A New Hilbert Invariant For A Pattern Of Points Under Affine Transformation

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ABSTRACT

This paper presents a new Hilbert Invariant for a pattern of points in an image. The Hilbert invariant is invariant under any affine transformation (translation, scaling, rotation and shearing). It is constructed based on the Hilbert transform. Hilbert transform is originally used for generating the imaginary part from the real part of a continuous or discrete complex signal to recover the original one in signal processing. In this paper, the real part is a discrete signal formed by a sequence of coordinates of image points. The Hilbert transformed signal is then used to construct a relative and an absolute affine invariant.

Keywords

Affine transformation, Image points, Hilbert transform, Hilbert invariant

1. INTRODUCTION

The Hilbert transformation is a process performed on a real continuous time-domain signal $u(t)$ yielding a new signal $v(t)$ to generate an analytic complex-valued signal $\psi(t) = u(t) + jv(t)$. Given a time function $u(t)$, the Hilbert transform is [Han96a]:

$$H[s] = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{u(t)}{s-t}\,dt; \quad -\infty < t < \infty \ldots(1)$$

The variable s here is a time variable, so the Hilbert transform of a time function is another time function.

The following section presents the derivation of Hilbert invariant based on the Hilbert transform of the coordinates of image points.

2. AFFINE HILBERT INVARIANT

Given a sequence $x_R[n]$ of finite length N taken to be the real part of a complex sequence. Then, the imaginary part $x_I[n]$ of finite length N is defined as the circular convolution of $x_R[n]$ with the impulse response $h[n]$ of the Hilbert transform [Joh99a]:

WSCG 2005 POSTERS proceedings, ISBN 80-903100-8-7
WSCG'2005, January 31-February 4, 2005
Plzen, Czech Republic.

$$x_I[k] = x_R[k] \otimes h[k] = \sum_{m=0}^{N-1} x_R[m]h[k-m],\ 0 \le k \le N-1 \ldots(2)$$

Now let $x_R[k]$ and $y_R[k]$ be two sequences of finite length N which are the x-coordinates and y-coordinates respectively of the ordered point set:

$$x_R[k] = [x_0, \ldots\ldots, x_{N-1}],\ y_R[k] = [y_0, \ldots\ldots, y_{N-1}]$$

Then $$x_I[k] = \sum_{m=0}^{N-1} x_R[m]h[k-m] \ldots (3)$$

$$y_I[k] = \sum_{m=0}^{N-1} y_R[m]h[k-m] \ldots (4)$$

Let $A = \begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ 0 & 0 & 1 \end{bmatrix}$ be an affine transformation and $x'_R[k]$ and $y'_R[k]$ be two sequences of finite length N which are the x-coordinates and y-coordinates respectively of the affine transformed ordered point set:

$$x'_R[k] = [x'_0, \ldots\ldots, x'_{N-1}],\ y'_R[k] = [y'_0, \ldots\ldots, y'_{N-1}]$$

where $\begin{bmatrix} x'_i \\ y'_i \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}\begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}, \quad i = 0, \ldots, N-1$.

$$\begin{aligned} \therefore x'_I[k] &= \sum_{m=0}^{N-1} x'_R[m]h[k-m] \\ &= \sum_{m=0}^{N-1} \left[r_{11}x_R[m] + r_{12}y_R[m] + t_1\right]h[k-m] \\ &= r_{11}(x_I[k]) + r_{12}(y_I[k]) + t_1\sum_{m=0}^{N-1} h[k-m] \quad \ldots(5) \end{aligned}$$

It can be shown that in both odd and even N cases:

$$\sum_{m=0}^{N-1} h[k-m] = 0 \quad \text{for } 0 \le k \le N-1 \ ...(6)$$

Hence, substituting (6) into (5):

$$x_I'[k] = r_{11}(x_I[k]) + r_{12}(y_I[k]) \ ...(7)$$

Similarly, $\quad y_I'[k] = r_{21}(x_I[k]) + r_{22}(y_I[k]) \ \ldots \ (8)$

Combining (7) & (8), $\begin{bmatrix} x_I'[k] \\ y_I'[k] \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} x_I[k] \\ y_I[k] \end{bmatrix} ...(9)$

Therefore, for $0 \le k_1, k_2 \le N-1$,

$$\begin{bmatrix} x_I'[k_1] & x_I'[k_2] \\ y_I'[k_1] & y_I'[k_2] \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} x_I[k_1] & x_I[k_2] \\ y_I[k_1] & y_I[k_2] \end{bmatrix} \ \ldots \ (10)$$

Taking determinant on both sides:

$$\begin{vmatrix} x_I'[k_1] & x_I'[k_2] \\ y_I'[k_1] & y_I'[k_2] \end{vmatrix} = \alpha \begin{vmatrix} x_I[k_1] & x_I[k_2] \\ y_I[k_1] & y_I[k_2] \end{vmatrix} ...(11), \ \alpha = \begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix}$$

Equation (11) is a relative affine invariant. An absolute affine invariant can be constructed by taking the ratio of two relative affine invariants:

$$HI = \frac{\begin{vmatrix} x_I'[k_1] & x_I'[k_2] \\ y_I'[k_1] & y_I'[k_2] \end{vmatrix}}{\begin{vmatrix} x_I'[k_3] & x_I'[k_4] \\ y_I'[k_3] & y_I'[k_4] \end{vmatrix}} = \frac{\begin{vmatrix} x_I[k_1] & x_I[k_2] \\ y_I[k_1] & y_I[k_2] \end{vmatrix}}{\begin{vmatrix} x_I[k_3] & x_I[k_4] \\ y_I[k_3] & y_I[k_4] \end{vmatrix}} \ \ldots(12)$$

for $0 \le k_1, k_2, k_3, k_4 \le N-1$, which is the Hilbert Invariant.

