setting α=-2. Fig.11 is α=-3 and Fig.12 is α=-4. Below characteristics are concluded from these images:

1. There is regular polygon structure with curve edge in the central of DRM.

2. Layer petal structure exists outside of the curve polygon, so is 0<α<1.

3. Central planet exists inside the curve polygon; the planet has some protuberances, which are tangent with curve edge of outside polygon in the middle points.

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4. The planet is surrounded by many satellites. Each satellite is surrounded by further satellites, and so on.

Fig.13 is the DRM by setting α=-2. Fig.14 is a partially zoomed in image of Fig.13; and so is Fig.15 and Fig.16. We can see the planet is surrounded by many satellites with buds near their boundaries and the satellites are surrounded by further satellites. The image has obvious self-similarity and complexity.

Fig.17 is the M-set generated by escape time algorithm for α=-2 (M-set for other exponent see [Guj91a]). Compared with Fig.13, the border of Fig.17 is the same as curve polygon of DRM. If render the M-set and DRM in one image, we will obtain Fig.18. We can see they are a perfect match except the central planet is partly overlapped. If we increase the escape value of escape time algorithm, the size of central planet will reduce. Under this condition, DRM is the complement set of M-set and they cover the whole complex plane. Therefore, distance ratio iteration method can be used to render "outer" of the M-set.

Based on above discussion, we have the following conjecture:

Conjecture: DRM for α<0 is composed of two parts: outer curve polygon and inner constellation. The curve polygon has |α|+1 edges and is covered by petal structures in outer layer; Inner constellation is composed of a central planet and surrounding satellites. The central planet has |α|+1 protuberances and tangent with curve edge of outside polygon in the middle points. The DRM is the complement of M-set in complex plane.