3. DISCUSSION

Given an input image of a set of ordered points of an oriented polygon extracted from the contour of an object after edge detection and thinning, we apply the Hilbert transform to them. As Hilbert invariant is invariant to the translation component of an affine transformation (see (7), (8)), there is no need to move the coordinate system to the area center. Let (x_i, y_i), $i = 0,\ldots,N\text{-}1$ be the coordinates of the N Hilbert transformed points, where $(x_0, y_0) = (x_N, y_N)$. We calculate the Hilbert invariant for each point along the polygon. Finally, a vector of invariants is constructed to represent the object for further processes such as recognition or matching.

Since area ratio is one of the commonly used affine invariants for matching, the noise performance of the area ratio is compared with the Hilbert invariant.

Given $0 \le k_1, k_2, k_3, k_4, k_5, k_6 \le N-1$, the area ratio Δ_{125}:Δ_{346} is defined as:

$$\frac{\begin{vmatrix} x_R'[k_1]-x_R'[k_5] & x_R'[k_2]-x_R'[k_5] \\ y_R'[k_1]-y_R'[k_5] & y_R'[k_2]-y_R'[k_5] \end{vmatrix}}{\begin{vmatrix} x_R'[k_3]-x_R'[k_6] & x_R'[k_4]-x_R'[k_6] \\ y_R'[k_3]-y_R'[k_6] & y_R'[k_4]-y_R'[k_6] \end{vmatrix}} = \frac{\begin{vmatrix} x_R[k_1]-x_R[k_5] & x_R[k_2]-x_R[k_5] \\ y_R[k_1]-y_R[k_5] & y_R[k_2]-y_R[k_5] \end{vmatrix}}{\begin{vmatrix} x_R[k_3]-x_R[k_6] & x_R[k_4]-x_R[k_6] \\ y_R[k_3]-y_R[k_6] & y_R[k_4]-y_R[k_6] \end{vmatrix}}$$

To have a better comparison, the Hilbert invariant defined in (12) is modified as follow:

$$\frac{\begin{vmatrix} x_I'[k_1]-x_I'[k_5] & x_I'[k_2]-x_I'[k_5] \\ y_I'[k_1]-y_I'[k_5] & y_I'[k_2]-y_I'[k_5] \end{vmatrix}}{\begin{vmatrix} x_I'[k_3]-x_I'[k_6] & x_I'[k_4]-x_I'[k_6] \\ y_I'[k_3]-y_I'[k_6] & y_I'[k_4]-y_I'[k_6] \end{vmatrix}} = \frac{\begin{vmatrix} x_I[k_1]-x_I[k_5] & x_I[k_2]-x_I[k_5] \\ y_I[k_1]-y_I[k_5] & y_I[k_2]-y_I[k_5] \end{vmatrix}}{\begin{vmatrix} x_I[k_3]-x_I[k_6] & x_I[k_4]-x_I[k_6] \\ y_I[k_3]-y_I[k_6] & y_I[k_4]-y_I[k_6] \end{vmatrix}}$$

Let $k_1 = k_3 = i$, $k_2 = i$+1, $k_4 = i$+2, $k_5 = k_6 = i$+3. Both invariants are then tested on 50 sets of 100 randomly generated points. Gaussian noise is added to each coordinate of the points. The standard deviation of noise is from 0.2 to 3 with 0.2 increments. The error percentage of an invariant for each point in each set is calculated and the average error percentage is found. The following figure shows the comparison result:

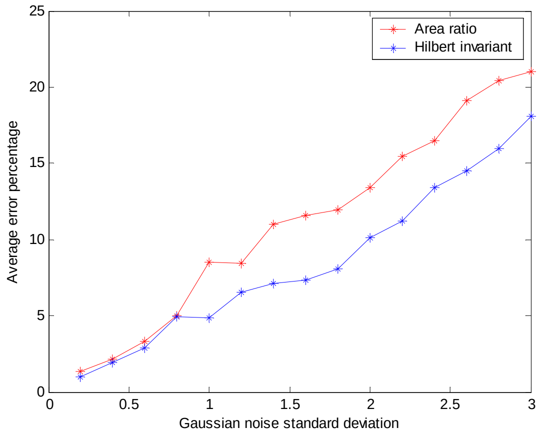

Figure 1. Comparison of robustness to noise.

From the figure, it can be seen that the performances of both invariants are similar in a low noise condition, but the Hilbert invariant performs better in a higher noise condition, so the Hilbert invariant is more robust to noise.

4. CONCLUSION

A new Hilbert Invariant for a pattern of points in an image is constructed based on the Hilbert transform. The Hilbert transformed sequence of coordinates of image points is then used to construct a relative and an absolute affine invariant.

5. ACKNOWLEDGMENTS

The work described in this paper was supported by a grant from the Research Grants Council of Hong Kong SAR, China (Project No.HKU7058/02E) and partially by CRCG of the University of Hong Kong.

6. REFERENCES

[Han96a] Stefan L. Hahn, “Hilbert transforms in signal processing”, Artech House publishers, Boston, London, 1996.

[Joh99a] Mathias Johansson, “The Hilbert transform”, master thesis, Mathematics/Applied Mathematics, Vaxjo University, 1999.