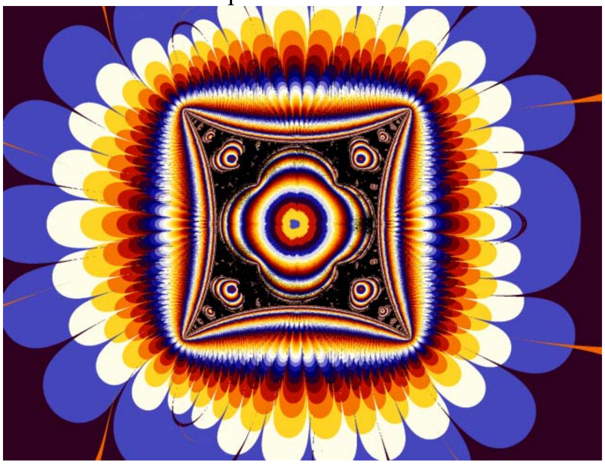

Figure 11. DRM for α=-3

Figure 12. DRM for α=-4

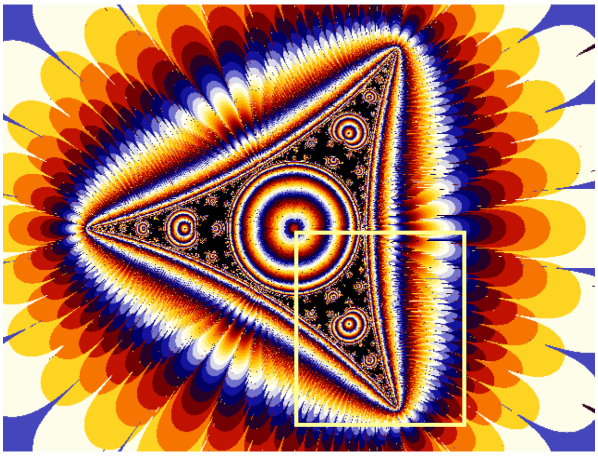

Figure 13. DRM for α=-2

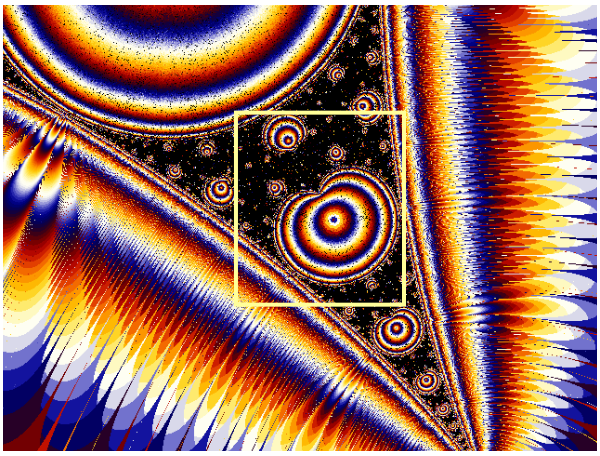

Figure 14. DRM for α=-2 zoomed

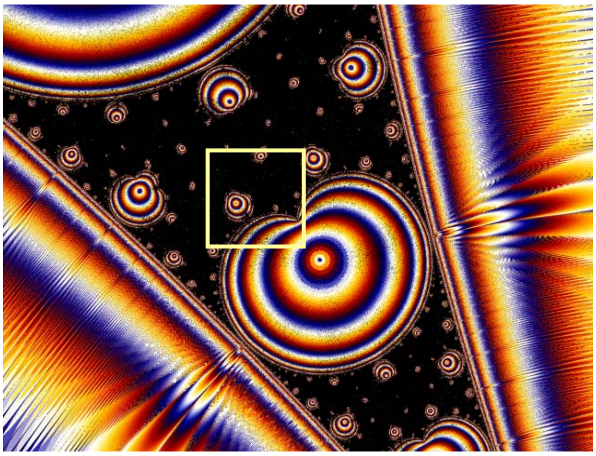

Figure 15. DRM for α=-2 zoomed

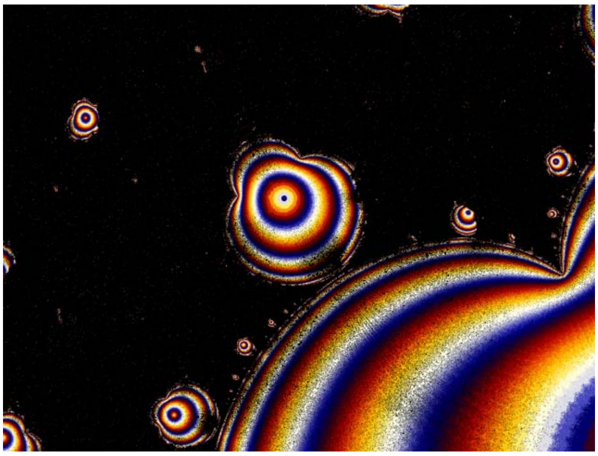

Figure 16. DRM for α=-2 zoomed

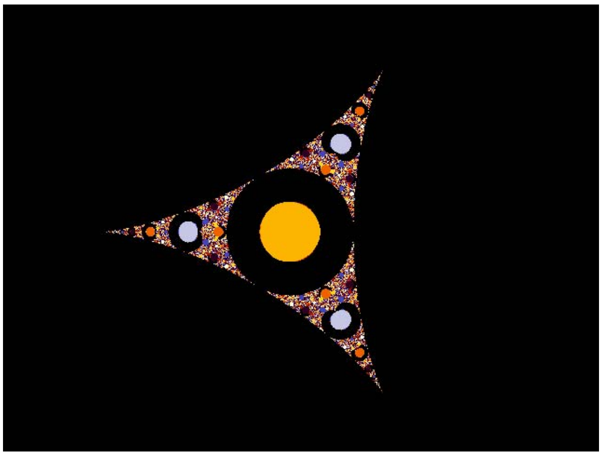

Figure 17. Mandelbrot set for α=-2

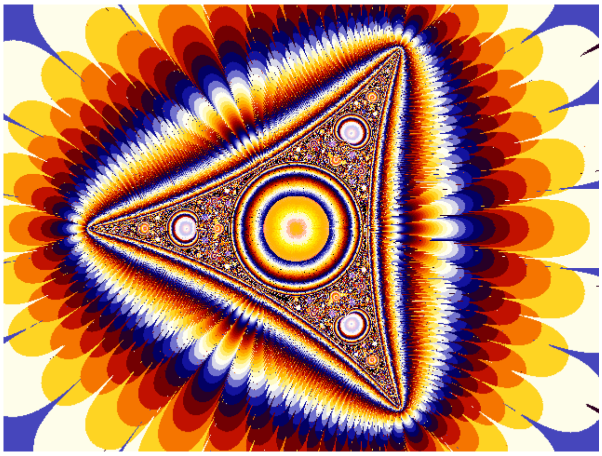

Figure 18. DRM and M-set for α=-2

4. CONCLUSION

This paper presents a new method based on iteration of distance ratio, and renders a generalized Mandelbrot set by using it. From these images we can see there are both difference and relation between DRM and M-set. When $\alpha>1$, DRM can generate the convergent region of mapping and be used as a method to render inner structure of M-set. When $0<\alpha<1$, distance ratio iteration method can render complex structure which escape time algorithm is not able to generate. When $\alpha<0$, DRM is the "outer" region and complement set of M-set, the two sets cover the whole complex plane.

This paper discusses only DRM of $f(z)=z^{\alpha}+c$. The rendering method can be used for other analytic mappings, such as trigonometric function, 3x+1 fractal [Pej04a]. Therefore, it is a new way to construct fractal images.

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